Existence of Walrasian Equilibrium:
Proof for the Two-Goods Case

Recall our definition of an equilibrium price list for a market net demand function:

Definition: Let $E = ((u^i, \mathring{\mathbf{x}}^i))_{i=1}^n$ be an economy consisting of n consumers, all of whose demand functions $\mathbf{x}^i(\cdot) : \mathbb{R}_+^\ell \to \mathbb{R}^\ell$ are well-defined and single-valued on a subset $\mathcal{P} \subseteq \mathbb{R}_+^\ell$, and let $\mathbf{z}(\cdot) : \mathbb{R}_+^\ell \to \mathbb{R}^\ell$ denote the corresponding market net demand function. A **Walrasian equilibrium** of E is a price-list $\mathbf{p} \in \mathcal{P}$ that satisfies the equilibrium condition

$$\forall k = 1, \ldots, \ell : \quad z_k(\mathbf{p}) \leqq 0 \quad \text{and} \quad z_k(\mathbf{p}) = 0 \text{ if } p_k > 0. \qquad (\mathbf{Clr})$$

If the market demand function $\mathbf{z}(\cdot)$ satisfies Walras's Law, then the first inequality in (**Clr**) — $\forall k = 1, \ldots, \ell : \ z_k(\mathbf{p}) \leqq 0$ — is enough to ensure that $\mathbf{p}$ is an equilibrium price-list.

Remark: If the function $\mathbf{z}(\cdot) : \mathbb{R}_+^\ell \to \mathbb{R}^\ell$ satisfies Walras's Law, and if $\forall k = 1, \ldots, \ell : \ z_k(\mathbf{p}) \leqq 0$, then $\mathbf{p}$ is an equilibrium price-list for $\mathbf{z}(\cdot)$.

Proof: Exercise.

Given any particular economy E, we would like to know whether E actually has a Walrasian equilibrium — *i.e.*, whether a Walrasian equilibrium exists for E. The way we'll proceed is to make some assumptions about market net demand functions, and then we'll show that any function that satisfies these assumptions will have an equilibrium price-list. However, with the tools we have at present, we'll provide a proof only for the case of two goods ($\ell = 2$); we'll give a proof for arbitrary ℓ later in the course.

Theorem: If a function $\mathbf{z}(\cdot) : \mathbb{R}_+^\ell \to \mathbb{R}^\ell$ is continuous and satisfies Walras's Law (WL), then it has an equilibrium price-list — *i.e.*, there is a $\mathbf{p}^* \in \mathbb{R}_+^\ell$ that satisfies (**Clr**).

Proof for $\ell = 2$: Define two functions $\tilde{z}_1$ and $\tilde{z}_2$ on the unit interval $[0, 1]$ as follows (see Figure 1):

$$\text{For each } p_1 \in [0,1] : \quad \tilde{z}_1(p_1) := z_1(p_1, 1 - p_1) \quad \text{and} \quad \tilde{z}_2(p_1) := z_2(p_1, 1 - p_1).$$

By considering only price-lists (p_1, p_2) that satisfy $p_1 + p_2 = 1$, we've converted the functions $\tilde{z}_1(p_1, p_2)$ and $\tilde{z}_2(p_1, p_2)$ into functions of only one variable. We'll show that at least one of the price-lists $\mathbf{p}$ that satisfies $p_1 + p_2 = 1$ also satisfies (**Clr**).

We begin by assuming that neither of the price-lists $\mathbf{p} = (0, 1)$ or $\mathbf{p} = (1, 0)$ satisfies (**Clr**): if either one actually does satisfy (**Clr**), then the theorem's conclusion is established. The fact that neither of these price-lists satisfies (**Clr**) provides some information about the values of the functions $\tilde{z}_1(\cdot)$ and $\tilde{z}_2(\cdot)$ at $p_1 = 0$ and at $p_1 = 1$, as follows:

At $p_1 = 0$, WL yields

$$0z_1(0, 1) + 1z_2(0, 1) = 0, \quad i.e., \; 0\tilde{z}_1(0) + 1\tilde{z}_2(0) = 0.$$

Therefore $\tilde{z}_2(0) = 0$, and since $\mathbf{p} = (0, 1)$ does not satisfy (**Clr**), we must have $\tilde{z}_1(0) > 0$.

Similarly, at $p_1 = 1$, WL yields

$$1z_1(1, 0) + 0z_2(1, 0) = 0\,, \quad i.e., \; 1\tilde{z}_1(1) + 0\tilde{z}_2(1) = 0.$$

Therefore $\tilde{z}_1(1) = 0$, and since $\mathbf{p} = (1, 0)$ does not satisfy (**Clr**), we must have $\tilde{z}_2(1) > 0$.

Summarizing so far, we have

$$\tilde{z}_1(0) > 0, \quad \tilde{z}_2(0) = 0, \quad \tilde{z}_1(1) = 0, \quad \text{and} \quad \tilde{z}_2(1) > 0,$$

as depicted in Figure 1.

Because $\tilde{z}_2(1) > 0$ and $\tilde{z}_2(\cdot)$ is continuous (which follows from the continuity of $\mathbf{z}(\cdot)$), there is a $\overline{p}_1 \in (0, 1)$ such that $\tilde{z}_2(\overline{p}_1) > 0$ as well. Applying WL at $\overline{p}_1$, we have

$$\overline{p}_1\tilde{z}_1(\overline{p}_1) + \overline{p}_2\tilde{z}_2(\overline{p}_1) = 0,$$

and $\overline{p}_1 > 0$ and $\overline{p}_2 = 1 - \overline{p}_1 > 0$. Because $\tilde{z}_2(\overline{p}_1) > 0$, this application of Walras's Law yields $\tilde{z}_1(\overline{p}_1) < 0$.

We now have $\tilde{z}_1(0) > 0$ and $\tilde{z}_1(\overline{p}_1) < 0$ for some $\overline{p}_1 > 0$. The function $\tilde{z}_1(\cdot)$ is continuous, so the Intermediate Value Theorem guarantees that there is a $p_1^* \in (0, \overline{p}_1)$ at which $\tilde{z}_1(p_1^*) = 0$, as depicted in Figure 1. Applying WL one more time, we have

$$p_1^*\tilde{z}_1(p_1^*) + p_2^*\tilde{z}_2(p_1^*) = 0, \tag{$*$}$$

and $p_1^* > 0$ and $p_2^* = 1 - p_1^* > 0$. Since $\tilde{z}_1(p_1^*) = 0$, the equation $(*)$ yields $\tilde{z}_2(p_1^*) = 0$ as well, thus establishing the theorem's conclusion: $z_1(p_1^*, p_2^*) = 0$ and $z_2(p_1^*, p_2^*) = 0$. ■

Unfortunately, this proof doesn't generalize to more than two goods. If we have three goods, for example, then we can reduce the problem to one in which there are only two independent prices, so the analogues of the $\tilde{z}_k$ functions have two variables instead of the one variable in our proof above. But the Intermediate Value Theorem is a theorem only about real functions of a single variable: there is no analogue for functions of more than one variable. Later in the course we will return to the problem of existence of equilibrium, when we have more powerful tools at hand.

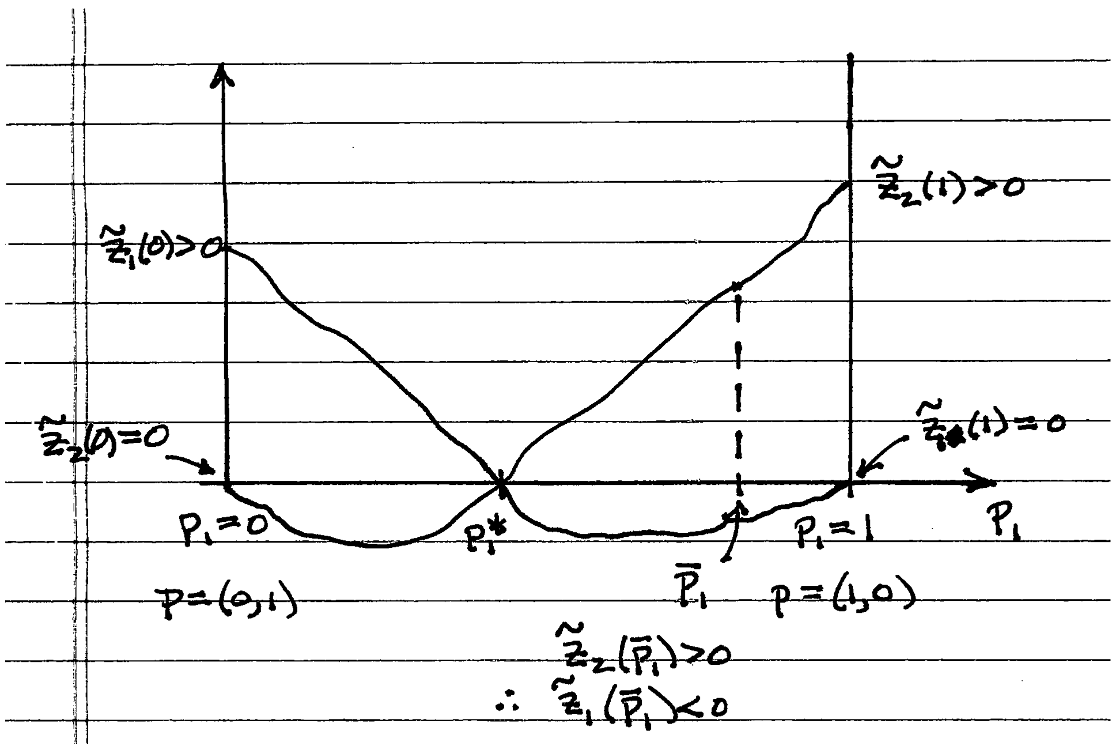

Figure 1