Arrow's Pricing Formula for Securities

Let S be a finite set of states of the world and let N be the index set for a finite set of consumers. Generic elements of S and N are denoted by $s \in S$ and $i \in N$. In a slight abuse of notation, we also use S and N to denote the number of elements in S and N. We assume that there is only one good; each consumer $i \in N$ is endowed with $\mathring{x}_0^i$ units of the good today and with $\mathring{x}_s^i$ units in state s tomorrow. It's convenient to think of the good as dollars. Let $\mathring{\mathbf{x}}^i = (\mathring{x}_1^i, \ldots, \mathring{x}_S^i)$. A *consumption plan* for consumer i is a $(1+S)$-tuple $(x_0^i, \mathbf{x}^i) \in \mathbb{R}_+^{1+S}$, and an *allocation* is an N-tuple of plans, $(x_0^i, \mathbf{x}^i)_{i \in N}$. Each consumer evaluates consumption plans according to a utility function $u^i : \mathbb{R}_+^{1+S} \to \mathbb{R}$. The economy is fully described by the set S of states and by the N-tuple $(u^i, \mathring{x}_0^i, \mathring{\mathbf{x}}^i)_{i \in N}$ of consumers.

A set of *securities* for this economy is an $S \times K$ matrix D. Each column of D is the $S \times 1$ *returns vector* or *dividends vector* of one of the securities: the element d_{sk} specifies how many dollars one unit of security k will return tomorrow if state s occurs. Note that d_{sk} may be positive, zero, or negative. The K columns of D are thus the K securities. Consumers purchase or sell units of the securities today and hold them until tomorrow, when one of the states $s \in S$ is realized and each security k returns d_{sk} dollars for every unit of the security a consumer owns. We denote by y_k^i the number of units of security k purchased by consumer i; y_k^i may be positive, zero, or negative. Consumer i's *portfolio* is the K-tuple $(y_1^i, \ldots, y_K^i)$, which we denote by $\mathbf{y}^i$. Note that if consumer i purchases the portfolio $\mathbf{y}^i$, then his vector of state-contingent returns will be the S-tuple $D\mathbf{y}^i$. (It's most convenient here to write $\mathbf{y}^i$ and $D\mathbf{y}^i$ as $K \times 1$ and $S \times 1$ column vectors.) We denote the price of security k by q_k, and we write $\mathbf{q} = (q_1, \ldots, q_K)$.

Definition: An **equilibrium** of the securities markets defined by the matrix D is a $(K + NK + N(1+S))$-tuple $(\mathbf{q}, (\mathbf{y}^i)_{i \in N}, (x_0^i, \mathbf{x}^i)_{i \in N}) \in \mathbb{R}_+^K \times \mathbb{R}^{NK} \times \mathbb{R}_+^{N(1+S)}$ that satisfies the utility-maximization and market-clearing conditions:

(U-M) $\forall i \in N : (\mathbf{y}^i, x_0^i, \mathbf{x}^i)$ maximizes $u^i(x_0^i, \mathbf{x}^i)$ subject to the constraints

$$x_0^i + \mathbf{q} \cdot \mathbf{y}^i \leqq \mathring{x}_0^i \quad \text{and}$$

$$x_s^i \leqq \mathring{x}_s^i + \textstyle\sum_{k=1}^K d_{sk} y_k^i, \ \ \forall s \in S, \qquad \textit{i.e.}, \quad \mathbf{x}^i \leqq \mathring{\mathbf{x}}^i + D\mathbf{y}^i$$

(M-C) $\sum_{i=1}^N x_0^i = \sum_{i=1}^N \mathring{x}_0^i$ and $\sum_{i=1}^N y_k^i = 0, \quad k = 1, \ldots, K.$

Examples: Our "Extended Example of Equilibrium Under Uncertainty" contains several examples of securities markets using this model. Part 3 of the example is a market with a single security, a credit instrument such as a saving account or a bond. Part 4 adds a second security, an insurance contract.

Example: Suppose there are only two states, $s = H$ and $s = L$, and one security, which returns a in state H and b in state L. By choosing y, the number of units of the security he will buy at today's security price q, a consumer can vary x_H and x_L, but not independently:

$$\begin{bmatrix} x_H - \mathring{x}_H \\ x_L - \mathring{x}_L \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} y \qquad \text{and} \qquad x_0 = \mathring{x}_0 - qy.$$

Thus, giving up y units of consumption today will only allow him to augment his consumption tomorrow by multiples of (a, b) across the two states.

Now suppose there's a second security, which returns c in state H and d in state L. If (c, d) is a multiple of (a, b), then nothing is gained by the introduction of the second security: choosing amounts y_1 and y_2 of the two securities still augments one's consumption tomorrow only by multiples of (a, b). But if (a, b) and (c, d) are not multiples of one another — *i.e.*, if they're linearly independent — then for any state-contingent consumptions x_H and x_L tomorrow, the equation

$$\begin{bmatrix} x_H - \mathring{x}_H \\ x_L - \mathring{x}_L \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

has a solution (y_1, y_2). Thus, in this case, state-contingent consumption tomorrow can be augmented by any amounts $x_H - \mathring{x}_H$ and $x_L - \mathring{x}_L$ by giving up some amount of consumption today in order to purchase some amounts y_1 and y_2 of the two securities. More securities would not add anything, but would not hurt either: as long as the securities returns matrix has two linearly independent columns (securities), any state-contingent consumptions can be achieved. More generally, with S states, the securities returns matrix D must have S linearly independent columns — *i.e.*, we must have $\mathit{rank}\, D = S$. We could equivalently say that the securities must *span* the space $\mathbb{R}^S$.

It seems intuitive that this spanning condition will be necessary and sufficient to ensure that the securities markets achieve the same outcome as with complete Arrow-Debreu contingent claims markets — that an equilibrium allocation attained via securities markets will coincide with an Arrow-Debreu allocation. We now verify this intuition.

To simplify notation, let's temporarily substitute z_0 for $x_0 - \mathring{x}_0$ and z_s for each $x_s - \mathring{x}_s$. The key to establishing the equivalence of equilibrium outcomes is the individual consumer's budget constraints: we show that if the securities span $\mathbb{R}^S$, then both market structures present the consumer with exactly the same budget sets at their respective equilibrium prices. In our z-notation, the consumer's Arrow-Debreu budget constraint is $z_0 + \mathbf{p} \cdot \mathbf{z} = 0$. We wish to be able to show that at some security prices $\mathbf{q}$ the constraint $z_0 + \mathbf{q} \cdot \mathbf{y} = 0$, together with the fact that $\mathbf{z} = D\mathbf{y}$, makes exactly the same set of $(z_0, \mathbf{z})$'s available as the constraint $z_0 + \mathbf{p} \cdot \mathbf{z} = 0$ does.

The following proposition establishes that this is so if the securities span $\mathbb{R}^S$ and if their prices are related to the contingent claims prices $\mathbf{p}$ according to $\mathbf{q} = \mathbf{p}D$. The proposition then leads to the subsequent theorem which establishes the equivalence between the securities markets equilibrium and the Arrow-Debreu equilibrium.

Proposition: Let $\mathbf{p} \in \mathbb{R}^S$; let D be an $S \times K$ matrix; let $\mathbf{q} = \mathbf{p}D \in \mathbb{R}^K$; and let

$$A = \{(z_0, \mathbf{z}) \in \mathbb{R}^{1+S} \mid z_0 + \mathbf{p} \cdot \mathbf{z} = 0\} \text{ and}$$
$$B = \{(z_0, \mathbf{z}) \in \mathbb{R}^{1+S} \mid \exists \mathbf{y} \in \mathbb{R}^K : z_0 + \mathbf{q} \cdot \mathbf{y} = 0 \text{ and } \mathbf{z} = D\mathbf{y}\}.$$

If $rank\, D = S$, then $A = B$.

Proof:
Note that if $\mathbf{z} = D\mathbf{y}$ then $\mathbf{p} \cdot \mathbf{z} = \mathbf{p} \cdot (D\mathbf{y}) = (\mathbf{p}D) \cdot \mathbf{y} = \mathbf{q} \cdot \mathbf{y}$. We show that $A \subseteq B$ and $B \subseteq A$.

(i) Let $(z_0, \mathbf{z}) \in A$. Since $rank\, D = S$, there is a $\mathbf{y} \in \mathbb{R}^K$ that satisfies $\mathbf{z} = D\mathbf{y}$. Since $z_0 + \mathbf{p} \cdot \mathbf{z} = 0$ (because $(z_0, \mathbf{z}) \in A$) and $\mathbf{p} \cdot \mathbf{z} = \mathbf{q} \cdot \mathbf{y}$ (because $\mathbf{z} = D\mathbf{y}$), we have $z_0 + \mathbf{q} \cdot \mathbf{y} = 0$, and therefore $(z_0, \mathbf{z}) \in B$.

(ii) Let $(z_0, \mathbf{z}) \in B$. Then, according to the definition of B, there is a $\mathbf{y} \in \mathbb{R}^K$ that satisfies both $z_0 + \mathbf{q} \cdot \mathbf{y} = 0$ and $\mathbf{z} = D\mathbf{y}$. Therefore $\mathbf{p} \cdot \mathbf{z} = \mathbf{q} \cdot \mathbf{y}$, and it follows that $z_0 + \mathbf{p} \cdot \mathbf{z} = 0$, and therefore $(z_0, \mathbf{z}) \in A$. ‖

Theorem: Let D be an $S \times K$ securities returns matrix that satisfies $rank\, D = S$, and let $\mathbf{q} = \mathbf{p}D$. If $(\mathbf{p}, (x_0^i, \mathbf{x}^i)_{i \in N})$ is an Arrow-Debreu equilibrium for the economy $E = (S, (u^i, (\mathring{x}_0^i, \mathring{\mathbf{x}}^i))_{i \in N})$, then there is a profile $(\mathbf{y}^i)_{i \in N}$ of portfolios for which $(\mathbf{q}, (\mathbf{y}^i)_{i \in N}, (x_0^i, \mathbf{x}^i)_{i \in N})$ is an equilibrium of the securities markets defined by D for the economy E. Conversely, if $(\mathbf{q}, (\mathbf{y}^i)_{i \in N}, (x_0^i, \mathbf{x}^i)_{i \in N})$ is a securities-markets equilibrium, then $(\mathbf{p}, (x_0^i, \mathbf{x}^i)_{i \in N})$ is an Arrow-Debreu equilibrium for E.

Remark: Note that the allocation $(x_0^i, \mathbf{x}^i)_{i \in N}$ is the same in both equilibria — *i.e.*, everyone's state-contingent consumption is the same in both equilibria.

Proof of the Theorem: This is a simple corollary of the preceding proposition. For each $i \in N$, we let $x_0^i - \mathring{x}_0^i$ and $\mathbf{x}^i - \mathring{\mathbf{x}}^i$ play the roles of z_0 and $\mathbf{z}$ in the proposition. The set A in the proposition is therefore the set of plans $(x_0^i, \mathbf{x}^i)$ available to consumer i — consumer i's budget constraint — at the equilibrium price-list $\mathbf{p}$ in the Arrow-Debreu equilibrium, and the set B is the set of plans available to him at the securities prices $\mathbf{q} = \mathbf{p}D$ in the corresponding securities markets. If $rank\, D = S$, then the two sets of available plans $(x_0^i, \mathbf{x}^i)$ are identical, and the consumer will therefore choose the same plan when facing either price-list. Therefore the utility-maximization and market-clearing conditions are satisfied in one case if and only if they are satisfied in the other case. ‖