The Intertemporal General Equilibrium Model

The power of the Walrasian general equilibrium model is that it can be applied to so many economic situations by a simple reinterpretation of the elements of the model. An excellent example is the application of the model to intertemporal consumption and production. Intertemporal production is (real) investment.

Recall that when we introduced production into the model, we did it in the simplest case, where there were just two goods — a good used as an input in the production process (also desired for its consumption value), and a second good produced by using the first good as input. In the simplest case there was only one consumer and only one producer. The elements of the model were

$x:$ consumption of the input good; $\mathring{x}$ is the consumer's endowment
$y:$ consumption of the output good; $\mathring{y}$ is the consumer's endowment
$\mathring{x} - x:$ forgone consumption available to be used as input
$z:$ amount of the x-good used as input to the production process
$f:$ production function
$q = f(z):$ amount of output produced
$p_x,\ p_y,\ p_x/p_y:$ prices and relative prices
$p_x x + p_y y:$ value of a consumption bundle (x, y)

Let's convert this model to a two-period consumption-investment model, in which consumption and (real) investment take place today; the investment yields a return tomorrow (an amount of the consumption good); and then consumption takes place again tomorrow. The elements of the model, corresponding to the elements above, are:

$x_0:$ consumption today; $\mathring{x}_0$ is the consumer's endowment today
$x_1:$ consumption tomorrow; $\mathring{x}_1$ is the consumer's endowment tomorrow
$s = \mathring{x}_0 - x_0:$ forgone consumption today (saving)
$z:$ amount of the good invested today
$f:$ function specifying investment yield
$q = f(z):$ yield tomorrow from today's investment
$r,\ \frac{1}{1+r}:$ interest rate and discount rate
$x_0 + \frac{1}{1+r}x_1:$ present value of a consumption plan (x_0, x_1)

Note that this model can accommodate

multiple periods: $(x_0, x_1, \ldots, x_T)$

an infinite horizon: $(x_0, x_1, x_2, \ldots)$

continuous time: $x : \mathbb{R}_+ \to \mathbb{R}_+$.

It can also accommodate multiple goods in each period, by replacing each number x_t in the above lists with a bundle $\mathbf{x}_t \in \mathbb{R}_+^{l_t}$.

Example: There are two consumers and only one good. The good will be available to be consumed in two periods, "today" and "tomorrow." To begin with, let's assume production is not possible, and it's not possible to store the good today for consumption tomorrow: all consumption tomorrow must come from the amount the economy will be endowed with tomorrow. Let (x_{A0}, x_{A1}) denote consumer A's consumption plan: consumption today, x_{A0}, and consumption tomorrow, x_{A1}. Let (x_{B0}, x_{B1}) denote B's consumption plan. Suppose their endowments are $(\mathring{x}_{A0}, \mathring{x}_{A1}) = (48, 0)$ and $(\mathring{x}_{B0}, \mathring{x}_{B1}) = (16, 32)$ and that the consumers both have the same preference, described by the utility function $u(x_0, x_1) = x_0^3 x_1$, which has $MRS = 3(x_1/x_0)$.

It's easy to verify that the Pareto allocations are the ones on the diagonal of the Edgeworth box — the ones that satisfy $x_{i0} = 2x_{i1}$ for each i — and that the efficiency-price ratio is $\sigma_0/\sigma_1 = 3/2$ at all the Pareto allocations. The First Welfare Theorem tells us that if there is an interior equilibrium the equilibrium allocation must be one of these Pareto allocations, and the equilibrium price ratio will therefore be $p_0/p_1 = 3/2$. At prices $p_0 = 3/2$ and $p_1 = 1$, for example, the values of the consumers' endowments would be $w_A = 72$ and $w_B = 56$. The equilibrium allocation is therefore $(x_{A0}, x_{A1}) = (36, 18)$ and $(x_{B0}, x_{B1}) = (28, 14)$ — these are the consumption plans the consumers choose in equilibrium. Note that each consumer's plan has the same market value as his endowment.

At the equilibrium, consumer A is saving 12 units of the good today (he is foregoing 12 units of consumption) in order to receive 18 units in return tomorrow. We clearly want to introduce the idea of an interest rate here: we denote the interest rate by r and we say that A is saving the amount $s_A = 12$ and that he receives $(1 + r)s_A = 18$ units tomorrow in return. Conversely, consumer B is dissaving, or borrowing, today: his saving is the negative number $s_B = -12$ and he has to repay 18 tomorrow — *i.e.*, his endowment tomorrow is augmented by the negative amount $(1 + r)s_B = -18$.

Thus, the equilibrium interest rate in this economy is $1/2 = 50\%$. We could express a consumer's budget equation in any of the following equivalent forms:

$$x_1 = \mathring{x}_1 + (1+r)s \tag{1}$$

$$x_1 = \mathring{x}_1 + (1+r)(\mathring{x}_0 - x_0) \tag{2}$$

$$(1+r)x_0 + x_1 = (1+r)\mathring{x}_0 + \mathring{x}_1 \tag{3}$$

$$x_0 + \frac{1}{1+r}x_1 = \mathring{x}_0 + \frac{1}{1+r}\mathring{x}_1. \tag{4}$$

Equation (4) says that the present value of the consumer's plan (*i.e.*, the present value of his planned consumption stream), has to be equal to the present value of his wealth (his endowment stream), where each stream is discounted at the rate r. Equation (3) expresses the same thing in terms of the "future value" of each stream. Note that in terms of the interest rate, the consumer's marginal condition for utility maximization in the two-period model becomes $MRS = 1 + r$. Note too that we could express the market-clearing equilibrium condition as $s_A + s_B = 0$, or $s_A(r) + s_B(r) = 0$, which we could solve to obtain the equilibrium interest rate had we not already determined the interest rate by applying the First Welfare Theorem and using the efficiency prices we obtained for the Pareto allocations.

Later in the course this intertemporal model will be the foundation for general equilibrium analysis in the presence of uncertainty. Here we extend the example to demonstrate how (real) investment is incorporated into the model by reinterpreting the Walrasian model with production.

Intertemporal Production (Investment): Now suppose that it's possible to invest today's foregone consumption in a process that adds to tomorrow's stock of the consumption good — an intertemporal production process. Let z denote the amount invested today, and let q denote the amount the investment produces tomorrow, according to the production function $q = f(z) = 2z$. Applying the Walrasian model with production, the marginal condition for Pareto efficiency in this economy is $f'(z) = MRS_A = MRS_B$. Therefore the efficiency price-ratio is $\sigma_0/\sigma_1 = 2$ at all the Pareto allocations, and just as before, the First Welfare Theorem tells us that if there is an interior equilibrium the equilibrium allocation must be one of the Pareto allocations and the equilibrium price ratio will be $p_0/p_1 = 2$. In terms of the interest rate, we have $1 + r = 2$, so the equilibrium interest rate is 100%.

Denoting the present value of a consumption stream (x_0, x_1) as $V_0(x_0, x_1)$, we have

$$V_0(x_0, x_1) = x_0 + \frac{1}{1+r} x_1,$$

and the present values of the consumers' endowments are

$$V_0(48, 0) = 48 + \frac{1}{1+r} 0 = 48 \quad \text{and} \quad V_0(16, 32) = 16 + \frac{1}{1+r} 32 = 32.$$

Recall from above that the marginal condition necessary for utility maximization is $MRS = 1 + r$. Combining this with the two consumers' budget constraints,

$$V_0(x_{A0}, x_{A1}) = V_0(48, 0) \quad \text{and} \quad V_0(x_{B0}, x_{B1}) = V_0(16, 32),$$

yields their chosen consumption plans: $(x_{A0}, x_{A1}) = (36, 24)$ and $(x_{B0}, x_{B1}) = (24, 16)$.

An interesting phenomenon shows up in this example. If you calculate the consumers' utilities, first when production/investment is not available, and then when it is available, you'll find that consumer A is better off when production is available, but consumer B is worse off. That is, consumer B is worse off *in the Walrasian equilibrium that includes the production being carried out by one or more price-taking firms, each of which is maximizing its profit in the same input and output markets as the consumers are participating in.*

Exercise: Instead of the no-production outcome or the Walrasian equilibrium outcome, suppose each consumer has access to the production technology f; the consumers first trade without any producer in the market; and then each consumer, starting with his Walrasian equilibrium consumption stream from that no-production market, uses f to unilaterally attain a better consumption stream for himself. Compare the outcomes (prices, production decisions, consumption streams, and welfare) under these three alternative situations. Is any one of the three outcomes Pareto superior to any of the others? Which ones are Pareto optimal? Can you explain why this phenomenon — that consumer B is worse off when the market includes producers — occurs here? What if the decisions are carried out in the reverse order, *i.e.*, if each consumer first unilaterally uses f to alter the endowment with which he will participate in the subsequent Walrasian (price-taking) market?