GENERAL EQUILIBRIUM EXAMPLE

(TWO PERSONS, LATER GENERALIZED TO n; TWO GOODS; PURE EXCHANGE; COBB-DOUGLAS UTILITIES.)

INDIVIDUAL DEMAND:

(x,y) MAXIMIZES $u(x,y) = x^{\alpha} y^{\beta}$ s.t. $x, y \geqq 0$ AND TO $p_x x + p_y y \leqq M$, WHERE $M = p_x \mathring{x} + p_y \mathring{y}$.

SOLUTION (INDIVIDUAL DEMAND FUNCTION): [WLOG, $\alpha + \beta = 1$]

$$x = \alpha \frac{M}{p_x} \quad \text{AND} \quad y = \beta \frac{M}{p_y}$$

LET $\rho := \dfrac{p_x}{p_y}$; $\quad x = \alpha\left(\mathring{x} + \frac{1}{\rho}\mathring{y}\right), \quad y = \beta\left(\mathring{y} + \rho\mathring{x}\right)$.

NET DEMAND:

$$\overset{\Delta}{x} := x - \mathring{x} = (\alpha - 1)\mathring{x} + \alpha\frac{1}{\rho}\mathring{y} = -\beta\mathring{x} + \alpha\frac{1}{\rho}\mathring{y}$$

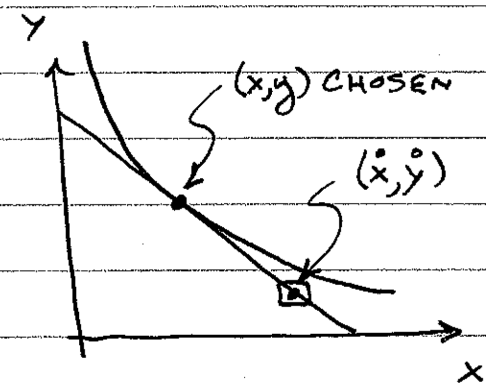
$$\overset{\Delta}{y} := y - \mathring{y} = (\beta - 1)\mathring{y} + \beta\rho\mathring{x} = \beta\rho\mathring{x} - \alpha\mathring{y}$$

NOTE: $\boxed{\overset{\Delta}{y} = -\rho\overset{\Delta}{x}.}$ (*) "WALRAS' LAW" WHEN AGGREGATED

i.e., $\rho\overset{\Delta}{x} + \overset{\Delta}{y} = 0$

$$\rho(x - \mathring{x}) + (y - \mathring{y}) = 0$$

$$p_x(x - \mathring{x}) + p_y(y - \mathring{y}) = 0$$



THE VALUE OF AN INDIVIDUAL'S NET DEMAND IS ZERO; HE CHOOSES ON HIS BUDGET CONSTRAINT.

AGGREGATE DEMAND:

(AGGREGATED ACROSS PERSONS, NOT ACROSS GOODS)

$$X = x_1 + x_2 = \alpha_1\left(\mathring{x}_1 + \frac{1}{p}\mathring{y}_1\right) + \alpha_2\left(\mathring{x}_2 + \frac{1}{p}\mathring{y}_2\right)$$

$$= \alpha_1\mathring{x}_1 + \alpha_2\mathring{x}_2 + \left(\alpha_1\mathring{y}_1 + \alpha_2\mathring{y}_2\right)\frac{1}{p}$$

$$\overset{\Delta}{X} := X - \mathring{X} = (\alpha_1 - 1)\mathring{x}_1 + (\alpha_2 - 1)\mathring{x}_2 + \left(\alpha_1\mathring{y}_1 + \alpha_2\mathring{y}_2\right)\frac{1}{p}$$

$$= -\left(\beta_1\mathring{x}_1 + \beta_2\mathring{x}_2\right) + \left(\alpha_1\mathring{y}_1 + \alpha_2\mathring{y}_2\right)\frac{1}{p}.$$

SIMILARLY FOR THE y-GOOD. $\overset{\Delta}{X}(p)$, A FUNCTION

EQUILIBRIUM:

$$X = \mathring{x}_1 + \mathring{x}_2 \quad \text{AND} \quad Y = \mathring{y}_1 + \mathring{y}_2$$

i.e., $\overset{\Delta}{X}(p) = 0$ AND $\overset{\Delta}{Y}(p) = 0$. ← [TWO EQUIL'M CONDITIONS (EQUATIONS)]

BUT BY SUMMING THE BUDGET-BALANCE CONDITION (*) ACROSS ALL PERSONS, WE KNOW THAT $\overset{\Delta}{Y}(p) = -p\overset{\Delta}{X}(p)$.

THUS, $\boxed{\overset{\Delta}{X}(p) = 0 \iff \overset{\Delta}{Y}(p) = 0}$ —

THE TWO EQUIL'M CONDITIONS ARE NOT INDEPENDENT. THERE IS REALLY ONLY <u>ONE</u> EQUILIBRIUM CONDITION:

$$\overset{\Delta}{X}(p) = 0 \; ; \qquad (**)$$

i.e., $-\left(\beta_1\mathring{x}_1 + \beta_2\mathring{x}_2\right) + \left(\alpha_1\mathring{y}_1 + \alpha_2\mathring{y}_2\right)\frac{1}{p} = 0$ (**)

THE CLOSED-FORM SOLUTION:

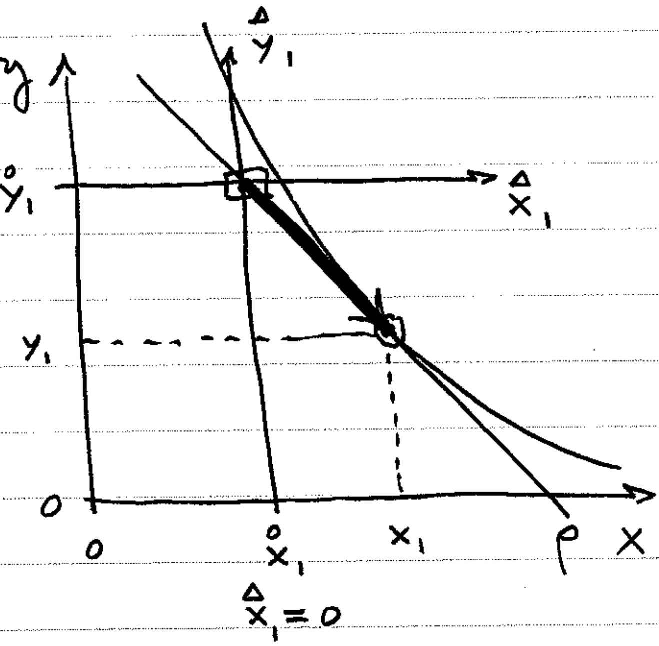
$$\rho = \frac{\alpha_1 \overset{\circ}{y}_1 + \alpha_2 \overset{\circ}{y}_2}{\beta_1 \overset{\circ}{x}_1 + \beta_2 \overset{\circ}{x}_2},$$

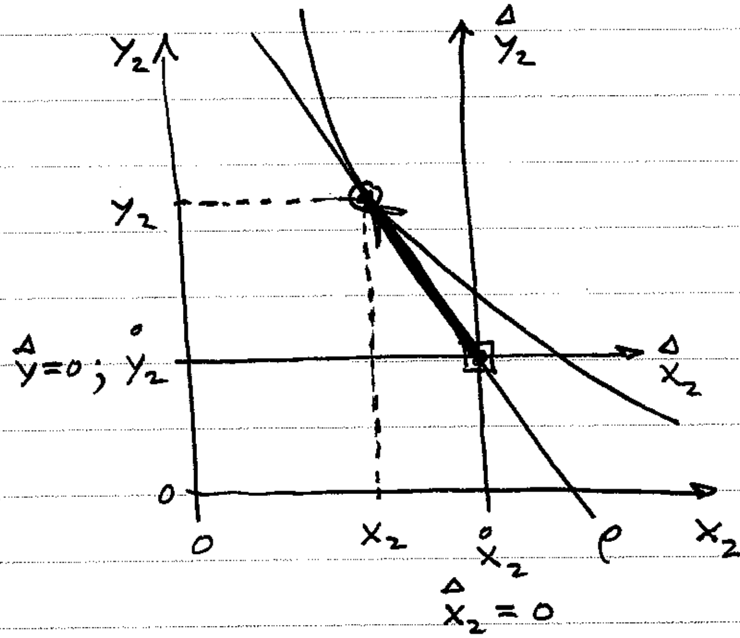
RATIO OF WEIGHTED SUMS OF ENDOWMENT AMOUNTS.

$$= \frac{\sum_{i=1}^{n} \alpha_i \overset{\circ}{y}_i}{\sum_{i=1}^{n} \beta_i \overset{\circ}{x}_i},$$

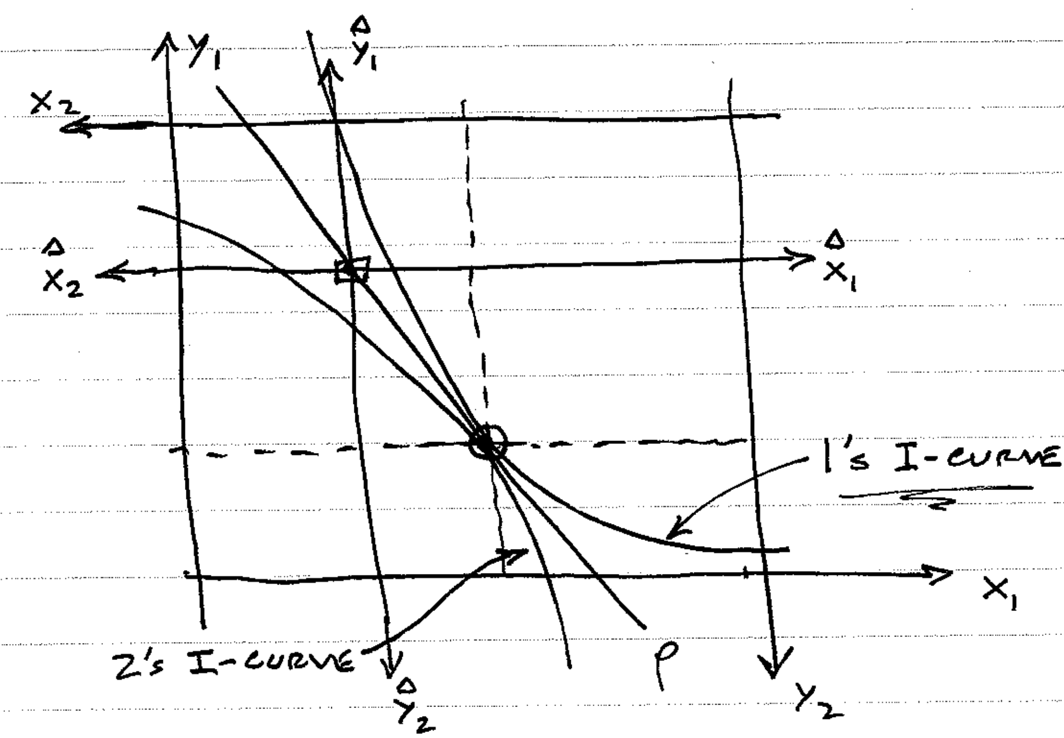
WHICH IS THE SOLUTION FOR n PERSONS (EVERYTHING WE'VE DONE CAN BE WRITTEN THE SAME WAY FOR n AS FOR 2).

THE GEOMETRY, INCLUDING THE EDGEWORTH BOX:





THE "PRICE LINE" IN BOTH INDIVIDUAL DIAGRAMS HAS $-$SLOPE $= \rho$. $\left(\rho = \frac{P_x}{P_y}\right)$.



A Numerical Example

DATA
$$\begin{cases} (\mathring{x}_1, \mathring{y}_1) = (40, 80), & \alpha_1 = \frac{7}{8} \\ (\mathring{x}_2, \mathring{y}_2) = (80, 40), & \alpha_2 = \frac{1}{2}. \end{cases}$$

$$\left(\therefore MRS_1 = 7\frac{y_1}{x_1}\right)$$
$$\left(\therefore MRS_2 = \frac{y_2}{x_2}\right)$$

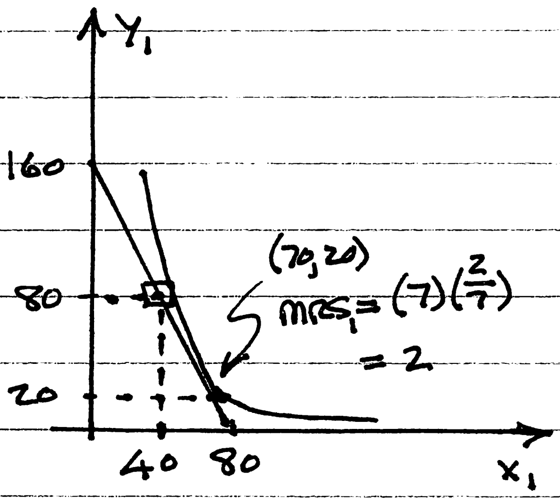
EQUILIBRIUM:

$$\rho = \frac{\frac{7}{8}(80) + \frac{1}{2}(40)}{\frac{1}{8}(40) + \frac{1}{2}(80)} = \frac{70+20}{5+40} = \frac{90}{45} = 2.$$

IF, FOR EXAMPLE, $p_x = 2$ AND $p_y = 1$, THEN

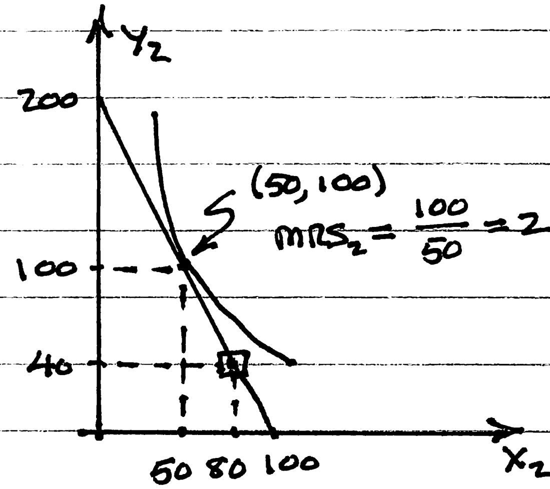
$$m_1 = (2)(40) + 80 = 160$$
$$m_2 = (2)(80) + 40 = 200.$$

$$\begin{array}{ll} x_1 = 70 & y_1 = 20 \\ \underline{x_2 = 50} & \underline{y_2 = 100} \\ X = 120 & Y = 120 \end{array} \qquad \begin{array}{ll} \overset{\Delta}{x}_1 = +30 & \overset{\Delta}{y}_1 = -60 \\ \underline{\overset{\Delta}{x}_2 = -30} & \underline{\overset{\Delta}{y}_2 = +60} \\ \overset{\Delta}{X} = 0 & \overset{\Delta}{Y} = 0. \end{array}$$



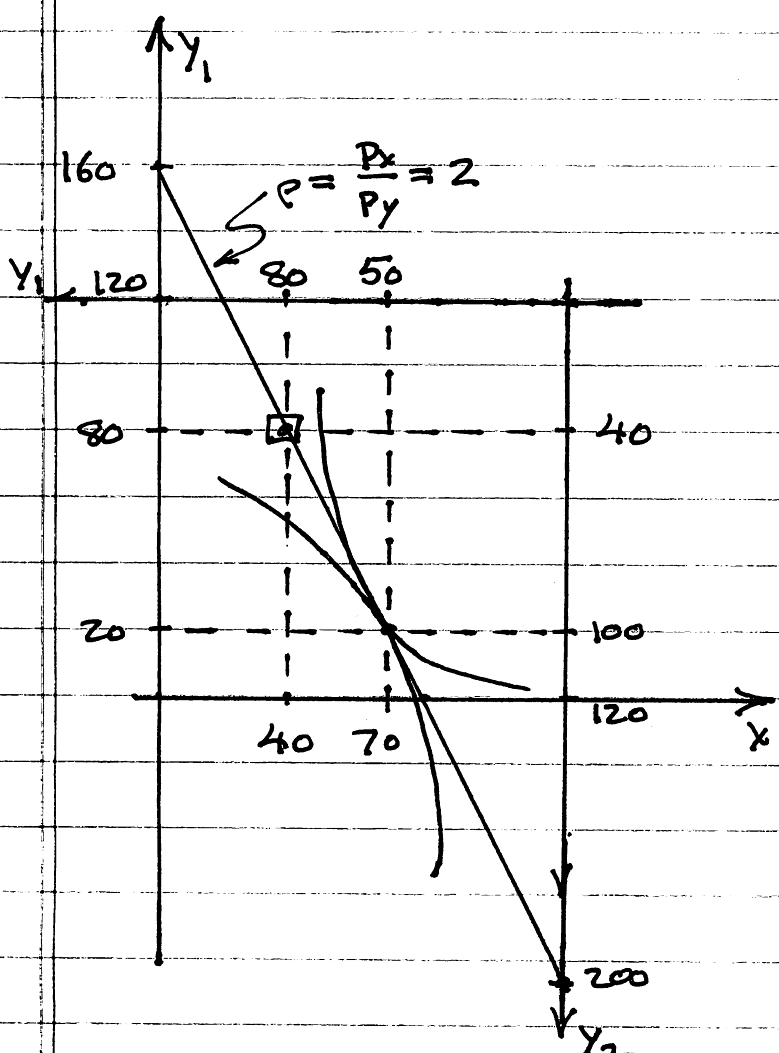
Y1
160
80
20
40
80
X1
(70, 20)
MRS1 = (7)(2/7)
= 2

Ms. 1



Y2
200
100
40
50
80
100
X2
(50, 100)
MRS2 = 100/50 = 2

Mr. 2



Y1
160
P = Px/Py = 2
Y1
120
80
50
80
40
20
100
40
70
120
X1
200
Y2

Another Numerical Cobb-Douglas Example

Data:

$n = 2$

$$(\mathring{x}_1, \mathring{y}_1) = (30, 60) \qquad \alpha_1 = \frac{2}{3}$$
$$(\mathring{x}_2, \mathring{y}_2) = (60, 30) \qquad \alpha_2 = \frac{2}{3}$$
$$(\mathring{X}, \mathring{Y}) = (90, 90)$$

Equilibrium:

$$\rho = \frac{\frac{2}{3}(60) + \frac{2}{3}(30)}{\frac{1}{3}(30) + \frac{1}{3}(60)} = \frac{40+20}{10+20} = \frac{60}{30} = 2$$

... FROM THE CLOSED-FORM EXPRESSION WE DERIVED FOR THE TWO-GOOD CASE.

[NOTE THAT IF $P_y = 1$ AND $P_x = 2$ (THIS IS ONE WAY TO HAVE $\rho = 2$), THEN

$$M_1 = (2)(30) + (1)(60) = 120$$
$$M_2 = (2)(60) + (1)(30) = 150$$

ARE THE VALUES OF THE CONSUMERS' INITIAL HOLDINGS (THEIR "ENDOWMENTS").]

$$(x_1, y_1) = (40, 40) \qquad (\overset{\Delta}{x}_1, \overset{\Delta}{y}_1) = (10, -20)$$
$$(x_2, y_2) = (50, 50) \qquad (\overset{\Delta}{x}_2, \overset{\Delta}{y}_2) = (-10, 20)$$
$$(X, Y) = (90, 90) = (\mathring{X}, \mathring{Y}) \qquad (\overset{\Delta}{X}, \overset{\Delta}{Y}) = (0, 0),$$

i.e., TOTAL <u>NET</u> TRADE IS ZERO FOR EACH GOOD.

IN EQUILIBRIUM,

MR. 1 CHOOSES TO GIVE UP 20 UNITS OF THE Y-GOOD TO OBTAIN A 10-UNIT ADDITION OF THE X-GOOD;

MR. 2 CHOOSES TO GIVE UP 10 UNITS OF THE X-GOOD (JUST THE AMOUNT MR. 1 WANTS TO BUY) TO OBTAIN AN ADDED 20 UNITS OF THE Y-GOOD (JUST THE AMOUNT MR. 1 WANTS TO SELL).

The Definition of Market Equilibrium

The concept of market equilibrium, like the notion of equilibrium in just about every other context, is supposed to capture the idea of a state of the system in which there are no forces tending to cause the state to change to a different state. For a market system, we think some prices are likely to change if there is excess demand or supply for any of the goods; and conversely that if all markets clear — *i.e.*, if no good is in excess demand or supply — then the prices will not change. And since the quantities that are transacted depend on the prices, the quantities should not change, either. So the natural definition of a general equilibrium of all markets is that all the markets clear — *i.e.*, that the price-list $\mathbf{p} \in \mathbb{R}^l_+$ satisfies the **equilibrium condition**

$$\overset{\Delta}{\mathbf{X}}(\mathbf{p}) = \mathbf{0} \qquad i.e.,\ \overset{\Delta}{X}_k(\mathbf{p}) = 0\,,\ k = 1, \ldots, l. \tag{$*$}$$

Provisional Definition: Let $E = ((u^i, \mathring{\mathbf{x}}^i))_{i=1}^n$ be an economy consisting of n consumers $(u^i, \mathring{\mathbf{x}}^i)$. Let $\mathbf{x}^i(\cdot) : \mathbb{R}^l_+ \to \mathbb{R}^l$ denote the demand function of consumer $(u^i, \mathring{\mathbf{x}}^i)$, and let $\overset{\Delta}{\mathbf{X}}(\cdot) : \mathbb{R}^l_+ \to \mathbb{R}^l$ denote the market net demand function $\overset{\Delta}{\mathbf{X}}(\mathbf{p}) := \sum_{i=1}^n (\mathbf{x}^i(\mathbf{p}) - \mathring{\mathbf{x}}^i)$. A **market equilibrium** of E is a price-list $\mathbf{p} \in \mathbb{R}^l_+$ that satisfies the equilibrium condition $(*)$.

There are many situations where this definition works just fine, but there are also many situations where it's not satisfactory. For example,

(1) If a price p_k is zero and there is excess supply of good k — *i.e.*, $\overset{\Delta}{X}_k(\mathbf{p}) < 0$ — it seems unlikely that this would lead to a change in any of the prices.

(2) What if the demand function $\mathbf{x}^i(\cdot)$ is not well-defined at some price-lists $\mathbf{p}$ for one or more consumers $(u^i, \mathring{\mathbf{x}}^i)$? For example, if $p_k = 0$, the CMP for some consumers may not have a solution.

(3) What if some consumer's demand function $\mathbf{x}^i(\cdot)$ is not single-valued at some price-lists? For example, a utility function u^i might have an indifference curve with a "flat spot" — an extreme example is a linear utility function $u(x, y) = ax + by$.

The following definition explicitly avoids issues (2) and (3) by including only situations in which all demand functions are well-defined and single-valued for every price-list $\mathbf{p} \in \mathbb{R}^l_+$. The definition takes account of issue (1) by allowing that excess supply of some goods is consistent with equilibrium if those goods have a price of zero.

Definition: Let $E = ((u^i, \mathring{\mathbf{x}}^i))_{i=1}^n$ be an economy consisting of n consumers, all of whose demand functions $\mathbf{x}^i(\cdot) : \mathbb{R}^l_+ \to \mathbb{R}^l$ are well-defined and single-valued on $\mathbb{R}^l_+$, and let $\overset{\Delta}{\mathbf{X}}(\cdot) : \mathbb{R}^l_+ \to \mathbb{R}^l$ denote the corresponding market net demand function. A **market equilibrium** of E is a price-list $\mathbf{p} \in \mathbb{R}^l_+$ that satisfies the equilibrium condition

$$\forall k = 1, \ldots, l: \quad \overset{\Delta}{X}_k(\mathbf{p}) \leqq 0 \quad \text{and} \quad \overset{\Delta}{X}_k(\mathbf{p}) = 0 \text{ if } p_k > 0. \qquad (\mathbf{Clr})$$

We'll also refer to a price-list that satisfies (**Clr**) as an equilibrium of the net demand function $\overset{\Delta}{X}(\cdot)$.

We'll use this equilibrium condition throughout the course, so we give it a name that we'll use to refer to it: (**Clr**), which is an abbreviation for *Clear*, since the condition says that all markets clear.

A market equilibrium is also called a **Walrasian equilibrium**. An essential feature of this equilibrium concept is the assumption — implicit in the definition — that all consumers are **price takers**. Each consumer, in solving his consumer maximization problem, treats the prices as *parameters* that will be unaffected by his decision about which consumption bundle he will choose.

Some Remarks

(1) Note the analogy with optimization: Here the *equilibrium conditions* are equations that determine the values of the variables, and in optimization the *first-order conditions* are equations that determine the values of the variables.

(2) With l goods we will have $l-1$ independent equilibrium conditions (equations), with Walras's Law accounting for the remaining market, so only $l-1$ **relative prices** are determined by equilibrium. (We could, for example, use one of the prices as "numeraire.") Because the demand functions are homogeneous of degree zero, they also depend only on the relative prices.

(3) What if, unlike in our Cobb-Douglas examples, we can't get a closed-form solution (*i.e.*, an explicit expression) for the state variables in terms of the parameters? How do we do comparative statics in that case? We can apply the Implicit Function Theorem to the equilibrium equations, just as we apply the IFT to the first-order equations to do comparative statics for optimization.

(4) The approach in (3) is often not good enough: for example, we often need to determine the actual equilibrium prices and/or quantities, not just the comparative statics derivatives. If we can't get closed-form solutions (which is the typical situation), we can try to *compute* the equilibrium values. How do we do that?

(5) What if there is no equilibrium? Under what conditions will there be an equilibrium?

(6) What if there are multiple equilibria? Under what conditions will there be a unique equilibrium?

(7) What if the system is not *in* equilibrium? What are the stability properties of the equilibrium?

(8) Is the equilibrium outcome a *good* outcome?

(9) What if markets and prices aren't used? Under what conditions will they be used?

(10) What if not everyone is a price-taker?

We will address all of these questions in the course, some in depth, and others only in passing.

Comparison of Individual Decision Analysis and Equilibrium Analysis

	Analysis of Individual Decision-Making (501A)	Analysis of Interaction Among Multiple Economic Units (501B)
Typical unit:	Household; Firm	Market, industry, economy, firm, club, organization, society electorate
Assumption:	Unit's choice is optimal, according to some objective function, perhaps subject to constraint(s)	Unit is in equilibrium
Analytical characterization:	First-order conditions -- i.e., equations	Equilibrium condition(s) -- i.e., equations
Typical question:	What is effect of an exogenous change -- e.g., in price of another good, weather, etc. ("comparative statics")	What is effect of exogenous change -- e.g., in expectations, weather, etc. ("comparative statics")
Method:	IFT applied to FOC	IFT on equilibrium conditions
Other questions:		Existence, optimality, stability. is it "the right model"?