CE 466 / 566 Highway Geometric Design
Spring 2002
Lecture Notes for Earthwork

Outline

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1 General concepts

In many cases, even when the design controls have been specified, there are a number of important design options available for a given highway geometric design. Given that these design criteria are met, one common objective is to minimize the overall cost of the roadway design. Primary costs include land acquisition, earthwork, pavements, structures, and operational features (signs, signals, etc.).

Common principles that apply which have the effect of minimizing costs include:

- Build the facility as straight as possible (minimize length of road and earthwork).
- Build the facility as close to the existing grade as possible (minimize earthwork).
- Build the facility with as few structural requirements as possible (bridges, etc.)

1.1 The importance of earthwork

It turns out that a significant amount of the cost of roadway construction is wrapped up in the necessary earthwork. By earthwork, we mean the amount of cut and fill that must be moved within, or imported to (borrow) and/or exported from (waste), a particular roadway site. For this reason, estimating the amount of cut and fill required for the particular roadway design becomes an important factor in estimating the cost of the roadway design.

The quantities for earthwork are typically the desired volumes of earth, either as cut or fill, for a particular segment of the roadway. In metric units, cubic meters are used; in standard units, cubic yards of material are used as the most common unit of measure.

There are a number of ways of calculating earthwork volumes; these include:

1. Using end areas and lengths to get volumes. Traditionally, these volumes are estimated using some combination of the roadway cross-section and the horizontal distance along the roadway (measured in stations or feet).

2. Using the existing surface and the proposed roadway surface directly. In this case, the differences in roadway volumes can be measured much more precisely.

1.2 Assumptions and notation for cross-sections

For roadway cross-sections, the transportation engineering profession has developed certain conventions in explaining the features of the existing cross-section and the desired roadway cross-section. First, we make a number of simplifying assumptions. These are, of course, bad assumptions, but typically the errors associated with these assumptions (as opposed to others we will make later) are small.

- Cross-sections are described from left to right, looking upstation.

- The differences in a cross-section begins at a point called a "catch point"; this is where the existing cross-section and the desired roadway cross-section begin to differ. If the existing cross-section intersects the planned roadway cross-section before reaching the other catch point, this is called a "hinge point."

- The difference in cross-sections (current vs. proposed) may be reasonably approximated using piecewise linear segments, connecting sampled points of reference in the cross-section.

- The road surface and/or subgrade, through the traveled way and shoulders, are assumed to be level (0% grade). This is generally not true for the road surface (small cross-slopes are introduced), but these errors are typically acceptable.

- Cross-slopes outside of the traveled way and shoulders are typically described using ratios of rise to run (e.g., 1:4).

- Similarly, the road surface for earthwork calculations is assumed to be at the elevation of the profile (i.e., the elevation of the centerline). When the subgrade level is used, the elevation at the center of the subgrade (directly beneath the centerline) is used.

For many instances, these approximations are perfectly fine – the small discrepancies are mere "noise" in the overall earthwork calculations. These at least provide a first guess at the level of earthwork and related costs that the roadway design will require.

As a prelude to the use of end areas, we will need to establish some basic nomenclature for points in the cross-section. The most common notation is to give the location of points in the *existing cross-section*, again moving from left to right as one looks upstation. Each point in this cross-section is referenced using:

$$\frac{\text{(F or C) } Y}{X}$$

with the following notation:

Y = distance above or below the profile (roadway) grade. Y = 0 is the elevation of the centerline
F = fill (literally), meaning that the Y value is *below* the profile elevation
C = cut (literally), or, the given Y value is *above* the profile elevation
X = distance left or right of the roadway centerline. This reference is ambiguous, in the sense that it is not directly obvious from a single point whether this is to the left or right of the centerline. However, most references to a cross–section will include a point where X = 0. Points to the left of this point are on the left, and points to the right are on the right.

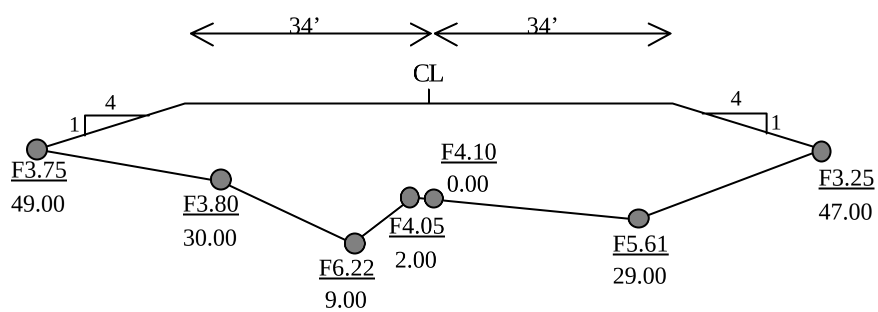
The proposed roadway cross-section must also be defined, using an appropriate description of the cross-slopes and the width of the roadway (the traveled way and shoulders).

The difference in cross-sections will form a polygon. At a minimum, the polygon has a left catch point, a right catch point, and some difference in cross-section at the centerline. In many cases, additional vertices of the polygon may be added. These are inserted in the appropriate locations, working left to right across the cross-section.

Consider the following example. A proposed roadway is 68' wide, with 1:4 side slopes. The existing cross-section can be described by:

F3.75	F3.80	F6.22	F4.05	F4.10	F5.61	F3.25
49.00	30.00	9.00	2.00	0.00	29.00	47.00

This has 4 points to the left of the centerline and two points to the right. The depth of fill at the centerline is 4.10 feet. This cross-section is illustrated below, with the proposed cross-section on top and the existing cross-section on the bottom.



2 Determining cross-sectional areas

One traditional method of determining volumes is to use the "end area" of this cross–section. That is, to determine earthwork volumes of cut and fill, we need to know the area of the difference between the existing and proposed cross-sections. If we have created a polygon that describes the difference in cross-section (see previous), then all we need to do is find the area of that polygon.

There are a number of methods that can be used to find the cross-section area. If we have plotted it out on graph paper, then we could use a planimeter to measure the area. Other methods use triangles, trapezoids, or simple coordinate geometry to determine these areas. For completeness, these methods are described and illustrated below.

Note that in any cross-section, one desires to identify areas of ***cut*** and areas of ***fill*** separately. This is important because we will treat cut and fill differently in earthwork calculations later.

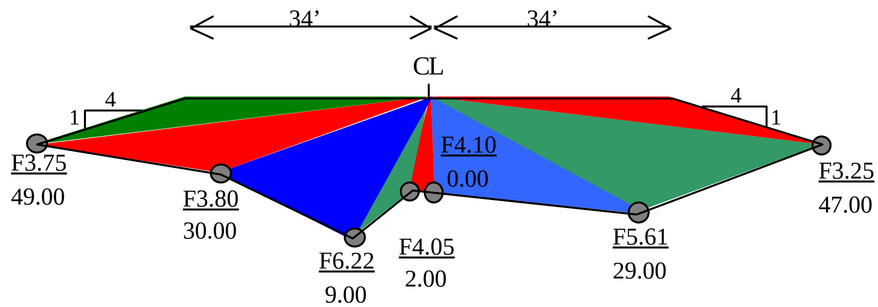
2.1 Triangles and trapezoids

In general, once the catch points and differences in elevation between the existing and planned cross-sections are known, the cross-sectional area can be determined easily by geometry.

Triangle technique

A simple technique divides the cross-section into triangles, and computes the cross-sectional area directly using the area of each triangle = ½*base*height [note that this must be at right angles]. We can exploit the fact that the X and Y dimensions of the cross-section are at right angles to make our life easier.

As an example, consider the cross-section shown earlier. One way of drawing triangles is simply to connect the centerline to two other adjacent vertices, like this:



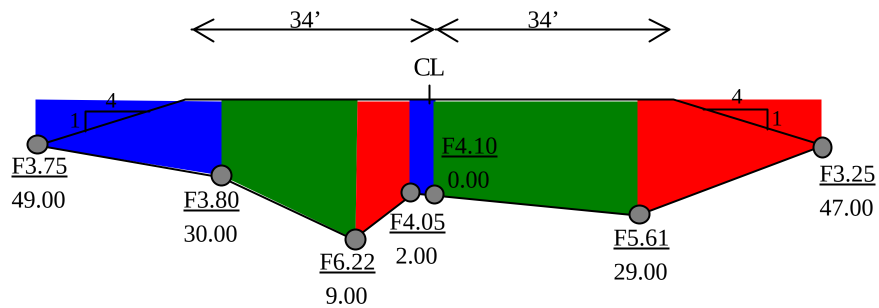
The area is calculated as the sum of triangular areas. To get orthogonal base and height, I dropped in perpendiculars from each point in the existing cross-section. Then, working clockwise from the top right,

$A = ½(34')*(3.25')$	Red
$+ ½(47')*(5.61' - {}^{29}/_{47}*3.25')$	Green
$+ ½(29')*(4.10')$	Blue
$+ ½(2')*(4.10')$	Red
$+ ½(9')*(4.05' - {}^{2}/_{9}*6.22')$	Green
$+ ½(30')*(6.22' - {}^{9}/_{30}*3.80')$	Blue
$+ ½(49')*(3.80' - {}^{30}/_{49}*3.75')$	Red
$+ ½(34')*(3.75')$	Green

$A = \underline{392.32\ ft^2}$

Trapezoid technique

Rather than using triangles in the above method, one might use trapezoids instead, recalling that the area of a trapezoid is ½h $(b_1 + b_2)$. This is applied directly. From right to left,



$A = ½*(18)*(3.25'+5.61') – ½*(3.25')*(13')$ Red, less a triangle
$+ ½*(29')*(4.10' + 5.61')$ Green
$+ ½*(2')*(4.10' + 4.05')$ Blue
$+ ½*(7')*(6.22' + 4.05')$ Red
$+ ½*(21')*(3.80'+6.22')$ Green
$+ ½*(19')*(3.75'+3.80') – ½*(3.75')*(15')$ Blue, less a triangle

A = <u>392.32 ft^2</u>

2.2 Coordinate method

Using the consecutive coordinates of points around the perimeter of a polygon, a mathematical technique essentially adds and subtracts squares to come up with the area of the polygon.

If the coordinates of a vertex *i* of the n-sided polygon are given as (X_i ,Y_i), then the area of the polygon is given as:

$$A = \left| \tfrac{1}{2}\Sigma_{i=1 \text{ to } n} X_i (Y_{i+1} - Y_{i-1}) \right| \quad \text{with } n + 1 \equiv 1, 1 - 1 \equiv n$$

An alternate form is:

$$A = \tfrac{1}{2}\left| \Sigma_{i=1 \text{ to } n} X_i Y_{i+1} - \Sigma_{i=1 \text{ to } n} X_i Y_{i-1} \right|$$

One method shown in class is simply to list the points X and Y in columns. The initial point is repeated at the bottom. Then, the down-and-right diagonal terms are multiplied, giving the first set of terms (positive values), and the down-and-left terms are multiplied, giving the second set of terms (minus values). In our case, the coordinates from the centerline around clockwise, and the resulting terms in the sum, are:

X	Y	Plus	Minus
0	0	0	0
34	0	0	110.5
47	3.25	94.25	263.67
29	5.61	0	118.9
0	4.1	-8.2	0
-2	4.05	-36.45	-12.44
-9	6.22	-186.6	-34.2
-30	3.8	-186.2	-112.5
-49	3.75	-127.5	0
-34	0	0	0
0	0		

Summing the "Plus" terms, subtracting the "Minus" terms, taking the absolute value, and dividing by two, yields A = <u>392.32 ft^2</u>.

3 Determining volumes

The real purpose of determining cross-sectional areas is to estimate cut and fill volumes. This method is presented first below. A second method, using the characteristics of a 3-dimensional surface, are described in the second section.

3.1 Cross-sectional area method

In this method, volumes of earthwork are associated with particular roadway segments. These volumes are measured between two cross-sections along the highway centerline. The total volume can be approximated as the total length L of the segment (again, along the centerline) multiplied by the "average" cross-sectional area. The average cross-sectional area is the average of the two end cross-sectional areas (A_1+A_2). This yields the volume calculation:

$$\text{Volume} = V = ½L\,(A_1+A_2)$$

This is only an approximation, of course, since differing cross-sectional areas along the length of the highway will result in changes to the volume. Also, there is some error associated with the fact that this volume is really the volume of a prismoid (see later) — the formula above assumes a cylindrical volume.

Note also that the cross-sectional areas used in this formula should be either in ***cut*** or in ***fill***. Volumes thus are calculated for ***each*** of cut and fill in a given road segment: for a given segment, one should calculate ***both*** V_{cut} ***and*** V_{fill}.

The approximate formula given above applies only when $A_1 > 0$ and $A_2 > 0$. That is, for V_{cut}, both of the cross-sections should have cut areas; for V_{fill}, both of the cross-sections should have fill areas. Violations of these assumptions are covered by the pyramid method.

Pyramid method

If one cross-sectional area is zero but the other is positive, then a better approximation of the total volume is as a pyramid. A pyramid has an area on the bottom and converges to a single point at the top. Recall that the total volume of a pyramid is one-third of the area times the length, or:

$$\text{Volume} = V = 1/3\ L\ A \qquad \text{where } A = \text{the non-zero cross-sectional area}$$

If one area is 100% cut and the other is 100% fill, a different calculation is necessary. This requires estimating where along the length the cross-sectional area goes to zero (or, approximately, where the current cross-section intersects the proposed roadway cross-section). Then one creates two new roadway segments: one up to this point of zero cross-sectional area, and the second to the next cross-sectional area. The volume is then calculated for both segments (separately) using the pyramid formula.

Prismoidal method

A more accurate method of calculating the total volume is to view the volume as a prismoid and to calculate the total volume using Simpson's formula. This yields:

$$\text{Volume} = V = L/6 * (A_1 + 4A_m + A_2)$$

where A_m is the cross-sectional area of the midpoint between A_1 and A_2 , along the centerline. This cross-sectional area is found by averaging the *linear* boundaries of the A_1 and A_2 polygons, then through direct calculation of the cross-sectional area.

As an example, consider the following areas and volume calculations. The shaded cells use the pyramid formula, and the remainder of the volumes use the standard end-area formula.

Station	Acut (sq ft)	Afill (sq ft)	Vcut (cu yd)	Vfill (cu yd)	Adjusted Vfill (cu yd)	Cumulative Cut Volume (cu yd)
02+00	760.0					10000.0
03+00	104.0		1600.0			11600.0
04+00	205.4	0.0	572.9			12172.9
05+00	13.2	11.3	404.9	14.0	16.1	12561.7
06+00	0.0	38.7	16.3	92.6	106.5	12471.5
07+00		280.7		591.5	680.2	11791.2
08+00		503.2		1451.8	1669.6	10121.7
09+00		1384.2		3495.2	4019.5	6102.2
10+00		708.3		3874.9	4456.1	1646.0

3.2 Surface method

Instead of the end-area method, one could calculate the volume more exactly as the difference between two surfaces. That is, the total volume is the difference between the original earth "surface" (think of a triangulated digital terrain model, for example) and a "surface" created by the roadway, as cross-sections are placed along the horizontal and vertical alignments. The total volume then is the difference between these two surfaces. "Cut" is the volume where the roadway surface is below the existing ground, and "Fill" is the volume where the roadway surface is above the existing ground.

In this method, cut and fill volumes are calculated separately. Each of these may be calculated by:

1. Over a given surface area A (viewed top-down), match points at the same X and Y (e.g., latitude and longitude) between the two surfaces. Common types of points X and Y include a "grid" of points (e.g., 10 m on square from a digital elevation model), or terrain points from a triangulated sruface.

2. Calculate the difference in elevation (Z) between the two surfaces (e.g., $h = Z_{earth} - Z_{roadway}$). In this case, the difference is positive for cut and negative for fill.

3. Calculate the volume between these two shapes on the two surfaces.

   $V_{cut} = A_{cut} * [\Sigma_{i=1 \text{ to } n} (Z_{earth,i} - Z_{roadway,i})] / n$ for points where $Z_{earth} > Z_{roadway}$

   $V_{fill} = A_{fill} * [\Sigma_{i=1 \text{ to } n} (Z_{earth,i} - Z_{roadway,i})] / n$ for points where $Z_{roadway} > Z_{earth}$

   If A and Z are in feet, one could also divide the volumes by 27 to convert them to cubic yards

As an example, in one small triangular area, the vertices of a triangulated surface are:

X	Y	$Z_{roadway}$	Z_{earth}	h
35.0	32.0	47.5	44.3	3.2
37.5	36.7	46.0	44.8	1.2
40.0	33.8	48.7	43.5	5.2

This particular triangle is all in fill. The surface area can be found using the X and Y coordinates in the coordinate method:

$A = ½| 35 * (36.7 – 33.8) + 37.5*(33.8 – 32.0) + 40.0*(32.0 – 36.7) | = \underline{19.0\ ft^2}$

$V_{fill} = 19.0\ ft^2 * (3.2 + 1.2 + 5.2) / 3 = 60.8\ ft^3 = \underline{2.25\ yd^3}$ (fill)

Such fill and cut volumes are then summed over the desired project area.

3.3 Shrinkage and swelling

Volumes of earth that are excavated from a segment will, in some cases, shrink or swell when they are placed in other locations. Depending on the type of soil, shrinkage or swelling may occur; as examples, clays will swell, while other soils typically shrink (become more compact). As a result, volumes in cut may shrink slightly when it is excavated and then used as fill in another location. This means, for purposes of transferring earth from a cut site to a fill site, one must account for this shrinking or swelling. In a simple mathematical representation,

$$V_{fill} = (1 + s)\ V_{cut}$$

where s is the shrinkage (negative) or swell (positive) associated with a certain soil. As an example, s may be about –10% to –15% for a particular project site. This means that a particular volume in cut is only worth only about 85-90% of that volume as fill.

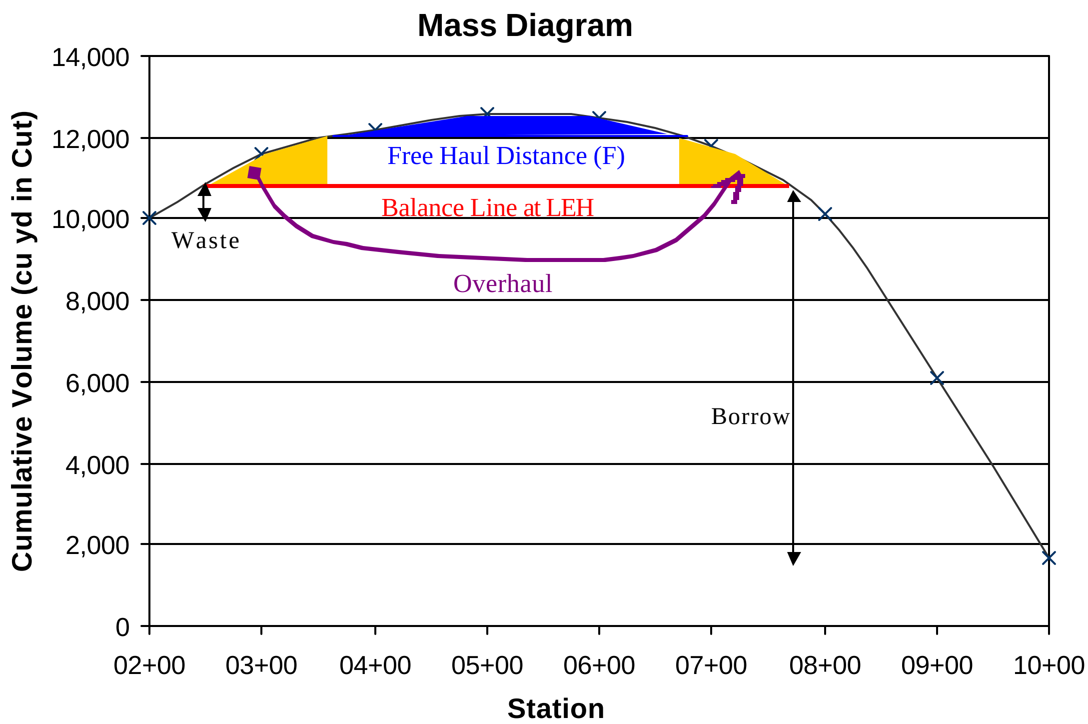
In the example shown previously in Section 3.1, the fill volumes are adjusted for 15% shrinkage (see the “Adjusted Fill” columns. This basically gives the equivalent cut necessary to provide these fill volumes.

4 Mass balance and haul requirements

4.1 Mass balance diagrams

Along a stretch of highway, the volumes of cut and fill are recorded in graphical form using something called a *mass diagram*. The mass diagram gives the *cumulative* volume of earthwork along this stretch of highway, summing cut (positive) and fill (negative) at each point. Hence, a rising curve means more cut is being excavated, and a falling curve means more fill is being added. A baseline value (e.g., 0 cubic yards) is chosen to begin the description over a particular length of highway.

For the example presented previously, the cumulative cut volume, with a baseline of 10,000 cubic yards, is shown below.



As a result of this being a *cumulative* curve, any horizontal line on this curve defines a section in which the (rising) amount of cut matches the (falling) amount of fill. Horizontal lines are thus called ***balance lines*** on this diagram.

Note that, in order to interpret the balance lines in the diagram properly, one must account for shrinkage or swelling of the soil as it is placed. That is, the fill volumes should be divided by (1+s) to get the resulting "equivalent" cut volumes.

With the mass diagram, one can then calculate the *haul*. Haul is the total amount of earth that needs to be transported, multiplied by the average distance that it must be hauled. Costs are typically quoted in terms of the *free haul distance* (F), which is not really "free" but represents the longest distance for which unit costs ($/cubic yard) of haul apply. Additional costs are incurred for *overhaul*, representing a unit cost per cubic yard per unit distance (e.g., in $/cubic yard/station) of hauling.

The problem of finding the most economic movement of earth, as it turns out, is the same as finding the most reasonable balance line(s) in such a diagram. The *length of economic haul* (LEH) is the maximum distance one would haul (as overhaul) before it becomes more economical to simply borrow earth from any other location. That is,

$$LEH = F + C_{borrow} / C_{overhaul}$$

where F = free haul distance, C_{borrow} is the cost of borrow ($/cubic yard), and $C_{overhaul}$ is the cost of overhaul ($/cubic yard/station). ***As a result, balance lines in a mass haul diagram, indicating where cut should be taken and used as fill, should always be equal to or less than LEH in length.***

Balance lines can then be drawn on the mass diagram. With the constraint that the length of the balance line be less than or equal to LEH, not all material will be hauled, requiring either waste or borrow. The amount of the overhaul is determined as the (cumulative) volume between the balance line and the free haul distance line in the loop. One will need to know both this volume, and the horizontal location of its center of mass, when calculating the cost of this overhaul. There are various techniques associated with calculating this center of mass, each of which is beyond the scope of this class.

4.2 Determining cost of haul

The total cost of earthwork, then, is calculated by summing:

1. The cost of excavation

   Excavation cost = cut volume * (Cost / cubic yard of excavation)

2. The cost of any borrow and waste

   Borrow cost = (Necessary fill volume outside the balance lines) * (Cost / cubic yard of borrow)
   Waste cost = (Necessary cut volume outside the balance lines) * (Cost / cubic yard of waste)

3. The cost of any overhaul

   Overhaul cost = (Volume of overhaul)
   * (Distance between centers of mass of overhaul sections above balance line)
   * Cost of overhaul (Cost per cubic yard per station)