The Debreu-Scarf Theorem:
The Core Converges to the Walrasian Allocations

We've shown that any Walrasian equilibrium allocation (any WEA) is in the core, but it's obvious that the converse is far from true: most core allocations are not WEAs for the given initial distribution of goods. (Core allocations *are* Pareto efficient, so the Second Welfare Theorem does tell us that they can be supported as WEAs *if* we first implement some kind of redistribution.) But we saw, at least in an example, that some core allocations — the ones that are "farthest" from being Walrasian — were eliminated as we added consumers to the economy. The more consumers we added, the more allocations we eliminated: the additional consumers provided more opportunity to improve upon any proposed allocation. It seems reasonable to conjecture, then, that when the economy is very large (*i.e.*, when it has very many consumers), the core may consist only of WEAs and allocations very near them — *i.e.*, that core allocations are very nearly WEAs. And that perhaps "in the limit," core allocations *are* Walrasian equilibrium allocations.

As we've seen, merely stating this idea formally is difficult. We'll take the approach that Edgeworth took when he first came up with this idea, and which Debreu and Scarf finally formalized and used to prove the conjecture many decades later — the idea of considering ever-larger replications of a basic economy. In this framework, the theorem we state and prove (for the 2×2 case) says that for any allocation that's not a WEA, if we make the economy large enough (*i.e.*, if we replicate it sufficiently many times), it will be so large that the non-WEA we started with will fail to be in the large economy's core.

Theorem: Let $E = (u^t, \mathring{\mathbf{x}}^t)_{t=1}^T$ be an economy in which each u^t is continuous, strictly quasiconcave, and strictly increasing, and in which $\mathring{\mathbf{x}}_k^t > 0$ for each $t \in T$ and each good $k = 1, \ldots, \ell$. If an allocation $(\mathbf{x}^t)_T \in \mathbb{R}_+^{T\ell}$ is not a Walrasian equilibrium allocation for E, then there is an integer $\widehat{r}$ such that, for all $r \geqq \widehat{r}$, the allocation $r * (\mathbf{x}^t)_T$ is not in the core of the replication economy $r * E$.

Proof: (For the 2×2 case — 2 persons, 2 goods)

(This proof assumes that each u^t is differentiable. This is not essential, but it makes the proof more transparent.)

Suppose that $((\widehat{x}^1, \widehat{y}^1), (\widehat{x}^2, \widehat{y}^2))$, or $(\widehat{x}^t, \widehat{y}^t)_T$ for short, is in the core, but is not a Walrasian equilibrium allocation (a WEA). We will show that if r is large enough, then the r-fold replication of $(\widehat{x}^t, \widehat{y}^t)_T$ — *i.e.*, $r * (\widehat{x}^t, \widehat{y}^t)_T$ — will *not* be in the core of the r-fold replication $r * E$.

First, notice that $(\widehat{x}^t, \widehat{y}^t)_T \neq (\mathring{x}^t, \mathring{y}^t)_T$: we've assumed that $(\widehat{x}^t, \widehat{y}^t)_T$ is in the core, so it is Pareto efficient; but if the endowment allocation is Pareto efficient, then the Second Welfare Theorem would ensure that it's a WEA, and we've assumed that $(\widehat{x}^t, \widehat{y}^t)_T$ is *not* a WEA.

Let L denote the line that passes through, say, $(\widehat{x}^1, \widehat{y}^1)$ and $(\mathring{x}^1, \mathring{y}^1)$, and let $-\tau$ be its slope:

$$\tau = -\frac{\widehat{y}^1 - \mathring{y}^1}{\widehat{x}^1 - \mathring{x}^1} = -\frac{\widehat{y}^2 - \mathring{y}^2}{\widehat{x}^2 - \mathring{x}^2},$$

which is the trading ratio defined by $(\widehat{x}^t, \widehat{y}^t)_T$ and $(\mathring{x}^t, \mathring{y}^t)_T$. [The two traders' trading ratios are equal because $\widehat{x}^1 + \widehat{x}^2 = \mathring{x}$ and $\widehat{y}^1 + \widehat{y}^2 = \mathring{y}$, which follows from the fact that each u^t is increasing.] Wlog, assume that the common MRS at $(\widehat{x}^t, \widehat{y}^t)_T$, denoted σ, satisfies $\sigma < \tau$, and assume that $\widehat{x}^1 > \mathring{x}^1$ and $\widehat{x}^2 < \mathring{x}^2$. [The common MRS exists because $(\widehat{x}^t, \widehat{y}^t)_T$ is Pareto efficient and preferences are quasiconcave.]

Since $\sigma < \tau$, each consumer would gain by giving up some (perhaps only very little) of the x-good in return for the y-good *at the rate* τ, as depicted in Figure 1. If we write

$$\mathbf{z}^t = (z_x^t, z_y^t) = (x^t - \mathring{x}^t,\ y^t - \mathring{y}^t)$$

for type t's net trades, and

$$\widetilde{u}^t(\mathbf{z}^t) := u^t(\mathring{x}^t + z_x^t,\ \mathring{y}^t + z_y^t)$$

for type t's utility from a net trade $\mathbf{z}^t$, then we have

(1) $\qquad \widehat{\mathbf{z}}^1 + \widehat{\mathbf{z}}^2 = (0,\, 0), \quad$ and

(2) $\qquad \widetilde{u}^1(\lambda_1 \widehat{\mathbf{z}}^1) > \widetilde{u}^1(\widehat{\mathbf{z}}^1)$ and $\widetilde{u}^2(\lambda_2 \widehat{\mathbf{z}}^2) > \widetilde{u}^2(\widehat{\mathbf{z}}^2)$

for some $\lambda_1 < 1$ and $\lambda_2 > 1$. We need to construct a coalition made up of α_1 members of type 1 and α_2 members of type 2, and give each member of the coalition the net trade $\lambda_t \widehat{\mathbf{z}}^t$ (depending on the member's type, $t = 1$ or $t = 2$), thereby making each member better off than at $\widehat{\mathbf{z}}^t$. The question is: How can we use the numbers $\lambda_1 < 1$ and $\lambda_2 > 1$ to determine the numbers α_1 and α_2 (which must be integers)?

If each member of the coalition receives the net trade $\widehat{\mathbf{z}}^t$ $(t = 1, 2)$, then the coalition's aggregate net trade will be $\alpha_1\lambda_1\widehat{\mathbf{z}}^1 + \alpha_2\lambda_2\widehat{\mathbf{z}}^2$. That aggregate net trade has to be $(0, 0)$ if the coalition is to implement it unilaterally. Therefore we need to have

$$\alpha_1\lambda_1\widehat{\mathbf{z}}^1 + \alpha_2\lambda_2\widehat{\mathbf{z}}^2 = (0, 0).$$

Since we do have $\widehat{\mathbf{z}}^1 + \widehat{\mathbf{z}}^2 = (0,\, 0)$, it will suffice to have

(3) $\qquad \alpha_1\lambda_1 = \alpha_2\lambda_2; \quad i.e., \ \dfrac{\alpha_2}{\alpha_1} = \dfrac{\lambda_1}{\lambda_2}.$

If λ_1 and λ_2 are rational numbers (each a ratio of integers), then we can choose integers α_1 and α_2 that satisfy (3), and then we let $\widehat{r} = \max\{\alpha_1,\, \alpha_2\}$. And it's clear from (2), together with continuity of each u^t, that we can indeed choose λ_1 and λ_2 to be rational. $\square$

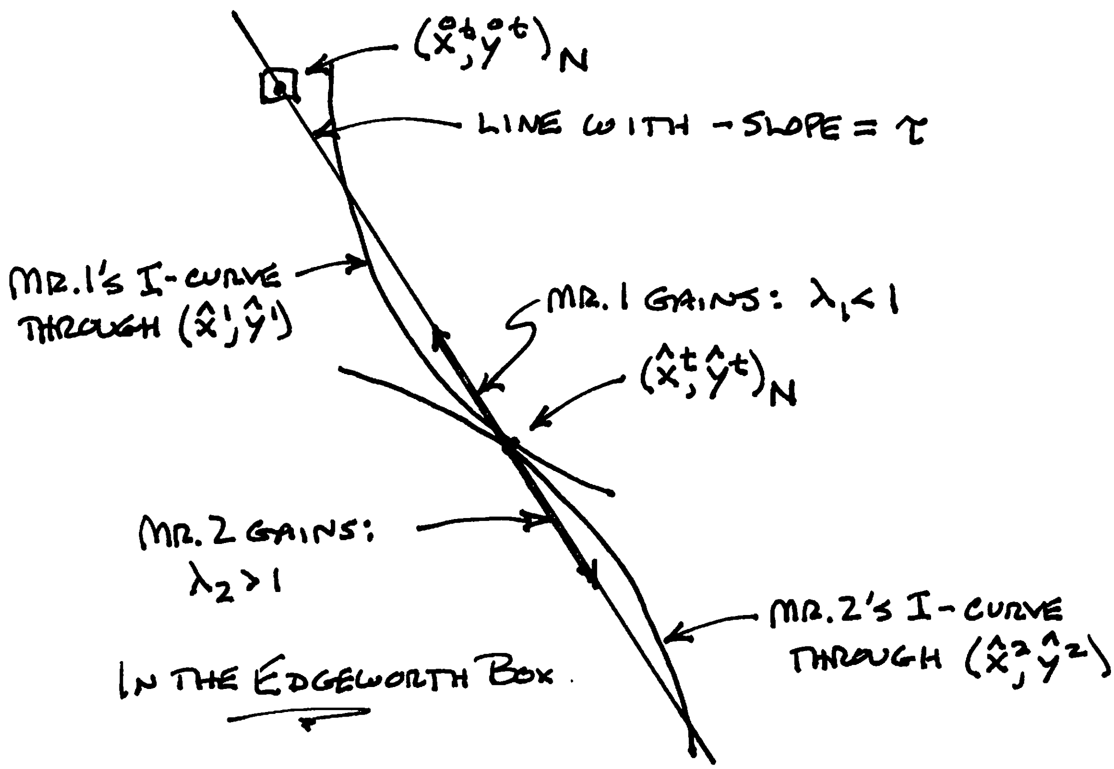

FIGURE 1

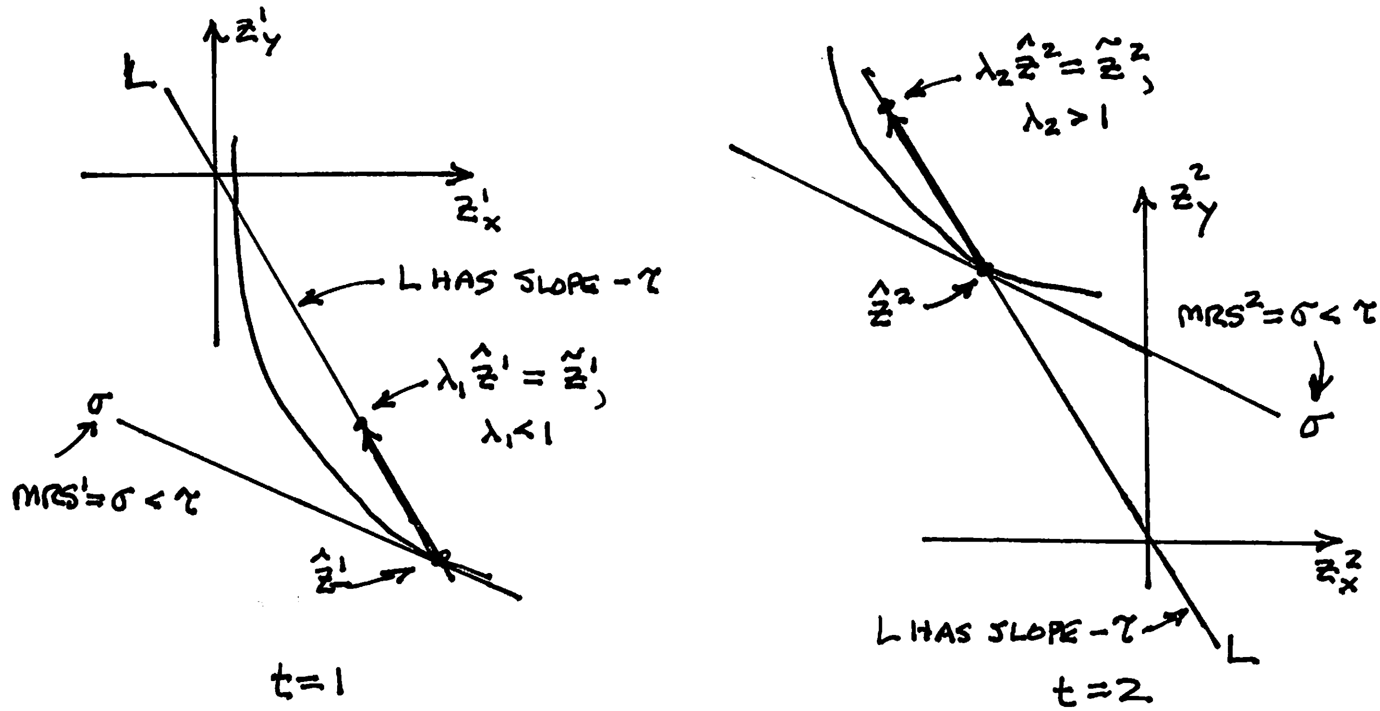

FIGURE 2

THE PIECES OF THE ARGUMENT

(1) $\tau := -\dfrac{\hat{z}_{1y}}{\hat{z}_{1x}} = -\dfrac{\hat{z}_{2y}}{\hat{z}_{2x}}$ OK, BECAUSE (a) $\hat{z}_{1x}, \hat{z}_{2x} \neq 0$ [IF THE ENDOWMENT IS IN THE CORE, IT'S COMPETITIVE], AND (b) $((\hat{x}_i, \hat{y}_i))_N$ USES ALL OF $(\mathring{x}, \mathring{y})$, BECAUSE EACH u_i INCREASING.

(2) $MRS_1 = MRS_2 = \sigma$, SAY BECAUSE $((\hat{x}_i, \hat{y}_i))_N$ IN THE CORE, AND EACH u_i QUASI-CONCAVE.

(3) $\tau \neq \sigma$ OTHERWISE $((\hat{x}_i, \hat{y}_i))_N$ IS COMPETITIVE.

(4) WLOG: $\sigma < \tau$; $\hat{x}_1 > \mathring{x}_1$;
$\therefore$ $\hat{x}_2 < \mathring{x}_2$; $\hat{y}_1 < \mathring{y}_1$, $\hat{y}_2 > \mathring{y}_2$.

(5) $\exists\, \lambda_1 < 1$ AND $\lambda_2 > 1$ s.t.
$\tilde{u}_1(\lambda_1 \hat{z}_1) > \tilde{u}_1(\hat{z}_1)$ AND $\tilde{u}_2(\lambda_2 \hat{z}_2) > \tilde{u}_2(\hat{z}_2)$.

(6) IF A COALITION WITH α_i OF TYPE i $(i=1,2)$ GIVES $\lambda_i \hat{z}_i$ TO EACH MEMBER, ALL ARE BETTER OFF THAN AT $\hat{z}_i$. THIS WILL BE FEASIBLE FOR THE COALITION IF $\alpha_1 \lambda_1 \hat{z}_1 + \alpha_2 \lambda_2 \hat{z}_2 = (0,0)$; i.e., IF $\alpha_1 \lambda_1 = \alpha_2 \lambda_2$ [SINCE $\hat{z}_1 + \hat{z}_2 = (0,0)$].

(7) WE WANT INTEGERS α_1, α_2 s.t. $\dfrac{\alpha_2}{\alpha_1} = \dfrac{\lambda_1}{\lambda_2}$, WHICH WE CAN DO IF λ_1/λ_2 IS RATIONAL. CONTINUITY OF EACH u_i ENSURES THERE WILL BE SUCH λ_i's.

EXAMPLE:

SUPPOSE $\lambda_1 = \frac{8}{9}$ AND $\lambda_2 = \frac{6}{5}$.

THEN $\frac{\lambda_1}{\lambda_2} = \frac{8/9}{6/5} = \frac{40}{54} = \frac{20}{27}$.

SO LET $\alpha_1 = 27$ AND $\alpha_2 = 20$, WHICH REQUIRES $r \geq 27$. THIS YIELDS

$$\begin{aligned} \alpha_1 \lambda_1 \hat{z}_1 + \alpha_2 \lambda_2 \hat{z}_2 &= 24 \hat{z}_1 + 24 \hat{z}_2 \\ &= 24 (\hat{z}_1 + \hat{z}_2) \\ &= 24\,(0,0) \\ &= (0,0), \end{aligned}$$

SO GIVING $\lambda_1 \hat{z}_1$ TO EACH TYPE 1 AND $\lambda_2 \hat{z}_2$ TO EACH TYPE 2

IS UNILATERALLY FEASIBLE FOR A COALITION WITH $\alpha_1 = 27$ TYPE 1'S AND $\alpha_2 = 20$ TYPE 2'S.

CONSEQUENTLY, IF $r \geq 27$ THEN THERE IS A COALITION THAT CAN UNILATERALLY IMPROVE UPON AN ALLOCATION GIVING NET TRADES $\hat{z}_1$ AND $\hat{z}_2$. (ASSUMING λ_1 AND λ_2 SATISFY $\tilde{u}_i(\lambda_i \hat{z}_i) > \tilde{u}_i(\hat{z}_i)$, $i = 1, 2$.)

Another Example:

Suppose $\lambda_1 = \frac{8}{9}$, $\lambda_2 = \sqrt{2} \cong 1.414$

If we want $\frac{\alpha_2}{\alpha_1} = \frac{\lambda_1}{\lambda_2}$, we have $\alpha_2 = \frac{8}{9\sqrt{2}}\alpha_1$,

i.e., $\alpha_1 = \frac{9}{8}\sqrt{2}\,\alpha_2$, and there ~~[illegible]~~ are no <u>integers</u> α_1 and α_2 that satisfy this equation.

[If α_2 is an integer, α_1 is clearly not.]

But there has to be some $\tilde{\lambda}_2$ <u>near</u> λ_2 satisfying

(1) $\tilde{\lambda}_2$ is rational (a ratio of integers),

and (2) type 2's prefer the trade $\tilde{\lambda}_2 \hat{z}_2$ to the proposal, $\hat{z}_2$.

[This $\tilde{\lambda}_2$ may have to be chosen <u>very</u> near λ_2.]

If, for example, $\tilde{\lambda}_2 = 1.4 = \frac{7}{5}$ yields a $\tilde{\lambda}_2 \hat{z}_2$ preferred to $\hat{z}_2$, then we can use

$$\frac{\alpha_2}{\alpha_1} = \frac{8/9}{7/5} = \frac{40}{63} : \quad r \geq 63.$$

But if 1.4 <u>doesn't</u> satisfy $\tilde{\lambda}_2 \hat{z}_2 \succ \hat{z}_2$, perhaps $\tilde{\lambda}_2 = 1.41$ does, etc.

Concluding Remarks:

The way we've modeled large economies is extremely special and unrealistic. An actual economy, if it's very large, isn't going to consist of only a small number of types of consumer, with every consumer being one of these few types. Even if this were a good approximation — even if there were a small number of types and every consumer were very close to one of those types — it would be astonishing if there were also *exactly the same number* of consumers of each type.

Shortly after Debreu and Scarf published their paper on the core convergence theorem, in 1963, Robert Aumann published a paper in which he took a remarkably innovative approach to formulating a model of a large economy in which individual consumers have negligible influence. Aumann modeled a large economy as one with an infinite set of consumers, endowed with a measure in which each individual consumer has measure zero. Within this model, Aumann used essentially the Debreu-Scarf method of proof to show that in a large economy the only core allocations are the Walrasian equilibrium allocations. Aumann's paper — especially the introductory section — is one of the most striking and elegant papers in economics. You should definitely read both the introductory and concluding sections, and make the effort to read the remaining five pages which contain the formal model and proof. The paper is available on the course website, in the Readings section.

After Aumann's paper, a great deal of work was devoted to these ideas over the subsequent two or three decades, in which Aumann's continuum model (and the core equivalence result, and others) was shown to be the limiting case, in a well-defined sense, of large but finite economies.

So what's the significance of the Core Convergence (Debreu-Scarf) Theorem? It tells us that if the economy is sufficiently large that individual consumers are negligible, then whatever institution we use to allocate resources, we will end up with the same outcome we would have attained via markets and prices. Of course, that assumes we have no externalities, consumers have complete information about the prices and the commodities, and consumers are free to "go their own way," using their own resources independently of other consumers. And note that we didn't allow production, which complicates things considerably, largely because of scale phenomena.

The concept of the core is important in contexts other than large economies. For one example, in auction theory, see the paper by Ausubel and Milgrom on the course website, especially Section 5 of the paper.

Exercise: The Core Shrinks Under Replication

We begin with a 2×2 "Edgeworth Box" exchange economy: each consumer has the same preference, described by the utility function $u(x, y) = xy$; Consumer 1 owns the bundle $(\mathring{x}_1, \mathring{y}_1) = (15, 30)$; and Consumer 2 owns the bundle $(\mathring{x}_2, \mathring{y}_2) = (75, 30)$.

(a) Verify that there is a unique Walrasian (competitive) equilibrium, in which the price ratio is $p_x/p_y = 2/3$ and the consumption bundles are $(x_1, y_1) = (30, 20)$ and $(x_2, y_2) = (60, 40)$.

(b) Verify that the Pareto allocations are the ones that allocate the entire resource endowment of $(\mathring{x}, \mathring{y}) = (90, 60)$ and satisfy $y_1/x_1 = y_2/x_2 = 2/3$.

(c) In the Edgeworth Box draw the competitive allocation, the Pareto allocations, and each consumer's budget constraint at the competitive prices. Draw each consumer's indifference curve containing his initial bundle and indicate the core allocations in the diagram.

(d) Verify that the Pareto allocations for which $x_1 < \sqrt{675}$ are not in the core. Note that $\sqrt{675}$ is approximately 26. Similarly, the Pareto allocations for which $x_2 < \sqrt{3375} \approx 58.1$ are not in the core.

(e) Consider a proposed allocation $(\hat{x}_1, \hat{y}_1) = (27, 18)$ and $(\hat{x}_2, \hat{y}_2) = (63, 42)$. Note that each consumer's marginal rate of substitution at the proposal is $2/3$. Verify that the proposal is in the core. Verify that the "trading ratio" τ defined by the proposal is $\tau = 1$. As in our lecture notes on the Debreu-Scarf Theorem, use the "shrinkage factor" $\lambda_1 = 2/3$ and the "expansion factor" $\lambda_2 = 4/3$ to verify that a coalition of just two "Type 1" consumers and one "Type 2" consumer can unilaterally allocate their initial bundles to make all three of them better off than in the proposal. Therefore the proposal is not in the core if there are two or more consumers of each type.

(f) Now consider the proposal $(\hat{x}_1, \hat{y}_1) = (28\frac{1}{2}, 19)$ and $(\hat{x}_2, \hat{y}_2) = (61\frac{1}{2}, 41)$, and use the same λ_1 and λ_2 as in (e) to establish that this proposal too is not in the core if there are two or more consumers of each type.

(g) Now consider the proposal $(\hat{x}_1, \hat{y}_1) = (29, 19\frac{1}{3})$ and $(\hat{x}_2, \hat{y}_2) = (61, 40\frac{2}{3})$, and use the factors $\lambda_1 = 4/5$ and $\lambda_2 = 6/5$ to establish that this proposal is not in the core if there are three or more consumers of each type.