More than one commodity: Spot markets

So far we've assumed that there is only one commodity at each period and in each state. Now let's redefine the model for multiple commodities. Let C denote both the set and the number of commodities, indexed by $c \in C$.

The presence of multiple commodities at $t = 0$ (*i.e.*, today) requires nothing that we haven't already done: we simply replace the single consumption variable $x_0 \in \mathbb{R}_+$ with the bundle $\mathbf{x}_0 = (x_{0c})_{c \in C} \in \mathbb{R}^C_+$, and add a price-list $\mathbf{p}_0 = (p_{0c})_{c \in C} \in \mathbb{R}^C_+$ of today's prices. But the presence of multiple commodities tomorrow introduces something new: spot markets and spot prices.

With only one commodity, no markets would be open tomorrow — there would be nothing to trade for that single commodity — so the only economic activity tomorrow would be delivery and consumption of the single commodity (contingent on which state has occurred). But if *more* than one commodity will be available when tomorrow arrives, we should expect that markets for the commodities will exist and trade will take place. The markets will be **spot markets**, in which transactions *and* delivery will take place "on the spot" — at the **spot prices** that are current tomorrow.

Of course, in the Arrow-Debreu contingent claims model these spot markets don't exist: all transactions take place today, at today's contingent-claim prices, for delivery tomorrow, contingent upon which state of the world has occurred. Therefore the elements of the Arrow-Debreu model with C commodities are:

Consumption plans: $\mathbf{x_0} \in \mathbb{R}^C_+$ and $\mathbf{x_1} \in \mathbb{R}^{S \times C}_+$.

Prices: $\mathbf{p_0} \in \mathbb{R}^C_+$ and $\mathbf{p_1} \in \mathbb{R}^{S \times C}_+$.

The consumer's budget constraint: $\mathbf{p_0} \cdot \mathbf{x_0} + \mathbf{p_1} \cdot \mathbf{x_1} \leqq \mathbf{p_0} \cdot \mathring{\mathbf{x}}_\mathbf{0} + \mathbf{p_1} \cdot \mathring{\mathbf{x}}_\mathbf{1}$.

If we let $l = C + SC$, we simply have the standard Walrasian model: l markets, with prices $\mathbf{p} \in \mathbb{R}^l_+$ and consumers' plans (bundles) $\mathbf{x} \in \mathbb{R}^l_+$, and with the single budget constraint $\mathbf{p} \cdot \mathbf{x} \leqq \mathbf{p} \cdot \mathring{\mathbf{x}}$.

The elements of the Arrow securities-and-spot-markets model are:

Consumption plans: $\mathbf{x_0} \in \mathbb{R}^C_+$ and $\mathbf{x_1} \in \mathbb{R}^{S \times C}_+$.

Commodity prices today: $\mathbf{p_0} \in \mathbb{R}^C_+$.

Securities: $\mathbf{d}_k \in \mathbb{R}^S, \quad k = 1, \ldots, K$.

Security prices today: $\mathbf{q} \in \mathbb{R}^K_+$.

Spot prices tomorrow: $\overline{\mathbf{p}} \in \mathbb{R}^{S \times C}_+$ — *i.e.*, $\overline{\mathbf{p}}_s \in \mathbb{R}^C_+, \quad \forall s \in S$.

Security holdings (portfolios): $\mathbf{y} \in \mathbb{R}^K$.

Budget constraint today: $\mathbf{p_0} \cdot \mathbf{x_0} + \mathbf{q} \cdot \mathbf{y} \leqq \mathbf{p_0} \cdot \mathring{\mathbf{x}}_\mathbf{0}$.

Budget constraints tomorrow: $\overline{\mathbf{p}}_s \cdot \mathbf{x_s} \leqq \overline{\mathbf{p}}_s \cdot \mathring{\mathbf{x}}_s + \sum_{k=1}^K d_{sk} y_k, \quad \forall s \in S$.

Arrow Securities with Multiple Commodities

Arrow-Debreu (Contingent Claims):

	One Good	Multiple Goods
Consumption plans:	$x_0 \in \mathbb{R}_+,\ (x_s)_s \in \mathbb{R}^S_+$	$x_0 \in \mathbb{R}^C_+,\ (x_s)_s \in \mathbb{R}^{S\times C}_+$
Prices:	$[p_0 \equiv 1]$ $p_0 \in \mathbb{R}_+,\ (p_s)_s \in \mathbb{R}^S_+$	$p_0 \in \mathbb{R}^C_+,\ (p_s)_s \in \mathbb{R}^{S\times C}_+$

Budget constraint:

One good:

$$x_0 + p\cdot x \leqq \mathring{x}_0 + p\cdot \mathring{x}$$

$$x_0 + \sum_{s\in S} p_s x_s \leqq \mathring{x}_0 + \sum_{s\in S} p_s \mathring{x}_s$$

Multiple goods:

$$p_0\cdot x_0 + \sum_{s\in S} p_s\cdot x_s \leqq p_0\cdot\mathring{x}_0 + \sum_{s\in S} p_s \mathring{x}_s$$

$$\sum_{c\in C} p_{0c} x_{0c} + \sum_{s\in S}\sum_{c\in C} p_{sc} x_{sc} \leqq \sum_{c\in C} p_{0c}\mathring{x}_{0c} + \sum_{s}\sum_{c} p_{sc}\mathring{x}_{sc}$$

Arrow (Securities & Spot Markets):

	One Good	Multiple Goods
Consumption plans:	$x_0 \in \mathbb{R}_+,\ (x_s)_s \in \mathbb{R}^S_+$	$x_0 \in \mathbb{R}^C_+,\ (x_s)_s \in \mathbb{R}^{S\times C}_+$
Commodity prices today:	$p_0 \in \mathbb{R}_+$	$p_0 \in \mathbb{R}^C_+$
Securities:	$d_k \in \mathbb{R}^S$	$d_k \in \mathbb{R}^S$ $(k = 1,\dots,K)$
Security prices today:	$q \in \mathbb{R}^K_+$	$q \in \mathbb{R}^K_+$
Security holdings:	$y \in \mathbb{R}^K$	$y \in \mathbb{R}^K$
Spot prices tomorrow:	NA	$\bar{p}_s \in \mathbb{R}^C_+$ $(s = 1,\dots,S)$

Budget constraints:

	One Good	Multiple Goods
Today:	$x_0 + q\cdot y \leqq \mathring{x}_0$	$p_0\cdot x_0 + q\cdot y \leqq p_0\cdot\mathring{x}_0$
Tomorrow:	$x_s \leqq \mathring{x}_s + \sum_{k=1}^{K} y_k d_{sk}$	$\bar{p}_s\cdot x_s \leqq \bar{p}_s\cdot\mathring{x}_s + \sum_{1}^{K} y_k d_{sk}$

A TYPICAL CONSUMER, NOT NECESSARILY IN EQUILIBRIUM

EXAMPLE: 3 GOODS, 2 STATES (cf. DUFFIE & SONNENSCHEIN)

$\bar{p}_R = (\$1, \$1, \$1)$, $\bar{p}_H = (\$3, \$2, \$1)$ ← EXPECTED SPOT MARKET PRICES

$z_R = (2, 2, 6)$, $z_H = (1, 1, 1)$ ← TARGET STATE-CONTINGENT CONSUMPTION PLANS

$\xi_R = \$2 + \$2 + \$6 = \$10$, $\xi_H = \$3 + \$2 + \$1 = \$6$. ← RETURNS REQUIRED FOR TARGET PLAN

$d_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $d_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\xi = \begin{bmatrix} \$10 \\ \$6 \end{bmatrix}$ ← TARGET STATE-CONTINGENT RETURNS

TO OBTAIN THE TARGET ξ VIA HOLDINGS $y = (y_1, y_2)$ OF THE TWO SECURITIES:

SOLVE $Dy = \xi$; i.e., $\begin{bmatrix} 3 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$.

WE HAVE THE SOLUTION $y = (y_1, y_2) = (4, -1)$.

CHECKING:

$$Dy = \begin{bmatrix} 3 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 12-2 \\ 12-6 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}.$$

IF $s = R$ OCCURS:

CONSUMER HAS \$10, SPOT MARKET BUDGET CONSTRAINT IS $\bar{p}_R \cdot x_R \leqq \$10$

i.e., $\$1 x_{R1} + \$1 x_{R2} + \$1 x_{R3} \leqq \$10$

HE CHOOSES $(x_{R1}, x_{R2}, x_{R3}) = (2, 2, 6)$, WHICH COSTS \$10.

SIMILARLY, IF STATE $s = H$ OCCURS:

$\bar{p}_H \cdot x_H \leqq \$6$; $(\$3)(1) + (\$2)(1) + (\$1)(1) = \$6$.

TODAY'S CHOICE AND CONSTRAINT:

$$(x_0, y) = (x_{01}, x_{02}, x_{03} ; y_1, y_2)$$

$$\text{s.t. } p_{01}x_{01} + p_{02}x_{02} + p_{03}x_{03} + q_1 y_1 + q_2 y_2 \leqq \sum_{c=1}^{3} p_{0c}\overset{o}{x}_{0c} .$$

$$(BC_0) \qquad \hookrightarrow p_0 \cdot x_0 + q \cdot y \leqq p_0 \cdot \overset{o}{x}_0$$

TOMORROW'S CHOICES AND CONSTRAINTS:

IF $s = R$: (x_{R1}, x_{R2}, x_{R3})

$$(BC_R) \qquad \text{s.t. } \bar{p}_{R1}x_{R1} + \bar{p}_{R2}x_{R2} + \bar{p}_{R3}x_{R3} \leqq \bar{p}_R \cdot \overset{o}{x}_R + y_1 d_{R1} + y_2 d_{R2}$$

IF $s = H$: (x_{H1}, x_{H2}, x_{H3})

$$(BC_H) \qquad \text{s.t. } \bar{p}_H \cdot x_H \leqq \bar{p}_H \cdot \overset{o}{x}_H + y_1 d_{H1} + y_2 d_{H2}$$

(A) BUT HE CAN MAKE HIS DECISIONS/PLANS ALL TODAY, EVEN THOUGH SPOT TRANSACTIONS WILL BE CARRIED OUT TOMORROW:

$$\text{CHOOSE } (x_0, y, x_R, x_H) \in \mathbb{R}^3_+ \times \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^3$$

s.t.

$$(BC_0) \qquad p_0 \cdot x_0 + q \cdot y \leqq p_0 \cdot \overset{o}{x}_0$$

$$(BC_R) \qquad \bar{p}_R \cdot x_R \leqq \bar{p}_R \cdot \overset{o}{x}_R + \sum_{k \in K} d_{Rk} y_k$$

$$(BC_H) \qquad \bar{p}_H \cdot x_H \leqq \bar{p}_H \cdot \overset{o}{x}_H + \sum_{k=1}^{K} y_k d_{Hk} .$$

(i.e., ALL THE CONSTRAINTS ABOVE).

IN THIS EXAMPLE:

ARROW-DEBREU (CONTINGENT CLAIMS):

MARKETS OPEN TODAY
- 3 MARKETS FOR TODAY'S GOODS
- 3 MARKETS FOR STATE R GOODS
- 3 MARKETS FOR STATE H GOODS

9 MARKETS AND PRICES TODAY

WITH SECURITIES:

OPEN TODAY
- 3 MARKETS FOR TODAY'S GOODS
- 2 SECURITIES MARKETS

OPEN TOMORROW 3 MARKETS FOR STATE s GOODS (THE s THAT OCCURS)

8 MARKETS AND PRICES (5 TODAY, 3 TOMORROW)

THE ONLY MARKETS THAT OPEN TOMORROW ARE FOR THE STATE s THAT ACTUALLY OCCURS.

$C=3$ AND $S=2$ IN THE EXAMPLE

ARROW-DEBREU: $SC=6$ $C+SC=9$ ALL TODAY

SECURITIES: $S+C=5$ $C+S+C=8$ $C+S=5$ TODAY

$C=3$ TOMORROW

MARKETS TO TAKE CARE OF TOMORROW

ADD C MARKETS FOR TODAY'S COMMODITIES

SECURITIES ECONOMIZE ON MARKETS.

Securities Economize on Markets

Assume $K = S$, the minimum number of securities necessary and sufficient to achieve Pareto efficiency.

Markets that must open:

Arrow-Debreu (Contingent Claims): $\underbrace{C + SC}_{\text{all today}}$

Arrow (Securities and Spot Markets): $\underbrace{C + S}_{\text{today}} + \underset{\uparrow \text{ tomorrow}}{C}$

Comparison:

Arrow-Debreu

$S = K$

Arrow

S	C	SC	$S+C$
2	2	4	4
3	100	300	103
100	2	200	102
1000	200	200,000	1200

WHAT HAPPENS WHEN TOMORROW ARRIVES?

SUPPOSE THE STATE THAT OCCURS IS $s = R$.

THE CONSUMER'S BUDGET CONSTRAINT IN OUR MODEL IS

$$\bar{p}_{R1}x_{R1} + \bar{p}_{R2}x_{R2} + \bar{p}_{R3}x_{R3} \leqq \bar{p}_{R1}\mathring{x}_{R1} + \bar{p}_{R2}\mathring{x}_{R2} + \bar{p}_{R3}\mathring{x}_{R3} + \sum_{k\in K} d_{Rk}y_k .$$

BUT THE PRICES $\bar{p}_{R\ell}$ ARE *TODAY'S* EXPECTED SPOT PRICES THAT WILL OCCUR *TOMORROW*. AN EQUILIBRIUM IS A *RATIONAL EXPECTATIONS EQUILIBRIUM*: TODAY'S EXPECTED PRICES $\bar{p}_{R\ell}$ TURN OUT TOMORROW TO BE THE PRICES THAT CLEAR THE MARKETS.

SOME PROBLEMS:

(1) EQUILIBRIUM REQUIRES THAT EVERYONE'S EXPECTATIONS ARE THE SAME — THE CORRECT EXPECTATIONS.

(2) THERE ARE NO FORCES TODAY TO DRIVE EXPECTED PRICES TO THEIR EQUILIBRIUM VALUES.

(3) MULTIPLE EQUILIBRIA ARE MUCH MORE LIKELY IN THIS CONTEXT.

(4) DISEQUILIBRIUM TYPICALLY INVOLVES SOME INDIVIDUALS BECOMING BANKRUPT TOMORROW. BUT THIS DOESN'T DRIVE TODAY'S EXPECTATIONS TOWARD EQUILIBRIUM. (A CONSUMER'S PLAN TODAY IS BASED ON HIS EXPECTATION OF TOMORROW'S PRICES.)