Math Meth Oper Res (2000) 52:65–77

Mathematical Methods
of Operations Research

Ideal equilibria in noncooperative multicriteria games[1]

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Manuscript received: October 1999/Final version received: March 2000

Abstract. Pareto equilibria in multicriteria games can be computed as the Nash equilibria of scalarized games, obtained by assigning weights to the separate criteria of a player. To analysts, these weights are usually unknown. This paper therefore proposes ideal equilibria, strategy profiles that are robust against unilateral deviations of the players no matter what importance is assigned to the criteria. Existence of ideal equilibria is not guaranteed, but several desirable properties are provided. As opposed to the computation of other solution concepts in noncooperative multicriteria games, the computation of the set of ideal equilibria is relatively simple: an exact upper bound for the number of scalarizations is the maximum number of criteria of the players. The ideal equilibrium concept is axiomatized. Moreover, the final section provides a non-trivial class of multicriteria games in which ideal equilibria exist, by establishing a link to the literature on potential games.

Key words: Multicriteria games, ideal equilibria, equilibrium concept

1 Introduction

The usual interpretation of a multicriteria game entails that a player is viewed as a single individual with multiple objectives. In this paper we take the alternative view that a player is an organization. In this organization, each of the criteria corresponds to the concerns of a different member of the organization:

[1] We thank Hannu Salonen for his comments.
[2] The hospitality of the Department of Economics at Stockholm University to this author during part of the preparation of this work is gratefully acknowledged.
[3] This author acknowledges the hospitality of the Department of Econometrics at Tilburg University during part of the research.

a player i with $r(i) \in \mathbb{N}$ criteria thus corresponds with an organization of $r(i)$ members. This view of players as organizations is realistic: in many real-life situations, a single decision is often influenced by several individuals with different objectives.

Several solution concepts for multicriteria games, like Pareto equilibria (Shapley, 1959, Voorneveld et al., 1999) or Pareto-Optimal Security Strategies (Ghose and Prasad, 1989, Ghose, 1991, Voorneveld, 1999) can be computed by assigning weights to the criteria of each player and solving the scalarized single-criterion game for its Nash equilibria. However, for an analyst the organizations typically act as "black boxes": he lacks the knowledge or resources to determine the exact weights assigned to the criteria or, equivalently, the importance assigned to the members of an organization. As a consequence, it is difficult to determine which outcomes are self-enforcing. In this paper we therefore propose a new equilibrium concept for noncooperative multicriteria games, so-called ideal equilibria. An ideal equilibrium is a strategy profile that is robust against unilateral deviations of the players, irrespective of the importance assigned to the criteria. The name of the concept is motivated by the notion of ideal points that play an important role in the theory of multicriteria optimization (cf. Zeleny, 1976). In a decision problem, the ideal point describes the vector of optima of all criteria simultaneously.

Several characterizations of ideal equilibria are provided. One of the characterizations establishes an important computational result. Not only is it shown that finitely many scalarizations suffice to determine the set of ideal equilibria, but we also provide a collection of weight vectors that can actually be used in their computation. Specifically, the computation of the set of ideal equilibria requires at most as many scalarizations as the maximum number of criteria of the players. To compare, it was shown by Borm et al. (1998) that infinitely many scalarizations might be necessary to compute the entire set of Pareto equilibria. Ghose (1991) proved that finitely many scalarizations suffice to determine the Pareto-Optimal Security Strategies, but it is not known exactly how many and which vectors of weights are needed.

In recent years, several articles appeared concerning the axiomatization of equilibrium concepts in single-criterion games and multicriteria games. See Peleg and Tijs (1996), Peleg, Potters, and Tijs (1996), Norde et al. (1996), Voorneveld et al. (1999). Using axioms that are similar to those in the literature, an axiomatization of ideal equilibria is provided.

The paper concludes with a section that formulates a class of games in which ideal equilibria always exist. This is done by establishing a formal link with the literature on ordinal potential games (Monderer and Shapley, 1996, Voorneveld and Norde, 1997).

Briefly, the structure of the paper is as follows. Section 2 provides several definitions and matters of notation. Ideal equilibria are introduced in Section 3. Their properties are studied in Section 4. Section 5 concludes with a class of multicriteria games in which existence of ideal equilibria is guaranteed.

2 Preliminaries

A finite multicriteria game is a tuple $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where $N \subset \mathbb{N}$ is a finite set of players, X_i is the finite set of pure strategies of player $i \in N$, and for each player $i \in N$, the function $u_i : \prod_{j \in N} X_j \to \mathbb{R}^{r(i)}$ maps each strategy combination to a point in $r(i)$-dimensional Euclidean space, where $r(i) \in \mathbb{N}$ is the number of criteria of player $i \in N$. We sometimes write $u_i = (u_{ik})_{k=1}^{r(i)}$ if the different coordinate functions need stressing.

The payoff functions are extended to mixed strategies in the obvious way. The set of mixed strategies of player $i \in N$, i.e., the set of all probability distributions over the set X_i of pure strategies, is denoted by $\Delta(X_i)$. For a mixed strategy $x_i \in \Delta(X_i)$, we denote by x_{ij} the probability that player i assigns to his pure strategy j. The set of mixed extensions of finite multicriteria games is denoted by Γ.

For a strategy profile $x = (x_i)_{i \in N} \in \prod_{i \in N} \Delta(X_i)$ and a coalition $S \subseteq N$ of players in a multicriteria game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, $x_S = (x_i)_{i \in S}$ denotes the strategy profile restricted to the players in coalition S.

Noncooperative game theory usually studies strategic games in which the players have only one criterion. This subclass of multicriteria games will be denoted Γ^*.

For $m \in \mathbb{N}$, $\Delta_m := \{\mu \in \mathbb{R}^m_+ \mid \sum_{i=1}^m \mu_i = 1\}$ is the unit simplex in $\mathbb{R}^m$. Its relative interior $\{\mu \in \mathbb{R}^m_{++} \mid \sum_{i=1}^m \mu_i = 1\}$ is denoted relint Δ_m. Consider a multicriteria game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ in which player $i \in N$ has $r(i) \in \mathbb{N}$ criteria. For each $i \in N$, let $\lambda_i \in \Delta_{r(i)}$ be a vector of weights for the criteria, $\lambda := (\lambda_i)_{i \in N}$. The *$\lambda$-weighted game* $G_\lambda = \langle N, (X_i)_{i \in N}, (v_i)_{i \in N} \rangle \in \Gamma^*$ is the (mixed extension of the) strategic game with payoff functions $(v_i)_{i \in N}$ defined for all $i \in N$ and $x \in \prod_{j \in N} X_j$ by $v_i(x) = \sum_{k=1}^{r(i)} \lambda_{ik} u_{ik}(x)$.

For $a, b \in \mathbb{R}^m$, define vector inequalities as follows:

$$a \geqq b \quad \text{if} \quad a_j \geqq b_j \text{ for all } j \in \{1, \ldots, m\},$$

$$a \geq b \quad \text{if} \quad a \geqq b \text{ and } a \neq b,$$

$$a > b \quad \text{if} \quad a_j > b_j \text{ for all } j \in \{1, \ldots, m\}.$$

The standard solution concept for strategic games $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma^*$ is the Nash equilibrium concept. A strategy profile $x \in \prod_{i \in N} \Delta(X_i)$ is a Nash equilibrium of a strategic game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma^*$ if there is no player $j \in N$ with an alternative strategy $y_j \in \Delta(X_j)$ for which $u_j(y_j, x_{-j}) > u_j(x)$. The set of Nash equilibria of a single-criterion game G is denoted $NE(G)$.

Shapley (1959) generalized the Nash equilibrium concept for strategic games to solution concepts for multicriteria games, here called weak and strong Pareto equilibrium. The definitions of equilibrium points by Shapley were for two-person zero-sum games, but can easily be extended to general multicriteria games.

Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. A strategy profile $x \in \prod_{i \in N} \Delta(X_i)$ is

- a *weak Pareto equilibrium* if for each player $i \in N$ there does not exist a strategy $\tilde{x}_i \in \Delta(X_i)$ such that

$$u_i(\tilde{x}_i, x_{-i}) > u_i(x_i, x_{-i}),$$

- a *strong Pareto equilibrium* if for each player $i \in N$ there does not exist a strategy $\tilde{x}_i \in \Delta(X_i)$ such that

$$u_i(\tilde{x}_i, x_{-i}) \geq u_i(x_i, x_{-i}).$$

The set of weak and strong Pareto equilibria of G are denoted by $WPE(G)$ and $SPE(G)$, respectively. Note that both WPE and SPE coincide with the Nash equilibrium concept in the single-criterion case.

Shapley (1959) proved that weak Pareto equilibria are exactly the Nash equilibria of weighted games where nonnegative weight is assigned to each of the criteria, whereas the strong Pareto equilibria coincide with the Nash equilibria of weighted games where positive weight is assigned to the criteria.

Proposition 2.1. *Let* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ *and* $x \in \prod_{i \in N} \Delta(X_i)$.

- $x \in WPE(G)$ *if and only if there exists a vector* $\lambda = (\lambda_i)_{i \in N} \in \prod_{i \in N} \Delta_{r(i)}$ *such that* $x \in NE(G_\lambda)$;
- $x \in SPE(G)$ *if and only if there exists a vector* $\lambda = (\lambda_i)_{i \in N} \in \prod_{i \in N} \text{relint}\, \Delta_{r(i)}$ *such that* $x \in NE(G_\lambda)$.

Let A, B be sets. Then A^B is the set of maps from B to A, 2^A is the collection of subsets of A, and $|A|$ is the cardinality of A. For instance, $(2^A)^B$ is the set of maps assigning to each element of B a subset of A. For a finite-dimensional vector space $\mathbb{R}^m$ and $x, y \in \mathbb{R}^m$, $\langle x, y \rangle := \sum_{k=1}^m x_k y_k$ is the inner product of x and y, $\|x\| := \sqrt{\langle x, x \rangle}$ is the Euclidean norm of x, e_k is the k-th standard basis vector of $\mathbb{R}^m$, and $e(m) = (1, \ldots, 1) \in \mathbb{R}^m$ is the vector of ones.

3 Ideal equilibria

In this paper, we take the view that the players of a multicriteria game are organizations and that each of the criteria corresponds to the concerns of a different organization member. Due to the lack of knowledge or resources to model the exact procedure by which an organization arrives at its strategic choice, the organization will often be a "black box" from the view point of an analyst. The uncertainty about what is going on inside an organization implies that the analyst does not know the importance that is assigned to the different organization members; therefore, he cannot say exactly what strategies an organization considers to be a best response against a certain strategy combination of the other players.

Given the problem of having organizations as black boxes, the approach of Shapley as mentioned in Proposition 2.1 appears unsatisfactory: for an analyst it is highly inefficient to compute all Nash equilibria of all weighted games. Unless the analyst is aware of the exact importance assigned to each of the criteria, the recommendations he will do by providing the set of Pareto equilibria will be very rough.

Therefore, it is desirable to look for a concept that is robust against deviations, despite the fact that we do not know which weight is assigned to each of the criteria. By necessity, such an equilibrium concept will be very restrictive. The new equilibrium concept for multicriteria games we propose satisfies this strong robustness condition: no matter what deviation an organization

T	(1, 0)
B	(0, 1)

Fig. 1. A one-player game without ideal equilibria

considers, the members would have been at least as well off in the old situation. Formally,

Definition 3.1. Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ be a multicriteria game. An *ideal equilibrium* is a strategy profile $x \in \prod_{i \in N} \Delta(X_i)$ such that for every player $i \in N$ and each $\tilde{x}_i \in \Delta(X_i)$:

$$u_i(x_i, x_{-i}) \geqq u_i(\tilde{x}_i, x_{-i}).$$

The set of ideal equilibria of G is denoted $IE(G)$.

It is immediate from the definitions that $IE(G) \subseteq SPE(G) \subseteq WPE(G)$ and that the set of ideal equilibria, weak Pareto equilibria and strong Pareto equilibria all coincide with the set of Nash equilibria if the players have only one criterion.

Given the strong robustness condition imposed on ideal equilibria, it is easy to show that ideal equilibria – as opposed to weak and strong Pareto equilibria – may not exist for games in Γ. See for instance the one-player game in Figure 1. In this game, the player has two criteria and two pure strategies, T and B. If he chooses T, for instance, his payoff is $(1, 0) \in \mathbb{R}^2$. If p is the probability the player assigns to playing the top strategy T, his expected payoff is $(p, 1-p)$. No $p \in [0, 1]$ yields an ideal equilibrium: increasing p would increase the payoff in the first criterion, while decreasing p would increase the payoff in the second criterion. The games in Figure 2 are two-player games. Both players have pure strategies: T and B for the first player, L and R for the second player. Both players take two criteria into account. For instance, in the game in Figure 2a, if the first player chooses B and the second chooses L, the payoff to the first player is $(1, 0)$, and to the second player $(0, 1)$. The game in Figure 2a has two ideal equilibria, both in pure strategies: (T, L) and (B, R). The game in Figure 2b has only one ideal equilibrium, namely $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

	L	R
T	(1, 1), (1, 1)	(0, 1), (1, 0)
B	(1, 0), (0, 1)	(1, 1), (1, 1)

a

	L	R
T	(1, 0), (0, 1)	(0, 1), (1, 0)
B	(0, 1), (1, 0)	(1, 0), (0, 1)

b

Fig. 2. Two games with ideal equilibria

4 Results

Given the strategy profile of the other organizations, a given organization can compute the optimal outcome for each goal function separately. In multicriteria problems, these optima will typically be achieved at different outcomes for the different criteria: there is no consensus over the criteria which outcome is best. Still, these separate optima give rise to an ideal point and decision makers will strive to be close to this ideal point. Zeleny (1976), states this as an axiom underlying human choice:

> "Alternatives that are closer to the ideal are preferred to those that are farther away. To be as close as possible to the perceived ideal is the rationale of human choice."

Voorneveld and van den Nouweland (1999) introduce compromise equilibria in multicriteria games based on the idea that given the strategy profile of the opponents each player wants to be as close to the ideal point as possible. Ideal equilibria are exactly those strategy profiles in which each player *achieves* his ideal point, given the strategy profile of his opponents. This and other characterizations of ideal equilibria are given in the first theorem of this section.

The characterizations of ideal equilibria stress the robustness with respect to the different weights that may be assigned to the criteria. Moreover, attention is paid to computational aspects. Compared to other solution concepts for multicriteria games, few weight vectors suffice to determine the set of ideal equilibria. The structure of the set of ideal equilibria for two-person multicriteria games coincides with the structure of the set of Nash equilibria for single-criterion games: it is a finite union of polytopes.

Several authors have provided axiomatizations for equilibrium concepts in single-criterion games. See for instance Peleg and Tijs (1996), Peleg, Potters, and Tijs (1996), and Norde et al. (1996). Voorneveld et al. (1999) provide axiomatizations of Pareto equilibria in multicriteria games. The final part of this section is devoted to giving an axiomatization of the ideal equilibrium concept.

Consider a multicriteria game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. The *ideal point* for player $i \in N$, given a strategy profile $x_{-i} \in \prod_{j \in N \setminus \{i\}} \Delta(X_j)$ of his opponents is defined as $\psi_i(x_{-i}) \in \mathbb{R}^{r(i)}$ with

$$\forall k \in \{1, \ldots, r(i)\} : \psi_{ik}(x_{-i}) = \max_{x_i \in \Delta(X_i)} u_{ik}(x_i, x_{-i}).$$

If G is a one-player game, i.e., if the unique player i has no opponents, the ideal point is simply defined to be $\psi_i \in \mathbb{R}^{r(i)}$ with

$$\forall k \in \{1, \ldots, r(i)\} : \psi_{ik} = \max_{x_i \in \Delta(X_i)} u_{ik}(x_i).$$

The following theorem gives several characterizations of the ideal equilibrium concept, implying some different interpretations. Some additional notation is required. A collection $\Lambda \subseteq \prod_{i \in N} \Delta_{r(i)}$ of weight vectors is called *representative* for the game G if for each organization $i \in N$ and each of its members $k \in \{1, \ldots, r(i)\}$, there exists a weight vector in Λ assigning weight one to this organization member:

$$\forall i \in N, \quad \forall k \in \{1, \ldots, r(i)\}, \quad \exists \lambda = (\lambda_j)_{j \in N} \in \Lambda : \lambda_i = e_k.$$

Theorem 4.1. *Let $G \in \Gamma, x \in \prod_{i \in N} \Delta(X_i)$, and Λ a representative collection for G. The following claims are equivalent:*

(a) $x \in IE(G)$;
(b) $\forall i \in N : u_i(x) = \psi_i(x_{-i})$;
(c) $x \in \bigcap_{\lambda \in \prod_{i \in N} \Delta_{r(i)}} NE(G_\lambda)$;
(d) $x \in \bigcap_{\lambda \in \Lambda} NE(G_\lambda)$.

Proof.
(a) ⇒ (b): Let $x \in IE(G), i \in N, k \in \{1, \ldots, r(i)\}, y_i \in \Delta(X_i)$ s.t. $u_{ik}(y_i, x_{-i}) = \psi_{ik}(x_{-i})$. Then

$$\begin{aligned} u_{ik}(x) &\geqq u_{ik}(y_i, x_{-i}) \\ &= \psi_{ik}(x_{-i}) \\ &= \max_{z_i \in \Delta(X_i)} u_{ik}(z_i, x_{-i}), \end{aligned}$$

implying that the weak inequality is in fact an equality. Hence $\forall i \in N : u_i(x) = \psi_i(x_{-i})$.

(b) ⇒ (a): Trivial from the definitions of ideal equilibria and ideal points.

(b) ⇒ (c): Let $u_i(x) = \psi_i(x_{-i})$ for all $i \in N$ and let $\lambda = (\lambda_i)_{i \in N} \in \prod_{i \in N} \Delta_{r(i)}$. To show: $x \in NE(G_\lambda)$. By nonnegativity of the vectors $(\lambda_i)_{i \in N}$, and the definition of ideal equilibria: for each $i \in N$ and each $y_i \in \Delta(X_i)$ we have that $u_i(x) \geqq u_i(y_i, x_{-i})$, which implies $\langle \lambda_i, u_i(x) \rangle \geqq \langle \lambda_i, u_i(y_i, x_{-i}) \rangle$. Hence $x \in NE(G_\lambda)$.

(c) ⇒ (d): Trivial, since $\Lambda \subset \prod_{i \in N} \Delta_{r(i)}$.

(d) ⇒ (b): Let $x \in \bigcap_{\lambda \in \Lambda} NE(G_\lambda), i \in N$, and $k \in \{1, \ldots, r(i)\}$. Choose $\lambda \in \Lambda$ such that $\lambda_i = e_k$. Since $x \in NE(G_\lambda)$:

$$\begin{aligned} u_{ik}(x) &= \langle e_k, u_i(x) \rangle \\ &= \max_{y_i \in \Delta(X_i)} \langle e_k, u_i(y_i, x_{-i}) \rangle \\ &= \max_{y_i \in \Delta(X_i)} u_{ik}(y_i, x_{-i}) \\ &= \psi_{ik}(x_{-i}). \end{aligned}$$

Hence $u_i(x) = \psi_i(x_{-i})$ for each $i \in N$. □

It is clear that claim (b) motivates the name of the equilibrium concept: in an ideal equilibrium each organization reaches its ideal point, given the strategy profile of the other organizations. If the ideal point, given the strategy profile

x_{-i} of the opponents of a player $i \in N$ is feasible, it is always a corner point of the payoff polytope $\{u_i(y_i, x_{-i}) \mid y_i \in \Delta(X_i)\}$ and as a consequence it is achieved at a pure strategy $x_i \in X_i$ of player i (possibly also at a non-pure strategy). Claim (c) indicates the robustness of the ideal equilibrium concept: it is an equilibrium, no matter what importance is assigned to the criteria. Claim (d) is of computational importance: few weight vectors suffice to determine the set of ideal equilibria. The following proposition makes this statement more precise. By definition of a representative collection, there exists for each organization and each of its members a weight vector assigning weight one to this organization member. Since there exists an organization with $\max_{i \in N} r(i)$ members, this is the smallest possible number of elements of a representative collection. The proposition indicates that this lower bound is precise.

Proposition 4.2. *Let* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. *The smallest representative collection for G has* $\max_{i \in N} r(i)$ *elements. Hence,* $\max_{i \in N} r(i)$ *scalarizations suffice to determine* $IE(G)$.

Proof. Recall that for $k, l \in \mathbb{N}$, $k \bmod l$ is defined to be the remainder after division of k by l. One can construct a representative collection Λ with $\max_{i \in N} r(i)$ elements by defining the k-th element, with $k \in \{1, \ldots, \max_{i \in N} r(i)\}$, to be $\lambda \in \prod_{i \in N} \Delta_{r(i)}$ with for each $i \in N$: $\lambda_i := e_{1+[(k-1) \bmod r(i)]}$. By Theorem 4.1(d): $IE(G) = \bigcap_{\lambda \in \Lambda} NE(G_\lambda)$. □

Example 4.3. In a two-person multicriteria game with $r(1) = 3, r(2) = 7$, the representative collection as mentioned in the proof of Proposition 4.2 would be

$$\Lambda = \{(e_1, e_1), (e_2, e_2), (e_3, e_3), (e_1, e_4), (e_2, e_5), (e_3, e_6), (e_1, e_7)\}.$$

Let us briefly compare the result of Proposition 4.2 with the results on other solution concepts for multicriteria games. Borm, Vermeulen, and Voorneveld (1999) show that the computation of the set of Pareto equilibria typically requires infinitely many vectors of weights. For the Pareto-Optimal Security Strategies of Ghose and Prasad (1989) for multicriteria matrix games, it was shown by Ghose (1991) that finitely many weight vectors suffice, but the exact number is not clear. Our result specifies a bound on the number of weight vectors required for the computation of the set of ideal equilibria, and it is easy to see that this bound is sharp in the sense that one can always find games for which $\max_{i \in N} r(i)$ vectors are necessary.

This implies that for games with relatively few criteria, algorithms for finding Nash equilibria can quickly yield statements concerning the non-emptiness of the set of ideal equilibria. Moreover, we have an important result concerning the structure of the set of ideal equilibria in two-person multicriteria games.

Proposition 4.4. *Let* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. *If* $|N| = 2$, *then* $IE(G)$ *is a finite union of polytopes.*

Proof. Recall that the set of Nash equilibria of a two-person single-criterion game is a finite union of polytopes (cf. Winkels, 1979, Jansen, 1981). According to Theorem 4.1(d), $IE(G)$ is a finite intersection of finite unions of polytopes, hence a finite union of polytopes. □

We proceed by giving an axiomatization of the ideal equilibrium concept. Formally, a solution concept on Γ is a function φ which assigns to each game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ a subset $\varphi(G) \subseteq \prod_{i \in N} \Delta(X_i)$ of strategy combinations. Sometimes the player set N of the game G needs stressing, in which case we denote it by N^G.

Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$ be a multicriteria game, let $x \in \prod_{i \in N^G} \Delta(X_i)$ be a strategy profile in G, and let $S \in 2^{N^G} \backslash \{\emptyset, N^G\}$ be a proper subcoalition of the player set N^G. The *reduced game* $G^{S,x}$ of G with respect to S and x is the multicriteria game in which

- the player set is S;
- each player $i \in S$ has the same set X_i of pure strategies as in G;
- the payoff functions $(u_i')_{i \in S}$ are defined by $u_i'(y_S) := u_i(y_S, x_{N^G \backslash S})$ for all $y_S \in \prod_{i \in S} \Delta(X_i)$ and all $i \in S$.

Notice that this is the game that arises if the players in $N^G \backslash S$ commit to playing according to $x_{N^G \backslash S}$, the strategy combination restricted to the players in $N^G \backslash S$.

Three properties are used in the axiomatization. A solution concept φ on Γ satisfies

- **One-Person Ideal Selection (OPIS)**, if $\varphi(G) \subseteq \{x_i \in \Delta(X_i) \mid u_i(x_i) = \psi_i\}$ for each one-person game $G = \langle \{i\}, X_i, u_i \rangle \in \Gamma$;
- **Restricted Nonemptiness (r-NEM)**, if $\varphi(G) \neq \emptyset$ for each game $G \in \Gamma$ with $IE(G) \neq \emptyset$;
- **Consistency (CONS)**, if for each game $G \in \Gamma$, each proper subcoalition $S \subset N^G$, and each element $x \in \varphi(G)$, we have that $x_S \in \varphi(G^{S,x})$.

One-person ideal selection states that in games with one decision maker only outcomes that yield the ideal point are selected by the solution. This restrictive axiom reflects the 'black box' perspective adopted in this paper, and the lack of confidence with which an analyst can predict the choices of an organization whose members' interests are not concordant. Restricted nonemptiness is a very weak condition: whenever a game has an ideal equilibrium, the solution concept φ should also be nonempty. Peleg and Tijs (1996), Peleg, Potters, and Tijs (1996), Norde et al. (1996), and Voorneveld et al. (1999) all use consistency as one of the main axioms. Consistency essentially states that if a strategy profile is selected by the solution concept and some players commit to playing according to this strategy profile, there is no need for the remaining players to reconsider their strategy choice: it will still be selected in the reduced game.

We now prove that these three properties axiomatize the ideal equilibrium concept. The line of proof is inspired by ideas developed in Norde et al. (1996) in their axiomatization of the Nash equilibrium concept.

Theorem 4.5. *A solution concept φ on Γ satisfies OPIS, r-NEM, and CONS if and only if $\varphi = IE$.*

Proof. It is easy to see that IE satisfies the axioms. Let φ be a solution concept that also satisfies OPIS, r-NEM, and CONS. To show: $\varphi = IE$.

That $\varphi(G) \subseteq IE(G)$ for every $G \in \Gamma$ follows from OPIS and CONS of φ: Let $G \in \Gamma$. If $|N^G| = 1$, then $\varphi(G) \subseteq IE(G)$ by OPIS. So assume $|N^G| \geqq 2$. Let $x \in \varphi(G)$, $i \in N^G$. By CONS of φ: $x_i \in \varphi(G^{\{i\},x})$. By OPIS of φ, x_i yields the ideal point in $G^{\{i\},x}$, so by definition of the reduced game $G^{\{i\},x}$ and Theorem 4.1, $x \in IE(G)$. This shows that $\varphi(G) \subseteq IE(G)$.

Remains to show that $IE(G) \subseteq \varphi(G)$ for every $G \in \Gamma$. Again, let $G \in \Gamma$. If $IE(G) = \emptyset$ we are done. So let $\bar{x} \in IE(G)$. Define a new multicriteria game H as follows.

- The player set N^H has $3|N^G|$ elements: $N^H = \{i, i_1, i_2 \mid i \in N^G\}$;
- Players i, i_1, and i_2 have the same strategy set $\Delta(X_i)$ as player i has in the game G;
- Payoff functions of the players i, i_1, and i_2 are denoted by U_i, V_i, and W_i, and their strategies are called x_i, y_i, and z_i, respectively;
- Payoffs are defined as follows:

  $$U_i(x, y, z) := u_i(x_i, x_{-i} + \bar{x}_{-i} - y_{-i}) - \langle x_i - \bar{x}_i, y_i - \bar{x}_i \rangle e(r(i)),$$

  where $x_{-i} + \bar{x}_{-i} - y_{-i}$ is shorthand notation for $(x_j + \bar{x}_j - y_j)_{j \in N^G \setminus \{i\}}$. (Recall that $e(r(i)) \in \mathbb{R}^{r(i)}$ is the vector of ones.)
  Notice that $x_j + \bar{x}_j - y_j$ need not be in $\Delta(X_j)$, but is an element of its affine hull. The multilinearity of $(u_i)_{i \in N^G}$, the payoff functions in G, allows a unique multilinear extension of u_i to affine hulls. Furthermore,

  $$V_i(x, y, z) := \langle y_i, x_i - z_i \rangle \in \mathbb{R},$$

  $$W_i(x, y, z) := \langle z_i, y_i - x_i \rangle \in \mathbb{R}.$$

  So the players $\{i_1, i_2 \mid i \in N^G\}$ have only one criterion.

This finishes the definition of the game H. It is immediately clear that $H \in \Gamma$. We proceed to prove

(α) $IE(H) \neq \emptyset$.
(β) $\forall (x, y, z) \in IE(H) : x = y = \bar{x}$;

Ad (α): Since $\bar{x} \in IE(G)$, it follows easily that $(x, y, z) := (\bar{x}, \bar{x}, \bar{x})$ is an ideal equilibrium of H.
Ad (β): Let $(x, y, z) \in IE(H)$. Suppose that $x_i \neq y_i$ for some index $i \in N^G$.

$$\text{If } z_{ij} > 0 \text{ (player } i_2 \text{ puts positive weight on his pure strategy } j\text{), then } y_{ij} > x_{ij}. \tag{1}$$

Why? Player i_2 has only one criterion, so in $(x, y, z) \in IE(H)$ he must maximize his payoff by definition of ideal equilibria. Moreover, $x_i \neq y_i$ implies that $y_{ik} > x_{ik}$ for at least one pure strategy k. So, by definition of his payoff function W_i, player i_2 uses only strategies with $y_{ik} > x_{ik}$, proving (1). So, if $z_{ij} > 0$, then $y_{ij} > x_{ij} \geqq 0$, so $y_{ij} > 0$ and therefore

$$x_{ij} \geqq z_{ij} \tag{2}$$

by the same argument, but now applied to player i_1. Then

$$\sum_{j:z_{ij}>0} y_{ij} > \sum_{j:z_{ij}>0} x_{ij} \geqq \sum_{j:z_{ij}>0} z_{ij} = 1,$$

where the strict inequality follows from (1) and the weak inequality from (2). Since $\sum_j y_{ij} = 1$, this is impossible. So $x = y$.

Now suppose that $x_i = y_i \neq \bar{x}_i$ for some index $i \in N^G$. At x_i, player i's payoff is $u_i(x_i, x_{-i} + \bar{x}_{-i} - y_{-i}) - \langle x_i - \bar{x}_i, y_i - \bar{x}_i \rangle e(r(i)) = u_i(x_i, \bar{x}_{-i}) - \langle x_i - \bar{x}_i, x_i - \bar{x}_i \rangle e(r(i)) = u_i(x_i, \bar{x}_{-i}) - \|x_i - \bar{x}_i\|^2 e(r(i))$, with $\|x_i - \bar{x}_i\|^2 > 0$ since $x_i \neq \bar{x}_i$. Deviating to $\bar{x}_i$, player i gets $u_i(\bar{x})$. Since $\bar{x} \in IE(G)$: $u_i(\bar{x}) \geqq u_i(x_i, \bar{x}_{-i}) > u_i(x_i, \bar{x}_{-i}) - \|x_i - \bar{x}_i\|^2 e(r(i))$. This contradicts $(x, y, z) \in IE(H)$. Therefore $x = y = \bar{x}$, finishing the proof of claim (β).

This preliminary work enables us to show that $\bar{x} \in IE(G)$ is also an element of $\varphi(G)$. Since $IE(H) \neq \emptyset$ by (α), $\varphi(H) \neq \emptyset$ by r-NEM of φ. Let $(x, y, z) \in \varphi(H)$. Since $\varphi(H) \subseteq IE(H)$ by the previous part of the proof, we have $x = y = \bar{x}$ by (β). By construction the reduced game of H with respect to N^G and (x, y, z) equals the game G:

$$H^{N^G,(x,y,z)} = H^{N^G,(\bar{x},\bar{x},z)} = G,$$

so by CONS of φ:

$$(x, y, z)_{N^G} = (\bar{x}, \bar{x}, z)_{N^G} = \bar{x} \in \varphi(H^{N^G,(x,y,z)}) = \varphi(G),$$

finishing our proof. □

5 Application

As indicated before, ideal equilibria will fail to exist in many games, simply because the ideal outcomes typically turn out to be infeasible. It is therefore of interest to point out cases in which ideal equilibria *do* exist. In this section we construct a class of multicriteria games from ordinal potential games (Monderer and Shapley, 1996, Voorneveld and Norde, 1997) to provide such a class with a nonempty set of ideal equilibria.

Potential games are single-criterion games in which relevant information concerning the set of Nash equilibria can be summarized into a single real-valued function on the strategy space. Formally, a finite single-criterion game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is an *ordinal potential game* if there exists a function $P : \prod_{i \in N} X_i \to \mathbb{R}$ such that for each player $i \in N$, each strategy profile $x_{-i} \in \prod_{j \in N \setminus \{i\}} X_j$ of i's opponents, and each pair $x_i, y_i \in X_i$ of strategies of player i:

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) > 0 \Leftrightarrow P(y_i, x_{-i}) - P(x_i, x_{-i}) > 0. \tag{3}$$

The function P is called an ordinal potential of the game G. In other words, if P is an ordinal potential function of G, the sign of the change in payoff to a unilaterally deviating player matches the sign of the change in the value of P.

Suppose we take a finite single-criterion game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with ordinal potential P and construct a multicriteria game $G = \langle N, (X_i)_{i \in N}, (v_i)_{i \in N} \rangle$ such that each player $i \in N$ in the game G has two criteria ($\forall i \in N$: $r(i) = 2$). The first criterion of player i equals u_i, the second criterion equals the potential function P:

$$\forall i \in N : v_{i1} = u_i \quad \text{and} \quad v_{i2} = P.$$

Theorem 5.1. *The multicriteria game G described above has a pure-strategy ideal equilibrium.*

Proof. By assumption, $\prod_{i \in N} X_i$ is finite, so P achieves its maximum over this set, say at strategy profile $x \in \prod_{i \in N} X_i$. Then x is a pure-strategy ideal equilibrium. Since P is maximal at x, it is clear that each player $i \in N$ achieves his maximal payoff in his second criterion. Suppose there is a player $i \in N$ who can reach a higher payoff in his first coordinate by unilaterally deviating to a different strategy $y_i \in X_i$:

$$v_{i1}(y_i, x_{-i}) - v_{i1}(x) = u_i(y_i, x_{-i}) - u_i(x) > 0.$$

Then (3) implies that

$$P(y_i, x_{-i}) - P(x) = v_{i2}(y_i, x_{-i}) - v_{i2}(x) > 0,$$

contradicting that P achieves its maximum at x. This shows that x is indeed an ideal equilibrium. □

This theoretical result gets a more practical flavor when we consider models in which the ordinal potential function has intuitive appeal. Koster, Reijnierse, and Voorneveld (1999), for instance, introduce a class of games in which players jointly contribute to the financing of a collection of public goods or facilities. It is possible that different players require different public goods. By formulating an appealing realization scheme that assigns to each profile of contributions the set of facilities that is provided, they obtain a 'contribution game' in which the utilitarian welfare function (i.e., the sum of the individual payoff functions) is an ordinal potential. Theorem 5.1 indicates that the corresponding multicriteria game in which each organization cares about two criteria, namely their private payoff and social welfare as measured by the utilitarian welfare function, has an ideal equilibrium. This provides a nontrivial class of games in which ideal equilibria exist.

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