A Simple Supermodular Mechanism that Implements Lindahl Allocations

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Abstract

This paper introduces a new incentive compatible mechanism which for general preference environments implements Lindahl allocations as Nash equilibria. We show via an example that having a mechanism induce a supermodular game (in the sense of Chen (2002)) is not typically sufficient to guarantee dynamic stability of equilibrium. However, for the new mechanism, inducing a supermodular game guarantees that the best reply mapping is a contraction. Thus, this new mechanism provides a connection between the desirable welfare properties of Lindahl allocations and the theoretical/convergence properties of games whose best reply mappings are a contraction.

1 Introduction

The reliance on unregulated markets for the provision of public goods presents well known challenges to efficiency. For economists, this problem continues to motivate the search for alternative institutions which yield Pareto optimal outcomes. One problem with this approach is that some Pareto outcomes may not be desirable for everyone involved. Some participants could end up being worse off than they were with their original endowment, a common

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critique, for example, of the Groves-Ledyard (G-L) mechanism. G-L is an institution that overcomes the free riding problem – the incentive to enjoy the public good's benefits while not sharing in its cost – but the mechanism may leave some participants worse off than before they participated. In contrast, Lindahl allocations, while also Pareto optimal, make no one worse off than he was to begin with (i.e., are individually rational). Lindahl allocations are therefore attractive outcomes when there are public goods.

In addition to being Pareto optimal and individually rational, Lindahl allocations share another important property of Walrasian ("competitive") allocations of private goods: every individual's payment for each unit of the public good is equal to the marginal value he places on the good. In the Walrasian setting consumers all face the same price and they demand potentially different quantities of a good; in the Lindahl setting they face distinct "personalized" prices and, in equilibrium, each consumer demands the same quantity of the public good. Actually implementing a Lindahl scheme, however is problematic, since it is not exactly clear how the personalized prices are to be determined. Perhaps one could use surveys, but there may be incentives for participants in the surveys to misrepresent their preferences in order to pay a lower price. This has led to the development of *incentive compatible* public goods mechanisms.

The purpose of this paper is to introduce a new incentive compatible mechanism which attains Lindahl allocations as Nash equilibria. This is true for economies with an arbitrary number of consumers and general preference environments. In addition to this Nash "implementation" result, the mechanism has several other attractive properties, which have been motivated by experimental research: the mechanism retains its structural simplicity as the number of consumers increases–there are no new penalty terms added to the mechanism when the economy increases in size; the minimum dimension of data needed to compute payoffs is smaller than other mechanisms with comparable properties; the components of the mechanism have a clear economic interpretation; and for quasi-linear preference environments the unique equilibrium is stable under a wide variety of learning behavior.

The mechanism introduced here is not the first to implement Lindahl allocations. Hurwicz (1979) and Walker (1981) were the first to present such mechanisms, but Kim (1987) has shown that both mechanisms are quite unstable. While some sort of dynamic stability is desired, there is no agreement about how people's behavior adjusts when out of equilibrium. In mechanism-design experiments, however, a common empirical finding is that in mechanisms with theoretically robust dynamic stability properties, subjects' behavior tends to converge. Supermodular mechanisms have been particularly successful.[1] This empirical regularity was presaged by the theoretical stability results established by Milgrom and Roberts (1990a) for supermodular games. In this light, Chen's (2002) theoretical contribution is of particular interest. She presented the first Lindahl mechanism that is supermodular in quasi-linear environments for some values of the mechanism parameters. Thus, motivated by the observation that supermodular mechanisms tend to perform better in the laboratory, she creates a Lindahl mechanism that induces a supermodular game. Two fundamental issues in her paper are worth noting.

First, in Chen's paper, the environment used to establish that the game is supermodular and the uniqueness of Nash equilibrium does not satisfy the conditions stated in the Milgrom and Roberts stability result. Thus, no stability of equilibrium results can be inferred from the fact that her mechanism induces a supermodular game with a unique equilibrium. As a cautionary example in this paper, we show that there is a simple incentive compatible Lindahl mechanism that, in the same environment as Chen's, always induces a supermodular game with a unique, dynamically *unstable* equilibrium. This observation suggests that we need to appeal to an alternative technique to guarantee that the game induced by our new mechanism yields a unique, dynamically stable Nash equilibrium. We do this by introducing a mechanism that, under some choices of the mechanism parameters, induces a game whose best reply mapping is a contraction mapping. Interestingly, we also show that if the game induced by our mechanism is supermodular, the sufficient conditions for the best reply mapping to be a contraction are satisfied giving us a relatively simple condition to check for stability.[2]

The second issue with Chen's mechanism is a practical one – the severe method in which it taxes out-of-equilibrium behavior. Van Essen, Lazzati, and Walker (2007), hereafter VLW, experimentally tested three Lindahl mechanisms, including the Chen mechanism, in a laboratory environment. The experimental evidence suggests that while the Chen mechanism produces public good levels near the Pareto optimal level, when not in equilibrium the mechanism generates large amounts of tax waste and participants frequently

[1]For example see Chen and Tang (1998), Chen and Gazzale (2004), and Healy (2004).

[2]Our mechanism induces a supermodular game that does not fit the Milgrom and Roberts result, however, for this mechanism, this is sufficient to guarantee dynamic stability.

did worse than their initial endowment. Furthermore, the subjects in the experiment did not show any signs of getting close to their equilibrium messages, so these poor out-of-equilibrium properties do not diminish much over time. VLW attribute this divergence to the complexity of the mechanism and the number of penalty terms the Chen mechanism needs to induce the supermodular game. Additionally, since the Chen mechanism adds more of these penalty terms (one term per consumer) as more consumers are added to the economy, one would conjecture that these large tax payments would only get worse for larger group sizes. The Lindahl mechanism presented in this paper addresses these concerns by arranging the mechanism so the number of penalty terms is fixed at two for any number of players.

The remainder of the paper will proceed as follows: Section 2 provides a simple definition of a supermodular game and summarizes some of the important properties these games exhibit, including the Milgrom and Roberts result; Section 3 outlines the basic public goods problem and mechanism environments; Section 4 contains the bulk of the paper's theoretical results concerning implementation and stability; finally, Section 5 discusses the actual implementation of the new mechanism and how the out-of-equilibrium tax penalties of the new mechanism compare to the penalties from the Chen mechanism.

2 Preliminaries

Supermodularity and contraction mappings play a significant role in several of the results we will develop. In this section we review some definitions, framed in terms of the strategy spaces and the payoff functions we will use. The strategy spaces are subsets of Euclidean spaces and the payoff functions are twice continuously differentiable (or C^2). More general definitions of a supermodular game can be found in Topkis (1998) or Milgrom and Roberts (1990a). A general treatment of contraction mappings can be found in Ortega and Rheinboldt (1970).

A normal form game is defined by a set of players, a strategy set for each player, and a payoff function for each player. Denote the set of players I, where $I = \{1, ..., N\}$. Let $M_i \subseteq \mathbb{R}^2$ be player i's strategy space with an arbitrary element $\mathbf{m}_i = (m_{i1}, m_{i2})$, where $M = \times_{i=1}^{N} M_i$ is the collection of all players' strategy spaces. For each player i let $u^i : M \to \mathbb{R}$ be a payoff function which maps strategy profiles into a numerical payoff.

A supermodular game is characterized by payoff functions that satisfy both the *supermodularity property* and the *increasing differences property.*

Definition 1: A C^2 payoff function u^i is *supermodular* if a player's own actions are strategic complements–i.e. for each i

$$\frac{\partial u^i(\mathbf{m})}{\partial m_{i1} \partial m_{i2}} \geq 0.$$

Definition 2: A C^2 payoff function u^i has *increasing differences* if a player's own actions are strategic complements with the actions of all other players–i.e. for each i

$$\frac{\partial u^i(\mathbf{m})}{\partial m_{in} \partial m_{jl}} \geq 0$$

for $n = 1, 2$ and $l = 1, 2$.

A game is supermodular if the payoffs for all players satisfy both properties:

Definition 3: A game is *supermodular* if for each player i, M_i is a non-empty subset of $\mathbb{R}^2$ and u^i has the supermodularity and increasing difference properties.

Supermodular games have properties that make them attractive for mechanism design. If the strategy space is *compact* and the payoff function is C^2, then Milgrom and Roberts (1990a) show that:

(1) The set of serially undominated strategy profiles has a maximum and a minimum element, and these elements are Nash equilibria;

(2) Under a wide class of dynamic adjustment processes the predicted behavior converges to the set of profiles bounded by the two extreme Nash equilibria. These dynamic processes include best-response dynamics, fictitious play, Bayesian learning, and others.

When the Nash equilibrium is unique, the predictive power of these results is increased: property (1) implies that the game is dominance solvable and property (2) says that the unique Nash equilibrium is "stable" under a wide range of adaptive behavior. While these properties are attractive, as mentioned in the introduction, the environment in which we will be working does not satisfy the above Milgrom and Roberts stability conditions – i.e., the players' message space for the new mechanism (and the Chen mechanism) is $\mathbb{R}^2$, which is not compact. We will therefore instead appeal to the Contraction Mapping Theorem for our stability results.

Definition 4: Let (X, d) be a metric space. A self map ζ on X is said to be a *contraction* if there exists a real number $0 < k < 1$ such that

$$d(\zeta(x), \zeta(y)) \leq kd(x, y)$$

for all x , $y \in X$.

Contraction Mapping Theorem: Let (X, d) be a non-empty complete metric space and let $\zeta : X \to X$ be a contraction. Then there exists a unique point $x^* \in X$ such that $\zeta(x^*) = x^*$. Furthermore if x_0 is any point of X and $x_1 = \zeta(x_0)$, $x_2 = \zeta(x_1)$, $x_3 = \zeta(x_2)$, etc., then

$$\lim_{n \to \infty} x_n = x^*.$$

3 The Public Good Economy

Our setting applies to $N \geq 2$ consumers. For simplicity of exposition, we restrict attention to economies with one private good, one public good, and a constant returns to scale production technology. However, it it straightforward to generalize the results to include economies with many private and public goods. The quantity of the public good will be denoted by x, and the private good for consumer i by y_i, where consumers are indexed by subscript i. Each consumer is characterized by the convex consumption set $C_i = \mathbb{R}^2_+$, an initial endowment of the private good $\omega_i > 0$, and no initial endowment of the public good. The public good is produced, using the private good as an input (quantity denoted z), with a constant returns to scale production technology $f(z) = \frac{z}{\beta}$ — i.e., each unit of the public good x requires β units ($\beta > 0$) of the private good. Thus β is the constant (real) marginal cost of production. An allocation in this simple economy is an $(N + 1)$- tuple $(x, y_1, ..., y_N) \in \mathbb{R}^{N+1}_+$.

3.1 The Mechanism

A mechanism maps consumers' strategies (or messages) into an outcome (an allocation). We consider a mechanism in which consumers report messages to a "planner" who uses this information to determine an amount of the public good to produce and a tax for each consumer. The message space of consumer i is $M_i = \mathbb{R}^2$ with generic element $\mathbf{m}_i = (r_i, s_i)$. Let $\mathbf{m} = (\mathbf{m}_1, ..., \mathbf{m}_N)$ denote the profile of all players' messages. Consumer i's action r_i should be interpreted as a *request* from the consumer to the planner for r_i units of the public good. Notice that negative requests are allowed. Consumer i's other action, s_i, is interpreted as his *statement* about the amount of the public good that will be produced. Rather than write $(r_1, s_1, r_2, s_2, ...)$ for a strategy profile, we write $(r_1, r_2, ..., r_N, s_1, ..., s_N) = (\mathbf{r}, \mathbf{s})$. These messages are collected by the planner and used to determine an amount of the public good and a tax for each player i according to outcome functions $\chi(\mathbf{r}, \mathbf{s})$ and $\tau^i(\mathbf{r}, \mathbf{s})$ respectively. For positive real numbers ξ, γ, and δ, let $\varphi^{\xi,\gamma,\delta}(\mathbf{r}, \mathbf{s}) = \left(\chi(\mathbf{r}, \mathbf{s}), (\omega_i - \tau^i(\mathbf{r}, \mathbf{s}))_{i=1}^N\right)$ be a mechanism with outcome functions defined as follows:

$$\begin{aligned} \chi(\mathbf{r}, \mathbf{s}) &= \frac{1}{N} \sum_{i=1}^{N} r_i \\ \tau^i(\mathbf{r}, \mathbf{s}) &= P^i(\mathbf{r}, \mathbf{s}) \cdot \chi(\mathbf{r}, \mathbf{s}) + \frac{\gamma}{2} \left(s_i - \chi(\mathbf{r}, \mathbf{s})\right)^2 + \frac{\delta}{2} \left(s_{i+1} - \chi(\mathbf{r}, \mathbf{s})\right)^2 \end{aligned}$$

where

$$P^i(\mathbf{r}, \mathbf{s}) = \frac{\beta}{N} - \xi \left(\frac{1}{N-1} \sum_{j \neq i} r_j - s_{i+1} \right),$$

where $s_{N+1} = s_1$.[3] $P^i(\mathbf{r}, \mathbf{s})$ can be thought of as i's personalized price for the public good and the remaining two terms as *statement penalties* i must pay.

The mechanism works as follows: the planner collects each consumer's request and produces an amount of the public good equal to the average request. In addition, the requests and statements are used to determine each

[3] For some non-equilibrium messages the payoffs are not completely well defined. That is they will take consumers outside of their consumption set C_i. This same weakness is shared by the Groves-Ledyard, Hurwicz, Walker, Kim, and Chen mechanisms. However it should be noted that at each interior equilibrium there is a neighborhood on which feasibility is assured.

consumer's tax, which is the sum of the two penalty terms and the term involving the personalized price. The first statement penalty for consumer i is increasing in the amount by which his own statement differs from the actual amount of the public good produced, and the other statement penalty is increasing in the amount by which his *neighbor's* (consumer $i+1$) statement, s_{i+1}, differs from the actual public good production. Since $\chi(\mathbf{r}, \mathbf{s})$ is independent of $\mathbf{s}$ and since preferences are increasing in y_i, it is clear that in a Nash equilibrium every consumer's statement will be correct: $s_i = \chi(\mathbf{r}, \mathbf{s})$. Consequently, in equilibrium both penalty terms will be zero for every consumer, and the consumer will therefore simply pay the price $P^i(\mathbf{r}, \mathbf{s})$ for each unit of the public good. Note that $P^i(\mathbf{r}, \mathbf{s})$is independent of both r_i and s_i.

The personalized price function, $P^i(\mathbf{r}, \mathbf{s})$, has an intuitive economic interpretation. The price is higher for a consumer who is perceived by his neighbor (consumer $i+1$) to demand more of the good than others and the price is lower if he himself is perceived to request less than others. The term $\sum_{j\neq i} \frac{r_j}{N-1}$ corresponds to the amount of the public good if consumer i did not participate in the mechanism. The term s_{i+1} represents consumer $i+1$'s statement about the level or quantity of the public good. Thus if $\sum_{j\neq i} \frac{r_j}{N-1} > s_{i+1}$, it means that consumer $i+1$ believes that consumer i's request will lower the level of the public good produced. As a consequence, i's personalized price is less than an equal share of the marginal cost. If $\sum_{j\neq i} \frac{r_j}{N-1} < s_{i+1}$, then the reverse is true and consumer i's personalized prize is greater than an equal share of the marginal cost. If $\sum_{j\neq i} \frac{r_j}{N-1} = s_{i+1}$, the personalized price is an even share of the marginal cost. The first term in $P^i(\mathbf{r}, \mathbf{s})$ is $\frac{\beta}{N}$, the per-capita cost of the public good.

3.2 Preference and Wealth Assumptions

The coupling of the mechanism $\varphi^{\xi,\gamma,\delta}(\mathbf{r}, \mathbf{s})$ and a preference environment defines a game. Our results require only that all of the Lindahl equilibria allocations be in the interior of the consumption set and that preference are continuos, convex, and strictly increasing in the private good. We introduce two types of preference environments: first, a "regular" environment E where preferences satisfy the usual set of consistency conditions; second, a "quasi-linear" environment E^Q that satisfies some additional properties. In both cases, the assumptions on E and E^Q are sufficient for Lindahl allocations to be in the interior of each consumer's consumption space. However, there are many other environments for which the Lindahl allocations will be interior and to which our implementation results will also apply.

Definition 5: A *regular preference environment* E is one in which for each player has a complete and transitive preference relation $\succeq_i$ that satisfies the following properties:

1. (Continuity): For every $(\bar{x}, \bar{y}_i) \in C_i$, the sets $\{(x, y_i)|(x, y_i) \succeq_i (\bar{x}, \bar{y}_i)\}$ and $\{(x, y_i)|(\bar{x}, \bar{y}_i) \succeq_i (x, y_i)\}$ are closed in C_i.

2. (Convexity): If $(x, y_i) \succeq_i (\bar{x}, \bar{y}_i)$, then $(\lambda x + (1-\lambda)\bar{x}, \lambda y_i + (1-\lambda)\bar{y}_i) \succeq_i (\bar{x}, \bar{y}_i)$ for any $\lambda \in [0, 1]$.

3. (Strictly Increasing in y_i): If $\bar{y}_i > y_i$, then for any $x > 0$, $(x, \bar{y}_i) \succ_i (x, y_i)$.

4. (Strict Preference of Interior Allocations to Boundary Allocations): If $(\bar{x}, \bar{y}_i) \in C_i^{++}$ and $(x, y_i) \in \partial C_i$, then $(\bar{x}, \bar{y}_i) \succ_i (x, y_i)$, where C_i^{++} and ∂C_i are the interior and the boundary of the consumption set C_i respectively (this assumption (together with 1) implies that all bundles on the boundary are indifferent to one another – i.e., the boundary comprises a single indifference set.)

These (Cobb-Douglas type) preferences (along with strictly positive income) ensure that each consumer's allocation in a Lindahl equilibrium in the interior of the consumption set.

Definition 6: E^Q denotes the set of standard C^2 *quasi-linear environments* – i.e., those in which,

1. For each i, there is a real-valued function v^i such that $u^i(x, y_i) = y_i + v^i(x)$.

2. v^i is C^2, where its second derivative is bounded from above and below by $\bar{K}_i$ and $\underline{K}_i$ respectively –i.e., $-\infty < \underline{K}_i \leq \frac{\partial^2 v^i(x)}{\partial x^2} \leq \bar{K}_i < 0$.

3. $\sum_i \frac{\partial v^i(0)}{\partial x} > \beta$ and $\sum_i \frac{\partial v^i(\frac{\Omega}{\beta})}{\partial x} < \beta$ – i.e., that there is unique, interior Pareto optimal level the public good that does not exhaust the economy's private good supply, where $\Omega = \sum \omega_i$.

4. For each i, $\omega_i - \frac{\partial v^i(x^{PO})}{\partial x} x^{PO} \geq 0$ – i.e., each consumer has enough wealth to cover his or her Lindahl taxes.

Items (1)-(4) in the definition of E^Q are sufficient to guarantee that for each $e \in E^Q$, there is a unique Lindahl equilibrium which is in the interior of each consumer's consumption set.

4 Implementation

The first result of the paper shows that the game induced by the mechanism $\varphi^{\xi,\gamma,\delta}(\mathbf{r}, \mathbf{s})$ implements Lindahl allocations as Nash equilibrium outcomes. Implementation is an exact correspondence between Lindahl and Nash outcomes. In other words, any Lindahl allocation can be achieved as the allocation of a Nash equilibrium; and at any Nash equilibrium the equilibrium allocation is Lindahl.

Theorem 1 *The mechanism $\varphi^{\xi,\gamma,\delta}$ implements the Lindahl allocations for any $e \in E$ and any $e \in E^Q$.*

Proof. See Appendix. ■

The conditions needed for the existence of Lindahl equilibria can be found in Milleron (1972) or Foley (1970). Theorem 1 does not impose any restrictions on the positive parameters ξ, γ, and δ. These are free parameters which will be manipulated later in the paper to create a family of stable Lindahl mechanisms. Furthermore, unlike many Lindahl mechanisms in the literature, the new mechanism applies to economies with two consumers.[4]

In order to illustrate the dual nature of the theorem the next example may be useful.

Consider a two-consumer economy, where each consumer is endowed with $\omega = 20$ units of the private good. Suppose it takes 4 units of the private good y to produce each unit of the public good x (i.e., $\beta = 4$) and that Consumer 1's and Consumer 2's preferences can be represented by the utility functions $u_1(x, y_1) = y_1 - \frac{1}{2}(6 - x)^2$ and $u_2(x, y_2) = y_2 - \frac{1}{2}(8 - x)^2$ respectively. The mechanism $\varphi^{1,1,1}(\mathbf{m})$ implements the Lindahl allocations of this economy.

[4] $N \geq 3$ is the usual restriction.

Implementation of Lindahl allocations requires first that any Lindahl allocation can be achieved as a Nash equilibrium of the mechanism, and at any Nash equilibrium, the equilibrium allocation is Lindahl. For this example, we start with the first requirement.

From the utility functions we solve for both Player 1's and Player 2's demand for the public good (or their marginal rates of substitution) which are $MRS_1 = 6 - x$ and $MRS_2 = 8 - x$ respectively. Using the Samuelson marginal condition (i.e., that at a Pareto optimal quantity of the public good $MRS_1 + MRS_2 = 4$), the Pareto optimal level of the public good for these two consumers is $x^{PO} = 5$. Inserting this quantity into each consumer's demand for the public good, we find that the corresponding Lindahl prices for Consumer 1 and 2 are $\bar{P}^1 = 1$ and $\bar{P}^2 = 3$ respectively. Therefore this example has a unique Lindahl allocation and it is in the interior of each consumer's consumption space.

Suppose $(\bar{r}_1, \bar{r}_2, \bar{s}_1, \bar{s}_2)$ is a Nash equilibrium of the game induced by mechanism $\varphi^{1,1,1}(\mathbf{m})$. If the Lindahl allocation is to be achieved as a Nash equilibrium, then two equations must hold: first, the average request must equal the Pareto optimal amount, i.e.,

$$\chi(\mathbf{r}, \mathbf{s}) = \frac{\bar{r}_1 + \bar{r}_2}{2} = 5;$$

second, Player 1's personalized price function must equal his Lindahl price $\bar{P}^1 = 1$, i.e.,

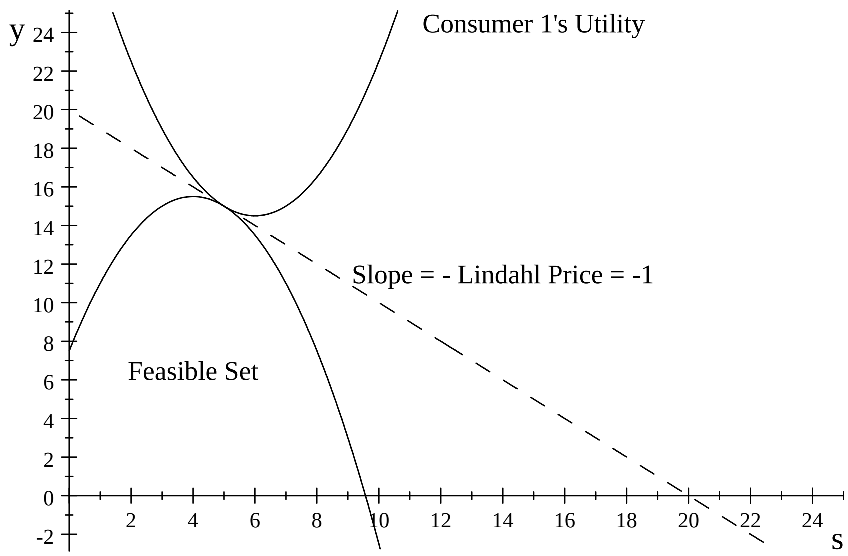
$$P^1(\mathbf{r}, \mathbf{s}) = \frac{\beta}{2} - \bar{r}_2 + \bar{s}_2 = 1.$$

Any equilibrium that achieves this allocation requires Consumer 2's statement to be correct (i.e., $\bar{s}_2 = 5$), it follows from the second equation that $\bar{r}_2 = 6$. Thus, the strategy profile $[(\bar{r}_1, \bar{s}_1), (\bar{r}_2, \bar{s}_2)] = [(4, 5), (6, 5)]$ is the only profile which could achieve the Lindahl outcome as an equilibrium. We now show that this profile is a Nash equilibrium by checking that Consumer 1 is best responding to Consumer 2's strategy and vice versa.

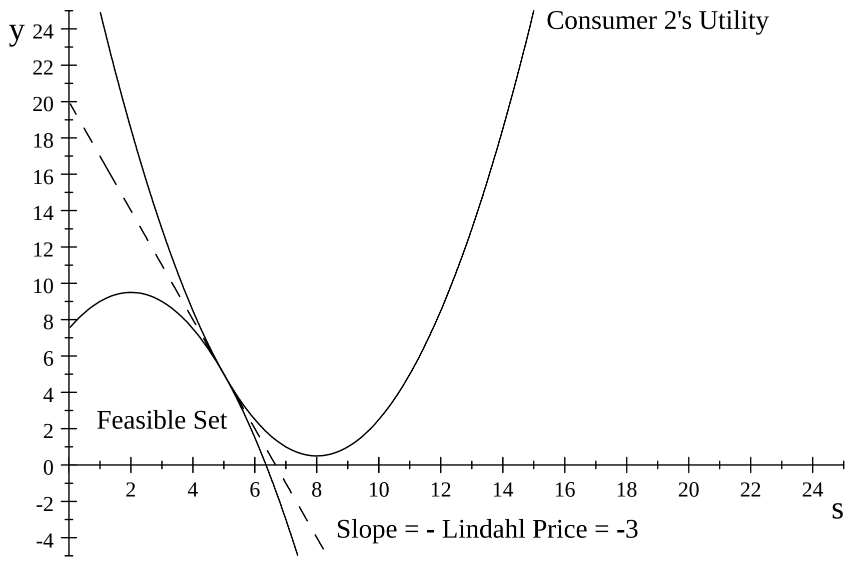
Consumer 1's best response problem is to maximize his utility subject to a feasible set defined by Consumer 2's strategy and the mechanism. Since in a best response Consumer 1's strategy satisfies $s_1 = \frac{1}{2}r_1 + \frac{1}{2}\bar{r}_2$ (i.e., $s_1 = \chi(r_1, \bar{r}_2, s_1, \bar{s}_2)$), Consumer 1's best response problem simplifies to

$$\max_{s_1} u_1(s_1, 20 - s_1 - \frac{1}{2}(5 - s_1)^2).$$

The first order condition yields $\bar{s}_1 = 5$, which implies $\bar{r}_1 = 4$ and verifies that Consumer 1's strategy $(4, 5)$ is his best response to Consumer 2's strategy. A graphical depiction of Consumer 1's best response problem is illustrated below.



A similar argument can be used to show that Consumer 2's best response to $(\bar{r}_1, \bar{s}_1) = (4, 5)$ is $(\bar{r}_2, \bar{s}_2) = (6, 5)$. Notice that Consumer 1's actions define a personalized price equal to the Lindahl price $\bar{P}_2 = 3$ for Consumer 2. The graphical depiction of Consumer 2's best response problem is given below.



Since both players are best responding to each others actions, the unique Lindahl allocation of this example is achieved as a Nash equilibrium.

The second implication of Theorem 1 says that it is also possible to go in the other direction. Namely, if $(\bar{r}_1, \bar{r}_2, \bar{s}_1, \bar{s}_2)$ is a Nash equilibrium of the mechanism, the equilibrium allocation is Lindahl. To demonstrate this in our example suppose $(\bar{r}_1, \bar{r}_2, \bar{s}_1, \bar{s}_2)$ is a Nash equilibrium. Then the first order condition (with respect to statement s_i) yields $\bar{s}_i = \frac{\bar{r}_1+\bar{r}_2}{2}$ for each i. Inserting this expression into each consumer's first order condition (with respect to their request) we have

$$6 - \frac{\bar{r}_1 + \bar{r}_2}{2} = 2 - \bar{r}_2 + \bar{s}_2$$

and

$$8 - \frac{\bar{r}_1 + \bar{r}_2}{2} = 2 - \bar{r}_1 + \bar{s}_1.$$

for Consumer 1 and Consumer 2 respectively. The unique solution of this pair of equations is $\bar{r}_1 = 4$, $\bar{r}_2 = 6$, which coupled with the optimal statements $\bar{s}_i = \frac{\bar{r}_1+\bar{r}_2}{2}$ for each i yields the Lindahl equilibrium $\bar{P}^1 = 1$, $\bar{P}^2 = 3$, and $x = 5$. Thus, the Nash allocation is Lindahl, completing the example.

4.1 Implementation in Quasi-Linear Environments

In this section, we show that the new Lindahl mechanism induces a supermodular game in quasi-linear E^Q environments for some values of the mechanism's parameters. Furthermore, we identify sufficient conditions for uniqueness and the stability of equilibrium in this environment. This aligns the desirable welfare properties of Lindahl equilibrium with a set of desirable behavioral properties one would like in practice. We begin however with the following useful implication of Theorem 1 which follows directly from the assumptions on primitives that give us of a unique Lindahl equilibrium in the E^Q environment.

Remark: For any $e \in E^Q$, the mechanism $\varphi^{\xi,\gamma,\delta}$ has a unique Nash equilibrium.

For N players in the E^Q environment and with an appropriate choice of mechanism parameters, the new mechanism induces a supermodular game. Recall from **Definition 6** that in the E^Q environment $\frac{\partial^2 v^i}{\partial x^2}$ is bounded from below by K_i for all $x \geq 0$. Theorem 2 therefore gives a *sufficient* condition for the game to be globally supermodular.

Theorem 2 *For any $e \in E^Q$, the mechanism $\varphi^{\xi,\gamma,\delta}$ induces a supermodular game if*

$$\begin{aligned} \gamma &\leq \frac{\delta}{N-1} + \min_{i \in I} K_i \\ \xi &\in \left[\frac{(N-1)}{N} \left(\gamma + \delta - \min_i K_i \right), \delta \right] \end{aligned}$$

Proof. See Appendix. ■

Theorem 2 provides conditions under which the mechanism induces a supermodular game. If the strategy set for each player is a compact rectangle in $\mathbb{R}^2$, then the game induced by the mechanism satisfies the Milgrom and Roberts conditions referred to above. However, simply compactifying the strategy set has a number of troubling consequences. Perhaps the most obvious of these is that the uniqueness result in the remark no longer applies. There may now exist boundary equilibria of the mechanism which are not Lindahl equilibria.[5] In the next section, we discuss a solution to this problem.

[5] This is an issue since Milgrom and Roberts only show that adaptive behavior converges

4.1.1 Stability

One of our goals is to find preference environments for which the new mechanism induces a game with a unique and stable equilibrium. Thus far it has been shown that, in quasi-linear environments, the mechanism φ has a unique Nash equilibrium and has the increasing difference and supermodular properties. These properties were also shown for the Chen mechanism in her 2002 paper. These two properties are typically not enough to guarantee stability of equilibrium. In fact, it is relatively straightforward to devise mechanisms (with an unbounded strategy space) that are supermodular with a unique, unstable equilibrium. For pedagogical reasons, we provide the following two-player, variation of the Walker mechanism as an example.

Specifically, each consumer chooses a 2 dimensional message in $\mathbb{R}^2$, where for each agent $m_i = (r_i, s_i)$ is an arbitrary element. These messages are collected by the planner and used to determine an amount of the public good and a tax for each player i according to outcome functions $\chi(\mathbf{r}, \mathbf{s})$ and $\tau^i(\mathbf{r}, \mathbf{s})$ respectively. For any positive real numbers ξ, γ, and δ, let $\varphi_{NS}^{\xi,\gamma,\delta}(\mathbf{r}, \mathbf{s}) = \Big(\chi(\mathbf{r}, \mathbf{s}), (\omega_i - \tau^i(\mathbf{r}, \mathbf{s}))_{i=1}^{N}\Big)$ be a mechanism with outcome functions defined as follows:

$$\begin{aligned}
\chi(\mathbf{r}, \mathbf{s}) &= r_1 - r_2 \\
\tau_1(\mathbf{r}, \mathbf{s}) &= (\frac{\beta}{N} - \xi r_2 - \xi s_2) \cdot \chi(r, s) + \frac{\gamma}{2}(s_1 - r_2)^2 + \frac{1}{2}(s_2 - r_1)^2 \\
\tau_2(\mathbf{r}, \mathbf{s}) &= (\frac{\beta}{N} + \xi r_1 + \xi s_1) \cdot \chi(r, s) + \frac{\gamma}{2}(s_1 - r_2)^2 + \frac{1}{2}(s_2 - r_1)^2.
\end{aligned}$$

To distinguish this mechanism from the new mechanism we refer to it as $\varphi_{NS}^{\xi,\gamma,\delta}$ (NS for "not stable"). It is relatively straightforward to show that, for *any* choice of the parameters, $\varphi_{NS}^{\xi,\gamma,\delta}$ will Nash implement the Lindahl allocations of a general environment and in quasi-linear environments the mechanism induces a supermodular game with a unique equilibrium, but the unique equilibrium is unstable.[6] In other words, we can replicate Theorem 1, Corollary 1, and Theorem 2 (minus the conditions on the mechanism parameters) for $\varphi_{NS}^{\xi,\gamma,\delta}$.

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to the bounds of the outermost Nash equilibria. If there are equilibria on the boundary then there is no predictive power.

[6] It has eigenvalues that are positive (which rules out stability in continuous time) and outside the unit circle (which rules out stability in discrete time).

The existence of such a mechanism would seem to contradict the Milgrom and Roberts stability theorem, but recall that an unbounded strategy space does not meet the criteria of their theorem. Specifically, the strategy space needs to be a complete lattice. If the strategy space is compactified for these problematic mechanisms, there would be boundary equilibria, and the Milgrom and Roberts stability result (which now applies) only predicts behavior will coincide between the extrema equilibria – which is not very useful. One way we can rule this sort of thing out is by creating conditions that ensure compacting the strategy space would not create new equilibria.

The most natural method to do this is to look for conditions that make the best reply mapping a contraction, for the following reasons. First, if the best reply mapping is a contraction, the equilibrium will be unique whether the strategy set is $\mathbb{R}^{2N}$ or a compact rectangle in $\mathbb{R}^{2N}$. This observation makes the theorem immediately relevant to the problems observed in the previous section. Second, the Contraction Mapping Theorem provides an algorithm for finding the unique fixed point of the game. We will elaborate on the application of this part of the theorem in Corollary 1. Clearly, a contraction mapping is a powerful tool. However, the sufficient conditions for such a mapping are sometimes difficult to use. In this section, we provide the somewhat surprising result that if the new mechanism induces a supermodular game, then the best reply map is always a contraction. Thus, we can replace the less tractable sufficient conditions found by brute force calculating the slopes of the reaction curves with the relatively simple parameter conditions from Theorem 2. Then, taking advantage of the Contraction Mapping Theorem, we show that the sufficient conditions for uniqueness and stability of the Nash equilibrium are satisfied even if the strategy space is compactified.[7] This can also be viewed as an alternative proof to the remark. While we later argue there is no need to compactify the strategy space, the discussion is useful since it highlights several issues in this literature.

The following theorem reports the contraction result.

Theorem 3 *If ξ, γ, and δ satisfy the supermodularity restrictions of Theorem 2, then the best reply mapping is a contraction.*

Proof. See Appendix. ■

[7] I am grateful to PJ Healy for many comments that have greatly improved this section of the paper.

Theorem 2 guarantees uniqueness of equilibrium so long as the strategy space is a complete metric space. Throughout the paper the complete metric space $\mathbb{R}^{2N}$ (with the usual metric) has been used. Consequently, Theorem 2 provides an alternative proof to the uniqueness result in Corollary 1. Since the best reply mapping is a contraction and since the compact rectangle in $\mathbb{R}^{2N}$ (with the usual metric) is still a complete metric space, we can compactify its strategy space and remain confident that our Nash equilibrium is unique (and finite). Therefore, if one were inclined to compactify the strategy space the mechanism φ will induce a game with a unique equilibrium that also satisfies the Milgrom and Roberts' dynamic stability properties. Thus, at the cost of shrinking the set of applicable preference environments to quasi-linear environments, we gain the property that only rationalizable strategies coincide with the Nash strategies and that the unique equilibrium is stable under "adaptive" learning dynamics such as fictitious play, k-period average best response, and Bayesian learning.

Unfortunately, compactifying the strategy set in this manner is an unacceptable way of guaranteeing stability for this class of mechanisms. Despite the fact that we always have a unique equilibrium, we cannot be sure that the equilibrium corresponds to the Lindahl outcome unless the strategy sets are compactified in such a way to keep the original equilibrium strategies in the strategy space. A planner would, in general, not have enough information to guarantee that equilibrium messages would be in the interior of the compactified message space. Thus, by arbitrarily compactifying the message space, we could actually eliminate the nice equilibrium outcome and prevent rational players from learning to achieve the Lindahl allocation. Fortunately, using the result from Theorem 3, stability of equilibrium can be ensured under some learning dynamics without resorting to compacting the strategy space. We formalize this statement in the following corollary.

Corollary 1 *If ξ, γ, and δ satisfy the supermodularity restrictions of Theorem 2, then the unique equilibrium of the induced game is stable under the myopic best reply learning algorithm.*

The corollary follows immediately from the "Successive Approximations" result of the Contraction Mapping Theorem. Thus, starting at any initial strategy profile and iterating the best reply mapping we are guaranteed by the Contraction Mapping Theorem to converge to the unique equilibrium –

i.e., the equilibrium is stable under the myopic best reply learning algorithm. Thus, in quasi-linear environments, without resorting to compactifying procedures, supermodularity of the game induced by the new mechanism actually ensures *existence*, *uniqueness,* and *global stability* of equilibrium.[8]

5 Out-of-Equilibrium Tax Penalties of the New Mechanism

In equilibrium, all Lindahl mechanisms yield the same, nice welfare properties. However, in practice, we do not expect people to immediately find equilibrium. We have argued throughout this paper that ensuring the dynamic stability of equilibrium is essential for participants in a Lindahl mechanism to find their way to equilibrium. The loss in welfare to consumers and the surplus/ deficit in tax revenue that the government incurs while consumers are learning to play equilibrium can be considered the cost of implementing the mechanism. These issues are highlighted in the VLW experiment where the Chen mechanism's out-of-equilibrium tax penalties are often quite severe. In this section, we briefly highlight a structural property of the new mechanism that was chosen specifically to provide a better alternative to the Chen mechanism – i.e., the manner in which the new mechanism penalizes incorrect statements.

In the new mechanism, for each consumer i, his personalized price function depends only on the statement of consumer $i+1$. The personalized price of Chen's mechanism depends on the statements off all the other players. This seemingly innocuous choice of personalized price function actually yields a potential welfare issue related to the statement penalties. In order to get the right complementarity between actions in quasi-linear environments, this choice of personalized price requires Chen to include a separate squared difference penalty for *each* consumer in the economy. In other words, a term $\frac{\delta}{2}\left(s_j - \chi\left(\mathbf{r},\mathbf{s}\right)\right)^2$ is added to the Chen tax function for each consumer $j \neq i$ in the economy. While in equilibrium each of these terms will be equal to

[8] Some non-equilibrium messages might take consumers outside of their consumption set C_i. This is a little troubling when considering out-of-equilibrium dynamics. Kim (1993) gets around this issue by taking the consumption set to be the whole of $\mathbb{R}^2$. For our environment, we can always find a neighborhood around equilibrium where all messages are feasible and equilibrium is locally feasible and stable.

zero and drop out of the tax function, when out of equilibrium, even small incorrect statements by each player can quickly increase the taxes each consumer has to pay (the magnitude of the penalties depends on the specific parameterization of the mechanism). This welfare issue was documented by Van Essen, Lazzati, and Walker (2008), where, in an experiment, subjects' incorrect statements often created large losses for all consumers, as well as generated large revenue swings to the government, and overall losses in welfare.

Since consumers in the new mechanism have only one penalty term connected to the statement of their neighbor, statement penalties for each consumer in a similar (parametric) situation to the situation mentioned above will also be significantly smaller. Additionally, from a welfare perspective, individuals are shielded from large incorrect guesses by everyone other than their partner. In actual implementation of the mechanism, it is easy to imagine that one consumer in a group may be a little slow to correct his statement. In the Chen mechanism, every participant pays for this slowness, while in the new mechanism only one other person is affected. Lastly, since for any economy size N, the new mechanism's personalized price for consumer i depends only on the statement of his neighbor $i+1$, it maintains this bilateral structure as N increases.[9] The degree to which this structural difference in penalties matters in actual implementation is an empirical question and is currently the subject of ongoing research.

6 Conclusion

We have introduced a new incentive compatible mechanism capable of implementing Lindahl allocations as Nash equilibria. While a simplified economy with two goods was used for the exposition, it is straightforward to generalize the mechanism to accommodate economies with an arbitrary number of private and public goods. We have seen in an example that a incentive compatible Lindahl mechanism inducing a supermodular game is not enough to get a dynamically stable Nash equilibrium. This observation led us to use the contraction mapping as our tool to produce stability. Ironically, for

[9] This circular ordering structure has been used by Hurwicz (1979) and Walker (1981) in their Lindahl mechanisms. Saijo (1988) also uses the same circular ordering structure to obtain a significant strategy space reduction of Maskin's Canonical mechanism.

this mechanism, we have shown that inducing a supermodular game is sufficient to guarantee that the best reply mapping is a contraction. Thus, supermodularity gives us a relatively simple condition to produce stability. Finally, we remark that the new mechanism has several desirable behavioral properties that suggest its out-of-equilibrium performance will improve on the difficulties with the Chen mechanism that were observed in the VLW experiment.

There are several interesting areas for future research. For example, it is known that two-dimensional stable Lindahl mechanisms can be found in quasi-linear preference environments (i.e., the mechanism introduced in this paper). And while the stability results in quasi-linear environments are important, it is unknown what is the maximum preference domain for stable environments. A natural extension of the quasi-linear environments could be those defined by generalized Bergstrom-Cornes preferences. It would also be nice to know if it is possible to find a Lindahl mechanism that is stable for some environments and always in budget balance; or a stable, one-choice-variable, Lindahl mechanism. Additional research on Lindahl and Walrasian contractive mechanisms is currently being explored by Healy and Mathevet (2009), who show, in a manner akin to the Milgrom and Roberts' results, contractive mechanisms induce games for which a wide variety of learning rules (other than myopic best reply) converge to the equilibrium bounds in this framework. Finally, we need more experiments on implementation theory. Experiments give us a better handle on what mechanism characteristics work or do not work in a more applied environment.

7 Appendix

The strategy for the proof of Theorem 1 will be as follows: first, we demonstrate that a Nash allocation is Pareto optimal via an argument similar to the one used by Groves and Ledyard (1979); second, using the fact that the Nash allocation is Pareto optimal, we use an "unbiasedness" proof similar to Foley (1970) p. 68-69 and Chen (2002) to establish that the outcome is Lindahl; finally, we show that any Lindahl allocation is achieved as a Nash allocation of the mechanism using a technique we believe was first used by Walker (1981).

Lemma 1 *Suppose the strategy profile* $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ *is a Nash equilibrium of* $\varphi^{\xi,\gamma,\delta}$

for $e \in E$, where $(\bar{x}, \bar{y}_i)$ is consumer i's Nash allocation, then the following statements are true:

1. *For any bundle $(x, y_i) \in C_i$, there is a pair (r_i, s_i) such that $x = \chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i})$.*

2. *The private good consumed by consumer i in equilibrium is $\bar{y}_i \equiv \omega_i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$.*

3. *Consumer i's statement and tax are $\bar{s}_i = \frac{1}{N}\sum_{i=1}^{N} \bar{r}_i$ and $\tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) \cdot \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ for all i.*

4. *If a feasible allocation $(x, y_i) \in C_i$ is weakly preferred to the Nash allocation $(\bar{x}, \bar{y}_i)$, then the preferred bundle is at least as expensive as consumer i's initial wealth (i.e, $y_i + \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) \geq \omega_i$).*

5. *If a feasible allocation $(x, y_i) \in C_i$ is strictly preferred to the Nash allocation $(\bar{x}, \bar{y}_i)$, then the preferred bundle is more expensive than consumer i's initial wealth (i.e., $y_i + \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) > \omega_i$).*

6. *Consumer i's equilibrium allocation is in the interior of his consumption set $(\bar{x}, \bar{y}_i) \in C_i^{++}$, and there exists (r_i, s_i) such that $\chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) > 0$ and $\tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) < \omega_i$.*

Proof.

L1.1 Since χ only depends on the requests of individuals, set $x = \frac{1}{N}(r_i + \sum_{j \neq i} \bar{r}_j)$ or $r_i = Nx - \sum_{j \neq i} \bar{r}_j$.

L1.2 Consider the following two bundles $(\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \bar{y}_i)$ and $(\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \hat{y}_i) \in C_i$, where $0 \leq \hat{y}_i < \omega_i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$. Since preferences are complete, transitive, and strictly increasing in y_i, we have $(\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \bar{y}_i) \succ_i (\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \hat{y}_i)$ for all $\hat{y}_i$.

L1.3 Since (r, s) is a Nash equilibrium, then for each consumer i

$$(\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \omega_i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})) \succeq_i (\chi(\bar{\mathbf{r}}, \mathbf{s}_{-i}, s_i), \omega_i - \tau^i(\bar{\mathbf{r}}, \mathbf{s}_{-i}, s_i)) \text{ for all } s_i.$$

From the functional form of the tax function and since preferences are complete, transitive, and strictly increasing in y_i, for each i, $\bar{s}_i = \frac{1}{N}\sum_{i=1}^{N} \bar{r}_i$. It follows directly that $\tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) \cdot \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$.

L1.4 Suppose not. Then $y_i+\tau^i(\bar{\mathbf{r}}_{-i},\bar{\mathbf{s}}_{-i},r_i,s_i) < \bar{y}_i+\tau^i(\bar{\mathbf{r}},\bar{\mathbf{s}}) = \omega_i$. Since τ^i is continuos and using the fact that preferences are continuos, convex, and strictly increasing in y_i, there exists $(\acute{y}_i,\acute{r}_i,\acute{s}_i)$ such that $(\chi(\acute{r}_i,\bar{\mathbf{r}}_{-i},\acute{s}_i,\bar{\mathbf{s}}_{-i}),\acute{y}_i) \in C_i$, $\acute{y}_i + \tau^i(\acute{r}_i,\bar{\mathbf{r}}_{-i},\acute{s}_i,\bar{\mathbf{s}}_{-i}) \leq \omega_i$, and $(\chi(\acute{r}_i,\bar{\mathbf{r}}_{-i},\acute{s}_i,\bar{\mathbf{s}}_{-i}),\acute{y}_i) \succ_i (\bar{x},\bar{y}_i)$. However, this means that there is an individually feasible bundle which is strictly preferred to the Nash allocation. This contradicts the assumption that $(\bar{\mathbf{r}},\bar{\mathbf{s}})$ is a Nash equilibrium.

L1.5 Suppose not. Then $(\bar{r}_i,\bar{s}_i)$ is not a best response which contradicts the assumption that $(\bar{\mathbf{r}},\bar{\mathbf{s}})$ is a Nash equilibrium.

L1.6 This lemma follows directly from continuity, strictly increasing preferences in the private good, and strict preference for interior allocation in the E environment.

■

Lemma 2 *Suppose consumer i could purchase units of the public good at a price of t_i, where t_i is defined as consumer i's equilibrium marginal tax rate (i.e., $t_i \equiv P^i(\bar{\mathbf{r}},\bar{\mathbf{s}})$), then the following statements are true:*

1. *If a feasible allocation $(x,y_i) \in C_i$ is weakly preferred to the Nash allocation $(\bar{x},\bar{y}_i)$ was then the preferred bundle is at least as expensive as the Nash allocation (i.e., $y_i + t_i \cdot x \geq \bar{y}_i + t_i \cdot \bar{x}$).*

2. *If an allocation achieved in the mechanism is less expensive than the Nash allocation (i.e., $y_i + \tau^i(r_i,\bar{\mathbf{r}}_{-i},s_i,\bar{\mathbf{s}}_{-i}) < \bar{y}_i + \tau^i(\bar{\mathbf{r}},\bar{\mathbf{s}})$) , then the same allocation is less expensive if the public good could be purchased at a price of t_i (i.e., $y_i + t_i \cdot x < \bar{y}_i + t_i \cdot \bar{x}$).*

3. *If a feasible allocation $(x,y_i) \in C_i$ is strictly preferred to the Nash allocation $(\bar{x},\bar{y}_i)$, then the preferred bundle is more expensive than the Nash allocation (i.e., $y_i + t_i \cdot x > \bar{y}_i + t_i \cdot \bar{x}$).*

Proof.

L2.1 By definition, $t_i = P^i(\bar{\mathbf{r}},\bar{\mathbf{s}})$. By assumption, preference are convex so certainly the set of bundles that are weakly preferred to $(\bar{x},\bar{y}_i)$ is convex and $(\bar{x},\bar{y}_i)$ is on the boundary of the set. Let the set of affordable bundles be denoted $B_i = \left\{(x, y_i) \in C_i | y_i + \hat{\tau}^i(x; \bar{\mathbf{r}}_{-i}, \bar{\mathbf{s}}_{-i}) \leq \omega_i\right\}$, where $\hat{\tau}^i(x; \bar{r}_{-i}, \mathbf{s}) = P^i(\mathbf{r}, \mathbf{s}) \cdot x + \frac{\gamma}{2}(s_i - x)^2 + \frac{\delta}{2}(\bar{\mathbf{s}}_{i+1} - x)^2$. B is convex since $\hat{\tau}^i$ is a convex function of x. By part (2) of Lemma 1, $(\bar{x}, \bar{y}_i)$ is on the boundary of set B_i. From part (5) of Lemma 1, we have the intersection of the set of weakly preferred bundles to $(\bar{x}, \bar{y}_i)$ (denote WP_i) and the budget set B_i is empty. From the Separating Hyperplane Theorem, there exists a hyperplane through $(\bar{x}, \bar{y}_i)$ that separates WP_i and B_i. The vector $(t_i, 1)$ defines this hyperplane. Also from the Separating Hyperplane Theorem, we have that $y_i + t_i \cdot x \geq c$ and $\bar{y}_i + t_i \cdot \bar{x} = c$, where $c \neq 0$. It follows that $y_i + t_i \cdot x \geq \bar{y}_i + t_i \cdot \bar{x}$.

L2.2 If $y_i + \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) < \bar{y}_i + \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = \omega_i$, we can expand each of these expressions to $y_i + P^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) \cdot \chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) + \frac{\gamma}{2}(s_i - x)^2 + \frac{\delta}{2}(\bar{s}_{i+1} - x)^2 < \bar{y}_i + P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) \cdot \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$. Let $x = \chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i})$ and $\bar{x} = \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$. By construction, the personalized price function $P^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = t_i$. we can subtract the two squared terms on the LHS to get $y_i + t_i \cdot x < \bar{y}_i + t_i \cdot \bar{x}$.

L2.3 Suppose not. By part (1) of Lemma 2, we have $y_i + t_i \cdot x = \bar{y}_i + t_i \cdot \bar{x}$. Since preferences are continuous there exists a neighborhood of (x, y_i), denoted $N(x, y_i)$ such that for all $(\hat{x}, \hat{y}_i) \in N(x, y_i) \cap C_i$, $(\hat{x}, \hat{y}_i) \succ_i (\bar{x}, \bar{y})$. Parts 1 and 6 of Lemma 1 and part 2 of Lemma 2 imply that there exists a bundle $(\acute{x}, \acute{y}_i) \in C_i$ such that $\acute{y}_i + t_i \cdot \acute{x} < \bar{y}_i + t_i \cdot \bar{x} = y_i + t_i \cdot x = \omega_i$. Let

$$G \equiv \left\{(\hat{x}, \hat{y}_i) \in C_i | \, (\hat{x}, \hat{y}_i) = (\lambda \acute{x} + (1 - \lambda)\, x, \lambda \acute{y}_i + (1 - \lambda)\, y_i) \text{ for all } \lambda \in (0, 1)\right\}.$$

All points in this line between $(\acute{x}, \acute{y}_i)$ and (x, y_i) have a value smaller than $(\bar{x}, \bar{y}_i)$. However since the consumption set is convex it follows that there exists a λ which is small enough such that $N(x, y_i) \cap G$ –i.e. there exists a bundle $(\hat{x}, \hat{y}_i)$ such that $(\hat{x}, \hat{y}_i) \succ_i (\bar{x}, \bar{y})$ and $\hat{y}_i + t_i \cdot \hat{x} < \bar{y}_i + t_i \cdot \bar{x}$ which leads to a contradiction of part (1) of Lemma 2.

■

The next lemma and its proof are almost identical to those in the First Fundamental Welfare Theorem for private good economies (see Debreu 1959).

Lemma 3 *Suppose* $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ *is a Nash equilibrium of* $\varphi^{\xi,\gamma,\delta}$ *for* $e \in E$ *, then the Nash allocation* $\left[\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), (\omega - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}))_{i=1}^N\right]$ *is Pareto optimal.*

Proof. Suppose $\left[\bar{x}, (\bar{y}_i)_{i=1}^N\right]$ is not a Pareto optimal allocation and that $\left[x, (y_i)_{i=1}^N\right]$ is a feasible, Pareto superior allocation. From part 3 of Lemma 2, we have that

$$y_i + t_i \cdot x > \bar{y}_i + t_i \cdot \bar{x} \qquad \text{for all } i.$$

Summing across all consumers, we have

$$\sum_{i=1}^N y_i + \sum_{i=1}^N t_i \cdot x > \sum_{i=1}^N \bar{y}_i + \sum_{i=1}^N t_i \cdot \bar{x}.$$

By construction, $\sum_{i=1}^N t_i = \sum_{i=1}^N P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = \beta$. Re-writing the above strict inequality, we have

$$\sum_{i=1}^N y_i + \beta \cdot x > \sum_{i=1}^N \bar{y}_i + \beta \cdot \bar{x} = \sum_{i=1}^N \omega_i.$$

Thus, the Pareto superior bundle is not feasible. ■

Lemma 4 *The affordable feasible set, denoted F, is a convex set, where*

$$F = \left\{ \begin{array}{c} (x_1, ..., x_N, y_1, ..., y_N) \,|\, (x_i, y_i) \in C_i \\ \text{where } x_i = x_j = x \text{ for all } j \neq i \text{ and } x \leq \frac{\sum_{i=1}^N (\omega_i - y_i)}{\beta} \end{array} \right\}.$$

Furthermore, the point $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$, associated with the Nash equilibrium, is on the boundary of F.

Proof. To show that F is convex choose two arbitrary profiles

$$(x_1, ..., x_N, y_1, ..., y_N), (\acute{x}_1, ..., \acute{x}_N, \acute{y}_1, ..., \acute{y}_N) \in F.$$

For $\lambda \in [0, 1]$, the convex combination of these two vectors is

$$(\lambda x_1 + (1-\lambda)\acute{x}_1, ..., \lambda x_N + (1-\lambda)\acute{x}_N, \lambda y_1 + (1-\lambda)\acute{y}_1, ..., \lambda y_N + (1-\lambda)\acute{y}_N).$$

First, since C_i is convex, $(\lambda x_i + (1-\lambda)\acute{x}_i, \lambda y_i + (1-\lambda)\acute{y}_i) \in C_i$ for all i. Second, because both $x_i = x_j = x$ and $\acute{x}_i = \acute{x}_j = \acute{x}$ for all $j \neq i$, then $\lambda x_i + (1-\lambda)\acute{x}_i = \lambda x_j + (1-\lambda)\acute{x}_j = \lambda x + (1-\lambda)\acute{x}$. Finally, $\beta x \leq \sum_{i=1}^N (\omega_i - y_i)$ implies $\lambda \beta x \leq \lambda \sum_{i=1}^N (\omega_i - y_i)$. Similarly, $\beta \acute{x} \leq \sum_{i=1}^N (\omega_i - \acute{y}_i)$ implies

$(1-\lambda)\,\beta\acute{x} \leq (1-\lambda)\sum_{i=1}^{N}(\omega_i - \acute{y}_i)$. Adding these two conditions together, we have the following inequality,

$$\lambda x + (1-\lambda)\,\acute{x} \leq \frac{\sum_{i=1}^{N}\left(\omega_i - \left(\lambda y_i + (1-\lambda)\,\acute{y}_i\right)\right)}{\beta}.$$

verifying that the set F is convex.

To see that $(\bar{x}_1,...,\bar{x}_N,\bar{y}_1,...,\bar{y}_N)$ is in the boundary of the set. Recall from the Lemma 3 that the Nash allocation is Pareto optimal–i.e. $\sum_{i=1}^{N}\bar{y}_i + \beta\cdot\bar{x} = \sum_{i=1}^{N}\omega_i$. Re-arranging this expression, we have $\bar{x} = \frac{\sum_{i=1}^{N}(\omega_i - \bar{y}_i)}{\beta}$, which is clearly on the boundary of F. ■

Proof of Theorem 1. The proof for Theorem 1 is done in three parts. In the first part of the proof we show that for any $e \in E$ if $(\bar{\mathbf{r}},\bar{\mathbf{s}})$ is a Nash equilibrium of $\varphi^{\xi,\gamma,\delta}$, the corresponding allocation $\left[\chi(\bar{\mathbf{r}},\bar{\mathbf{s}}),(\omega_i - \tau^i(\bar{\mathbf{r}},\bar{\mathbf{s}}))_{i=1}^{N}\right]$ is a Lindahl equilibrium and for each i, $P^i(\bar{\mathbf{r}},\bar{\mathbf{s}})$ is the corresponding Lindahl price. It is first shown that the personalized price associated with the Nash equilibrium per unit tax $P^i(\bar{\mathbf{r}},\bar{\mathbf{s}})$ defines a separating hyperplane between the feasible allocation set F and the preferred set D; second, we show that the Nash allocation is the allocation that maximizes a consumer's preferences subject to a budget constraint when facing the personalized price $P^i(\bar{\mathbf{r}},\bar{\mathbf{s}})$; finally, we show that the tax revenue equals the cost of producing the public good.

In the second part of the proof, we show that for $e \in E$ if $\left(\bar{P}^1,...,\bar{P}^N\right)$ is the profile of Lindahl prices and $\left(\bar{x},\left(\omega_i - \bar{P}^i\cdot\bar{x}\right)_{i=1}^{N}\right)$ is the corresponding Lindahl allocation, then it must correspond to a Nash equilibrium of the mechanism. we do this by first showing that the messages that could achieve this allocation in the mechanism are unique. Subsequently that this profile of strategies is a Nash equilibrium of the game induced by the mechanism.

Part 3 of the proof estabilishes Nash implementation for the E^Q environment.

(Part 1): Consider the point $(\bar{x}_1,...,\bar{x}_N,\bar{y}_1,...,\bar{y}_N)$ associated with the Nash allocation for each consumer. From Lemma 4, we have that the feasible set F is convex and that the point $(\bar{x}_1,...,\bar{x}_N,\bar{y}_1,...,\bar{y}_N)$ is on its boundary. Similarly from Lemma 5, we have that the set D is convex and point $(\bar{x}_1,...,\bar{x}_N,\bar{y}_1,...,\bar{y}_N)$ is on the boundary. Notice that the intersection of the interiors of F and D have no points in common. To see this suppose that these sets do have points in the interior that are common. Then there is a strictly cheaper feasible point that is weakly preferred by all consumers. However, this contradicts the fact that $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$ is Pareto optimal (Lemma 3). Therefore by the Separating Hyperplane Theorem, there exists a vector $(p_1^x, \cdots, p_N^x, p_1^y, \cdots, p_N^y) \neq 0$ and $c \in \mathbb{R}$ such that for all points in the weakly preferred set D,

$$(\sum_{i=1}^{N} p_i^x) \cdot x + \sum_{i=1}^{N} p_i^y \cdot y_i \geq c.$$

In addition, since the vector $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$ is in the boundary of both F and G,

$$(\sum_{i=1}^{N} p_i^x) \cdot \bar{x} + \sum_{i=1}^{N} p_i^y \cdot \bar{y}_i = c.$$

Since $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is a Nash equilibrium, the hyperplane that crosses through $(\bar{x}, \bar{y}_i)$ is defined by the vector of $(p_i^x, p_i^y) = (t_i, 1)$ for each i where $p_i^x = t_i = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ (Lemmas 1 and 2). This should be thought of as consumer i's personalized price.

Next, we show that the bundle $(\bar{x}, \bar{y}_i)$ maximizes the preferences of consumer i subject to i's budget constraint when facing $P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ as his personalized price.

Suppose $(x_i, y_i) \succ_i (\bar{x}, \bar{y}_i)$ while $x_j = \bar{x}$ and $y_j = \bar{y}_j$ for all $j \neq i$. This point is in set D. From the separating hyperplane defined above we have,

$$\left(\sum_{i=1}^{N} P^i(\bar{r}, \bar{s})\right) \cdot x + \sum_{i=1}^{N} y_i \geq \left(\sum_{i=1}^{N} P^i(\bar{r}, \bar{s})\right) \cdot \bar{x} + \sum_{i=1}^{N} \bar{y}_i.$$

All terms in this expression are the same except those belonging to consumer i. Thus the expression can be simplified to $y_i + P^i(\bar{r}, \bar{s}) \cdot x \geq \bar{y}_i + P^i(\bar{r}, \bar{s}) \cdot \bar{x}$. From part 3 of Lemma 2, since the bundle $(x_i, y_i) \succ_i (\bar{x}, \bar{y}_i)$ equality cannot hold so

$$P^i(\bar{r}, \bar{s}) \cdot x + y_i > P^i(\bar{r}, \bar{s}) \cdot \bar{x} + \bar{y}_i.$$

The personalized price for consumer i is independent i's actions–i.e., $P^i(\bar{r}, \bar{s}) = P^i(r_i, \bar{r}_{-i}, s_i, \bar{s}_{-i})$ for all r_i and s_i. Using this fact we am going to rewrite the above expression to be

$$P^i(r_i, \bar{r}_{-i}, s_i, \bar{s}_{-i}) \cdot x + y_i > P^i(\bar{r}, \bar{s}) \cdot \bar{x} + \bar{y}_i.$$

Now adding two appropriately chosen positive terms on the LHS, we have

$$P^i\left(r_i,\bar{r}_{-i},s_i,\bar{s}_{-i}\right)\cdot x+y_i+\frac{\gamma}{2}\left(s_i-x\right)^2+\frac{\delta}{2}\left(\bar{s}_{i+1}-x\right)^2>P^i\left(\bar{r},\bar{s}\right)\cdot\bar{x}+\bar{y}_i.$$

However, this is equivalent to $y_i+\tau^i\left(r_i,\bar{r}_{-i},s_i,\bar{s}_{-i}\right)>\bar{y}_i+\tau^i(\bar{r},\bar{s})=\omega_i$, where $x=\frac{1}{N}(r_i+\sum_{j\neq i}\bar{r}_j)$ and $\bar{x}=\frac{1}{N}(\bar{r}_i+\sum_{j\neq i}\bar{r}_j)$. Thus, any bundle that is strictly preferred to the Nash bundle is not affordable by the consumer–i.e., the Nash allocation maximizes consumer i's preferences subject to a budget constraint.

The last part of the argument requires tax revenue to equal the total cost of production.

If we add up the tax revenue, we have that

$$\begin{aligned}
\sum_{i=1}^{N}\tau^i(\bar{r},\bar{s}) &= \sum_{i=1}^{N}P^i(\bar{r},\bar{s})\cdot\ \chi(\bar{r},\bar{s}) \\
&= \sum_{i=1}^{N}\left(\frac{\beta}{N}-\xi\sum_{j\neq i}\frac{\bar{r}_j}{N-1}+\xi\bar{s}_{i+1}\right)\cdot\chi(\bar{r},\bar{s}) \\
&= \left(\beta-\xi\sum_{i=1}^{N}\sum_{j\neq i}\frac{\bar{r}_j}{N-1}+\xi\sum_{i=1}^{N}\bar{s}_{i+1}\right)\cdot\chi(\bar{r},\bar{s}) \\
&= \left(\beta-\xi N\chi(\bar{r},\bar{s})+\xi N\chi(\bar{r},\bar{s})\right)\cdot\chi(\bar{r},\bar{s}) \\
&= \beta\cdot\chi(\bar{r},\bar{s}).
\end{aligned}$$

Thus the allocation is feasible and this is a Lindahl allocation, where $\left(\bar{P}^1,...,\bar{P}^N\right)$ will be the profile of Lindahl prices.

(Part 2): For all i, let $\bar{s}_i=\bar{x}$. Consider the following system of N linear equations and N variables $(r_1,...,r_N)$

$$\begin{aligned}
r_1+r_2+\cdots+r_N &= N\cdot\bar{x} \\
-\sum_{j\neq i}r_j &= \left[\frac{1}{\xi}\left(\bar{P}^i-\frac{\beta}{N}\right)-\bar{s}_{i+1}\right](N-1) \qquad \text{for } i=1,...,N-1
\end{aligned}$$

It is straightforward to verify that the $N\times N$ coefficient matrix of this system of equations is non-singular with a rank of N. Thus, the system has a unique solution which we will call $(\bar{r},\bar{s})$. It remains to show that $(\bar{\mathbf{r}},\bar{\mathbf{s}})$ is a Nash equilibrium.

Since the allocation $(\bar{x}, (\omega_i - \bar{P}^i \cdot \bar{x})_{i=1}^N)$ is Lindahl, $(\bar{x}, \omega_i - \bar{P}^i \cdot \bar{x}) \succeq_i (x, \omega_i - \bar{P}^i \cdot x)$ for all x. Let $x = \frac{1}{N}(r_i + \sum_{j \neq i} \bar{r}_j) = \chi(r_i, \bar{\mathbf{r}}_{-i}, \bar{\mathbf{s}})$, then

$$(\bar{x}, \omega_i - \bar{P}^i \cdot \bar{x}) \succeq_i \left(\frac{1}{N}(r_i + \sum_{j \neq i} \bar{r}_j), \omega_i - \bar{P}^i \cdot \frac{1}{N}(r_i + \sum_{j \neq i} \bar{r}_j) \right)$$

for all r_i.

Similarly, since preferences are strictly increasing in y_i, it is also true that

$$\begin{aligned} & (\bar{x}, \omega_i - \bar{P}^i \cdot \bar{x}) \\ \succeq & {}_i \left(\frac{1}{N}(r_i + \sum_{j \neq i} \bar{r}_j), \omega_i - \bar{P}^i \cdot \frac{1}{N}(r_i + \sum_{j \neq i} \bar{r}_j) - \frac{\gamma}{2}(s_i - \frac{1}{N} \sum_{i=1}^N \bar{r}_i)^2 \right. \\ & \left. - \frac{\gamma}{2}(s_{i+1} - \frac{1}{N} r_i - \frac{1}{N} \sum_{j \neq i} \bar{r}_j)^2 \right) \end{aligned}$$

for all r_i, s_i.

By construction, the public good

$$\bar{x} = \frac{\bar{r}_1 + \cdots + \bar{r}_N}{N} = \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$$

consumer i's Lindahl price was

$$\bar{P}^i = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$$

and $\bar{s}_i = \frac{1}{N} \sum_{k=1}^N \bar{r}_k$ for all i.

Plugging in these expressions into the above inequality, we have

$$\begin{aligned} & (\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \omega_i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})) \\ \succeq & {}_i(\chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{s}_{-i}), \omega_i - \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i})) \end{aligned}$$

for all r_i, s_i.Therefore $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is a Nash equilibrium of the mechanism.

(Part 3) For any $e \in E^Q$, if $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is a Nash equilibrium, then the stictly increasing utility in the private good implies each i's statement is correct and first order conditions for the Nash equilibrium must be $\frac{\partial v^i(x)}{\partial x} = MRS_i = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$. Since, by construction $\sum MRS_i = \sum P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = \beta$, the equilibrium production of the public good is Pareto. Moreover, since in equilibrium i's private good allocation is $\omega_i - \frac{\partial v^i(x^{PO})}{\partial x} x^{PO} > 0$ the allocation is Lindahl. By assumption, the E^Q environment has a unique Lindahl allocation. In a manner similar to part 2 of above we can show that this unique Lindahl allocation is implemented by a unique Nash equilibrium. ■

Proof of Theorem 2. Since $M_i = \mathbb{R}^2$, it is a sublattice of $\mathbb{R}^2$. By definition of being in the E^Q environment u^i is C^2 and therefore trivially satisfies the continuity requirement. To see that u^i has the supermodularity property, we appeal to the fact that the utility function is C^2. we therefore need to check the following cross-partial derivative

$$\frac{\partial^2 u^i}{\partial r_i \partial s_i} \geq 0$$

Checking this, we see that

$$\frac{\partial^2 u^i}{\partial r_i \partial s_i} = \frac{\gamma}{N} \geq 0$$

for each consumer i. The increasing difference property requires checking the following five conditions:

$$\begin{array}{llll}
(1) & \frac{\partial^2 u^i}{\partial r_i \partial r_j} & \geq 0 & \text{for all } j \neq i \\
(2) & \frac{\partial^2 u^i}{\partial r_i \partial s_j} & \geq 0 & \text{for all } j \neq i \text{ and } j \neq i+1 \\
(3) & \frac{\partial^2 u^i}{\partial r_i \partial s_{i+1}} & \geq 0 & \\
(4) & \frac{\partial^2 u^i}{\partial s_i \partial r_j} & \geq 0 & \text{for all } j \neq i \\
(5) & \frac{\partial^2 u^i}{\partial s_i \partial s_{i+1}} & \geq 0 &
\end{array}$$

Checking each of these in turn we have

$$\frac{\partial^2 u^i}{\partial r_i \partial r_j} = \xi \frac{1}{N(N-1)} - \frac{\gamma}{N^2} - \frac{\delta}{N^2} + \frac{1}{N^2} \frac{\partial^2 v^i}{\partial x^2}.$$

In order for the above expression to be positive we need

$$\xi \geq \frac{N-1}{N} \left(\gamma + \delta - \frac{\partial^2 v^i}{\partial x^2} \right) \quad \text{for all } j \neq i.$$

A more compact way of writing this is

$$\xi \geq \frac{N-1}{N}\left(\gamma + \delta - \min_{i \in I} \frac{\partial^2 v^i}{\partial x^2}\right).$$

Condition 2 is trivially satisfied since

$$\frac{\partial^2 u^i}{\partial r_i \partial s_j} = 0 \qquad \text{for all} \qquad j \neq i \text{ and } j \neq i+1.$$

Checking Condition 3 we have

$$\frac{\partial^2 u^i}{\partial r_i \partial s_{i+1}} = -\frac{\xi}{N} + \frac{\delta}{N}.$$

This expression is positive for all i if and only if

$$\xi \leq \delta.$$

Condition 4 and 5 are always satisfied since

$$\frac{\partial^2 u^i}{\partial s_i \partial r_j} = \frac{\gamma}{N} > 0 \qquad \text{and} \qquad \frac{\partial^2 u^i}{\partial s_i \partial s_{i+1}} = 0$$

Therefore, for the mechanism to be supermodular the following is sufficient.

$$\xi \in \left[\frac{N-1}{N}\left(\gamma + \delta - \min_{i \in I} K_i\right), \delta\right]$$

Finally, this interval is non-empty if and only if $\gamma \leq \frac{\delta}{N-1} + \min_{i \in I} K_i$ is true. ■

Proof of Theorem 3. First, we characterize the best replies. Applying the mechanism to each consumer's utility we arrive at the augmented utility function

$$v^i\left(\frac{1}{N}\sum_{k=1}^{N} r_k\right) - \left(\frac{\beta}{N} - \frac{\xi}{N-1}\sum_{j \neq i} r_j + \xi s_{i+1}\right)\frac{1}{N}\sum_{k=1}^{N} r_k$$
$$-\frac{\gamma}{2}\left(s_i - \frac{1}{N}\sum_{k=1}^{N} r_k\right)^2 - \frac{\delta}{2}\left(s_{i+1} - \frac{1}{N}\sum r_k\right)^2.$$

Best responding requires consumers choices to satisfy first order conditions – i.e.,,

$$
\begin{aligned}
r_i &: \frac{v_1^i}{N} - \left(\frac{\beta}{N} - \frac{\xi}{N-1}\sum_{j\neq i} r_j + \xi s_{i+1}\right)\frac{1}{N} + \frac{\gamma}{N}(s_i - \frac{1}{N}\sum_{k=1}^{N} r_k) + \frac{\delta}{N}(s_{i+1} - \frac{1}{N}\sum r_k) = 0 \\
s_i &: -\gamma(s_i - \frac{1}{N}\sum_{k=1}^{N} r_k) = 0
\end{aligned}
$$

If we let $r_i^*(r_{-i}, s_{-i})$ and $s_i^*(r_{-i}, s_{-i})$ be the solutions to these first order conditions. Clearly, $s_i^*(r_{-i}, s_{-i}) = \frac{1}{N} r_i^*(r_{-i}, s_{-i}) + \frac{1}{N}\sum_{k=1}^{N} r_k$, if we plug s_i^* into the r_i condition. The new r_i condition is

$$
\frac{v_1^i(\cdot)}{N} - \left(\frac{\beta}{N} - \frac{\xi}{N-1}\sum_{j\neq i} r_j + \xi s_{i+1}\right)\frac{1}{N} + \frac{\delta}{N}(s_{i+1} - \frac{1}{N} r_i^*(r_{-i}, s_{-i})) - \frac{1}{N}\sum r_k) = 0
$$

We can think of each decision being chosen by a separate agent: 1 agent for $r_i^*(r_{-i}, s_{-i})$ and one agent for the $s_i^*(r_{-i}, s_{-i})$. Since we have already accounted for the interaction between own decisions in the determination of the best replies, we can think of the game as one with $2N$ independent players choosing according to the specified reaction functions. The problem of showing a contraction reduces to the one of Vives.[10] Therefore a sufficient condition for the best reply map to yield a contraction is that, for each i, the absolute total change in $r_i^*(r_{-i}, s_{-i})$ and $s_i^*(r_{-i}, s_{-i})$ (evaluated at any point (r_{-i}, s_{-i})) is bounded by 1. In other words, we require

$$
\begin{aligned}
\sum_{j\neq i}\left|\frac{\partial r_i^*(r_{-i}, s_{-i})}{\partial r_j}\right| + \sum_{j\neq i}\left|\frac{\partial r_i^*(r_{-i}, s_{-i})}{\partial s_j}\right| &< 1 \\
\sum_{j\neq i}\left|\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial r_j}\right| + \left|\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial s_{i+1}}\right| &< 1
\end{aligned}
$$

[10]See, for example, p. 47.

We compute these slopes directly. Differentiating the new r_i first order condition with respect to r_j.

$$\begin{aligned}
\frac{v_{11}^i}{N^2}(1+\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j})+\frac{\xi}{N(N-1)}-\frac{\delta}{N^2}\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j}-\frac{\delta}{N^2} &= 0\\
v_{11}^i(1+\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j})+\frac{N\xi}{(N-1)}-\delta\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j}-\delta &= 0\\
v_{11}^i+\frac{N\xi}{(N-1)}-\delta &= (\delta-v_{11}^i)\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j}\\
\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j} &= \frac{v_{11}^i+\frac{N\xi}{(N-1)}-\delta}{\delta-v_{11}^i}
\end{aligned}$$

Differentiating with respect to s_{i+1}

$$\begin{aligned}
\frac{v_{11}^i}{N^2}\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}}-\frac{\xi}{N}+\frac{\delta}{N}-\frac{\delta}{N^2}\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}} &= 0\\
-N\xi+N\delta &= \delta\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}}-v_{11}^i\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}}\\
\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}} &= \frac{-N\xi+N\delta}{\delta-v_{11}^i}
\end{aligned}$$

A sufficient condition for $r^*(r_{-i},s_{-i})$ to be a contraction is that

$$\sum_{j\neq i}\left|\frac{v_{11}^i+\frac{N\xi}{(N-1)}-\delta}{\delta-v_{11}^i}\right|+\left|\frac{-N\xi+N\delta}{\delta-v_{11}^i}\right|<1.$$

Suppose γ, δ, and ξ satisfy the supermodularity conditions from Theorem 1, then the slopes are all positive leaving

$$\begin{aligned}
(N-1)v_{11}^i+N\xi-(N-1)\delta-N\xi+N\delta &< \delta-v_{11}^i\\
0 &< -Nv_{11}^i.
\end{aligned}$$

This condition is always satisfied since $v_{11}^i<0$.

Now consider $s_i^*(r_{-i}, s_{-i}) = \frac{1}{N} r_i^*(r_{-i}, s_{-i}) + \frac{1}{N}\sum_{k=1}^{N} r_k$.

$$\begin{aligned}
\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial r_j} &= \frac{1}{N}\frac{\partial r_i^*}{\partial r_j} + \frac{1}{N} \\
&= \frac{1}{N}\left(\frac{v_{11}^i + \frac{N\xi}{(N-1)} - \delta}{\delta - v_{11}^i}\right) + \frac{1}{N} \\
&= \frac{1}{N}(\frac{-(\delta - v_{11}^i) + \frac{N\xi}{(N-1)}}{\delta - v_{11}^i} + 1) \\
&= (\frac{\xi}{(N-1)(\delta - v_{11}^i)})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial s_{i+1}} &= \frac{1}{N}\frac{\partial r_i^*}{\partial s_{i+1}} \\
&= \frac{1}{N}\left(\frac{-N\xi + N\delta}{\delta - v_{11}^i}\right) \\
&= \frac{-\xi + \delta}{\delta - v_{11}^i}
\end{aligned}$$

Adding up across each player and checking the sufficient condition.

$$\sum_{j \neq i}\left|\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial r_j}\right| + \left|\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial s_{i+1}}\right| < 1$$

Supermodularity ensures the slopes are all positive. Therefore,

$$\begin{aligned}
\frac{\xi}{\delta - v_{11}^i} + \frac{-\xi + \delta}{\delta - v_{11}^i} &< 1 \\
\delta &< \delta - v_{11}^i \\
v_{11} &< 0
\end{aligned}$$

and since we have $v_{11}^i < 0$ by assumption, the second condition is satisfied. Since this is true for each i, the best reply map is a contraction. ■

8 Resources

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