Frustration and Anger in Games. Online Appendix

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Abstract

This document contains the proofs of the results stated in "Frustration and Anger in Games."

1 Preliminaries

For each topological space X, we let $\Delta(X)$ denote the space of Borel probability measures on X endowed with the topology of weak convergence of measures. Every Cartesian product of topological spaces is endowed with the product topology. A topological space X is metrizable if there is a metric that induces its topology. A Cartesian product of a countable (finite, or denumerable) collection of metrizable spaces is metrizable.

To ease exposition, we report below some key definitions and equations contained in "Frustration and Anger in Games" (equation numbers may differ from those of the paper).

$\Delta_i^1 \subseteq \times_{h_i \in H_i} \Delta\left(Z(h_i)\right)$ is the set of first-order beliefs, that is, the set of $\alpha_i = \left(\alpha_i\left(\cdot|Z(h_i)\right)\right)_{h_i \in H_i}$ such that:

- for all $h_i, h_i' \in H_i$, if $h_i \prec h_i'$ then for every $Y \subseteq Z(h_i')$

$$\alpha_i(Z(h_i')|Z(h_i)) > 0 \Rightarrow \alpha_i\left(Y|Z(h_i')\right) = \frac{\alpha_i(Y|Z(h_i))}{\alpha_i(Z(h_i')|Z(h_i))}; \qquad (1)$$

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- for all $h \in H$, $a_i \in A_i(h)$, $a_{-i} \in A_{-i}(h)$ (using obvious abbreviations)

$$\alpha_{i,-i}(a_{-i}|h) = \alpha_{i,-i}(a_{-i}|h, a_i). \tag{2}$$

$\Delta_i^2 \subseteq \times_{h_i \in H_i} \Delta\left(Z(h_i) \times \Delta_{-i}^1\right)$ —where $\Delta_{-i}^1 = \times_{j \neq i} \Delta_j^1$— is the set of second-order beliefs, that is, the set of $\beta_i = (\beta_i(\cdot|h_i))_{h_i \in H_i}$ such that:

- if $h_i \prec h_i'$ then

$$\beta_i(h_i'|h_i) > 0 \Rightarrow \beta_i\left(E|h_i'\right) = \frac{\beta_i\left(E|h_i\right)}{\beta_i(h_i'|h_i)} \tag{3}$$

  for all $h_i, h_i' \in H_i$ and every event $E \subseteq Z(h_i') \times \Delta_{-i}^1$;

- i's beliefs satisfy an own-action independence property:

$$\beta_i\left(Z\left(h, (a_i, a_{-i})\right) \times E_\Delta|(h, a_i)\right) = \beta_i\left(Z\left(h, (a_i', a_{-i})\right) \times E_\Delta|(h, a_i')\right), \tag{4}$$

  for every $h \in H$, $a_i, a_i' \in A_i(h)$, $a_{-i} \in A_{-i}(h)$, and (measurable) $E_\Delta \subseteq \Delta_{-i}^1$. The space of second-order beliefs of i is denoted Δ_{-i}^2.

Note that (1) and (4) are given by equalities between marginal measures (on $A_{-i}(h)$ and $A_{-i}(h) \times \Delta_{-i}^1$ respectively):$\alpha_{i,-i}(a_{-i}|h) = \alpha_{i,-i}(a_{-i}|h, a_i)$.

Lemma 1 *For each player $i \in I$, Δ_i^2 is a compact metrizable space.*

Proof Let Θ be a non-empty, compact metrizable space. Lemma 1 in Battigalli & Siniscalchi (1999) (B&S) establishes that the set of arrays of probability measures $(\mu(\cdot|h_i))_{h_i \in H_i} \in \times_{h_i \in H_i} \Delta\left(Z(h_i) \times \Theta\right)$ such that

$$h_i \prec h_i' \wedge \mu(h_i'|h_i) > 0 \Rightarrow \mu\left(E|h_i'\right) = \frac{\mu\left(E|h_i\right)}{\mu(h_i'|h_i)}$$

is closed. Note that, in the special case where Θ is a singleton, each $\Delta\left(Z(h_i) \times \Theta\right)$ is isomorphic to $\Delta\left(Z(h_i)\right)$; hence, the set of first-order beliefs satisfying (1) is closed. Letting $\Theta = \Delta_{-i}^1$, we obtain that the set of second-order beliefs satisfying (3) is closed.

Since $\times_{h_i \in H_i} \Delta\left(Z(h_i)\right)$ is a compact subset of a Euclidean space and eq. (2) is a closed condition (equalities between marginal measures are preserved in the limit), Lemma 1 in B&S implies that Δ_i^1 is a closed subset of a compact metrizable space. Hence, Δ_i^1 is a compact metrizable space.

It is well known that if $X_1, ..., X_K$ are compact metrizable, so is $\times_{k=1}^K \Delta(X_k)$ (see Aliprantis & Border 2006, Theorem 15.11). Hence, by Lemma 1 in B&S, the set of second-order beliefs satisfying (3) is a closed subset of a compact metrizable space. Since eq. (4) is a closed condition (equalities between marginal measures are preserved in the limit), this implies that Δ_i^2 is compact metrizable. ■

Lemma 2 *For each profile of behavioral strategies $\sigma = (\sigma_i)_{i\in I}$ there is a unique profile of second-order beliefs $\beta^\sigma = (\beta_i^\sigma)_{i\in I}$ such that (σ, β^σ) is a consistent assessment. The map $\sigma \mapsto \beta^\sigma$ is continuous.*

Proof Write $\mathbb{P}^\sigma(h'|h)$ for the probability of reaching h' from h, *e.g.*,

$$\mathbb{P}^\sigma(a^1, a^2|\varnothing) = \left(\prod_{j\in I} \sigma_j(a_j^1|\varnothing)\right)\left(\prod_{j\in I} \sigma_j(a_j^2|a^1)\right).$$

Define α_i^σ as $\alpha_i^\sigma(z|h) = \mathbb{P}^\sigma(z|h)$ for all $i \in I$, $h \in H$, and $z \in Z$. Define β^σ as $\beta_i^\sigma(\cdot|h) = \alpha_i^\sigma(\cdot|h) \times \delta_{\alpha_{-i}^\sigma}$ for all $i \in I$, $h \in H$. It can be checked that (1) $\beta^\sigma \in \Delta_i^2$ for each $i \in I$, (2) (σ, β^σ) is a consistent assessment, and (3) if $\beta \neq \beta^\sigma$, then either (a) or (b) of the definition of consistency is violated. It is also apparent from the construction that the map $\sigma \mapsto \beta^\sigma$ is continuous, because $\sigma \mapsto \alpha^\sigma$ is obviously continuous, and the Dirac-measure map $\alpha_{-i} \mapsto \delta_{\alpha_{-i}}$ is continuous. ■

Lemma 3 *The set of consistent assessments is compact.*

Proof Lemma 1 implies that $\times_{i\in I}(\Sigma_i \times \Delta_i^2)$ is a compact metrizable space that contains the set of consistent assessments. Therefore, it is enough to show that the latter is closed. Let $(\sigma^n, \beta^n)_{n\in\mathbb{N}}$ be a converging sequence of consistent assessments with limit $(\sigma^\infty, \beta^\infty)$. For each $i \in I$, let α_i^n be the first-order belief derived from β_i^n ($n \in \mathbb{N} \cup \{\infty\}$), that is,

$$\alpha_i^n(Y|h) = \beta_i^n(Y \times \Delta_{-i}^1|h)$$

for all $h \in H$ and $Y \subseteq Z(h)$. By consistency, for all $n \in \mathbb{N}$, $i \in I$, $h \in H$, $a \in A(h)$, and $E_{-i} \subseteq \Delta_{-i}$ it holds that

- (a.n) $\alpha_i^n(a|h) = \beta_i^n\left(Z(h,a) \times \Delta_{-i}^1|h\right) = \prod_{j\in I}\sigma_j^n(a_j|h)$,
- (b.n) $\mathrm{marg}_{\Delta_{-i}^1}\beta_i^n(\cdot|h) = \delta_{\alpha_{-i}^n}$, where each α_j^n is determined as in (a.n).

Then,

$$\alpha_i^\infty(a|h) = \beta_i^\infty\left(Z(h,a) \times \Delta_{-i}^1|h\right) = \prod_{j\in I}\sigma_j^\infty(a_j|h)$$

for all $i \in I$, $h \in H$, $a \in A(h)$. Furthermore, $\mathrm{marg}_{\Delta_{-i}^1}\beta_i^\infty(\cdot|h) = \delta_{\alpha_{-i}^\infty}$ for all $i \in I$ and $h \in H$, because $\alpha_{-i}^n \to \alpha_{-i}^\infty$ and the marginalization and Dirac maps $\beta_i \mapsto \mathrm{marg}_{\Delta_{-i}^1}\beta_i$ and $\alpha_{-i} \mapsto \delta_{\alpha_{-i}}$ are continuous. ■

2 Proofs

2.1 Proof of Remark 2

Fix $i \in I$ arbitrarily. First-order belief α_i is derived from β_i and, by consistency, gives the behavioral strategies profile σ. Therefore, by assumption each $h' \preceq h$ has probability one under α_i, which implies that $\mathbb{E}[\pi_i|h';\alpha_i] = \mathbb{E}[\pi_i;\alpha_i]$, hence $\mathrm{F}_i(h';\alpha_i) = 0$. Since blame is capped by frustration, $u_i(h', a_i'; \beta_i) = \mathbb{E}[\pi_i|h';\alpha_i]$. Therefore, sequential rationality of the equilibrium assessment implies that $\mathrm{Supp}\sigma_i(\cdot|h') \subseteq \arg\max_{a_i' \in A_i(h')}\mathbb{E}[\pi_i|h';\alpha_i]$. If there is randomization only in the last stage (or none at all), then players maximize locally their expected material payoff on the equilibrium path. Hence, the second claim follows by inspection of the definitions of agent form of the material-payoff game and Nash equilibrium. ■

2.2 Proof of Proposition 1

Let $(\bar\sigma, \bar\beta) = \left(\bar\sigma_i, \bar\beta_i\right)_{i\in I}$ be the SE of the material payoff game, which is in pure strategies by the perfect information assumption. Fix decision-utility functions $u_i(h, a_i; \cdot)$ of the ABI, or ABB kind, and a sequence of real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$, with $\varepsilon_n \to 0$ and $0 < \varepsilon_n < \frac{1}{\max_{i\in I, h\in H}|A_i(h)|}$ for all $n \in \mathbb{N}$. Consider the constrained psychological game where players can choose mixed actions in the following sets:

$$\Sigma_i^n(h) = \{\sigma_i(\cdot|h) \in \Delta(A_i(h)) : \|\sigma_i(\cdot|h) - \bar\sigma_i(\cdot|h)\| \le \varepsilon_n\}$$

if h is on the $\bar{\sigma}$-path, and

$$\Sigma_i^n(h) = \{\sigma_i(\cdot|h) \in \Delta(A_i(h)) : \forall a_i \in A_i(h), \sigma_i(a_i|h) \geq \varepsilon_n\}$$

if h is off the $\bar{\sigma}$-path. By construction, these sets are non-empty, convex, and compact. Since the decision-utility functions are continuous, and the consistent assessment map $\sigma \longmapsto \beta^\sigma$ is continuous (Lemma 2), correspondence

$$\sigma \longmapsto \times_{h \in H} \times_{i \in I} \arg \max_{\sigma_i'(\cdot|h) \in \Sigma_i^n(h)} \sum_{a_i \in A_i(h)} \sigma_i'(a_i|h) u_i(h, a_i; \beta_i^\sigma)$$

is upper-hemicontinuous, non-empty, convex, and compact valued; therefore (by Kakutani's theorem), it has a fixed point σ^n. By Lemma 3, the sequence of consistent assessments $\left(\sigma^n, \beta^{\sigma^n}\right)_{n=1}^{\infty}$ has a limit point (σ^*, β^*), which is consistent too. By construction, $\bar{\sigma}(\cdot|h) = \sigma^*(\cdot|h)$ for h on the $\bar{\sigma}$-path, therefore $(\bar{\sigma}, \bar{\beta})$ and (σ^*, β^*) are realization-equivalent. We let $\bar{\alpha}_i$ (respectively, α_i^*) denote the first-order beliefs of i implied by $(\bar{\sigma}, \bar{\beta})$ (respectively, (σ^*, β^*)).

We claim that the consistent assessment (σ^*, β^*) is a SE of the psychological game with decision-utility functions $u_i(h, a_i; \cdot)$. We must show that (σ^*, β^*) satisfies sequential rationality. If h is off the $\bar{\sigma}$-path, sequential rationality is satisfied by construction. Since $\bar{\sigma}$ is deterministic and there are no chance moves, if h is on the $\bar{\sigma}$-path (*i.e.* on the σ^*-path) it must have unconditional probability one according to each player's beliefs and there cannot be any frustration; hence, $u_i(h, a_i; \beta_i^*) = \mathbb{E}[\pi_i|h, a_i; \alpha_i^*]$ $(i \in I)$ where α_i^* is determined by σ^*. If, furthermore, it is the second stage ($h = \bar{a}^1$, with $\bar{\sigma}(\bar{a}^1|\varnothing) = 1$), then —by construction— $\mathbb{E}[\pi_i|h, a_i; \alpha_i^*] = \mathbb{E}[\pi_i|h, a_i; \bar{\alpha}_i]$, where $\bar{\alpha}_i$ is determined by $\bar{\sigma}$. Since $\bar{\sigma}$ is a SE of the material-payoff game, sequential rationality is satisfied at h. Finally, we claim that (σ^*, β^*) satisfies sequential rationality also at the root $h = \varnothing$. Let $\iota(h)$ denote the active player at h. Since $\iota(\varnothing)$ cannot be frustrated at $\varnothing$, we must show that action $\bar{a}^1$ with $\bar{\sigma}(\bar{a}^1|\varnothing) = 1$ maximizes his expected material payoff given belief $\alpha_{\iota(\varnothing)}$. According to ABB and ABI, player $\iota(a^1)$ can only blame the first mover $\iota(\varnothing)$ and possibly hurt him, if he is frustrated. Therefore, in assessment (σ^*, β^*) at node a^1, either $\iota(a^1)$ plans to choose his (unique) payoff maximizing action, or he blames $\iota(\varnothing)$ strongly enough to give up some material payoff in order to bring down the payoff of $\iota(\varnothing)$. Hence, $\mathbb{E}[\pi_{\iota(\varnothing)}|a^1; \alpha^*_{\iota(a^1)}] \leq \mathbb{E}[\pi_{\iota(\varnothing)}|a^1; \bar{\alpha}_{\iota(a^1)}]$ (anger). By consistency of (σ^*, β^*) and $(\bar{\sigma}, \bar{\beta})$, $\alpha^*_{\iota(a^1)} = \alpha^*_{\iota(\varnothing)}$ and $\bar{\alpha}_{\iota(a^1)} = \bar{\alpha}_{\iota(\varnothing)}$ (cons.). Since (σ^*, β^*) is realization-equivalent to $(\bar{\sigma}, \bar{\beta})$ (r.e.), which is the material-payoff equilibrium (m.eq.), for each $a^1 \in A(\varnothing)$,

$$
\begin{aligned}
&\mathbb{E}[\pi_{\iota(\varnothing)}|\bar{a}^1; \alpha^*_{\iota(\varnothing)}] \overset{\text{(r.e.)}}{=} \mathbb{E}[\pi_{\iota(\varnothing)}|\bar{a}^1; \bar{\alpha}_{\iota(\varnothing)}] \overset{\text{(m.eq.)}}{\geq} \\
&\mathbb{E}[\pi_{\iota(\varnothing)}|a^1; \bar{\alpha}_{\iota(\varnothing)}] \overset{\text{(cons.)}}{=} \mathbb{E}[\pi_{\iota(\varnothing)}|a^1; \bar{\alpha}_{\iota(a^1)}] \overset{\text{(anger)}}{\geq} \\
&\mathbb{E}[\pi_{\iota(\varnothing)}|a^1; \alpha^*_{\iota(a^1)}] \overset{\text{(cons.)}}{=} \mathbb{E}[\pi_{\iota(\varnothing)}|a^1; \alpha^*_{\iota(\varnothing)}].
\end{aligned}
$$

This completes the proof for the ABB and ABI cases. If there are only two players, then we have a leader-follower game and SA is equivalent to ABB (Remark 1 of "Frustration and Anger in Games"), so (σ^*, β^*) is a SE in this case too. ■

2.3 Proof of Proposition 2

We denote the leader by $\iota(\varnothing)$. Let $(\sigma_i, \beta_i)_{i\in I}$ be a SE under ABB/SA with parameter profile $(\theta_i)_{i\in I}$, and suppose that the leader's strategy has full support: $\text{Supp}\sigma_{\iota(\varnothing)}(\cdot|\varnothing) = A_{\iota(\varnothing)}(\varnothing)$. Construct a polymorphic consistent assessment $\bar{\lambda}$ as follows: For each follower i, $T_i(\bar{\lambda}_i) = \{t_i\}$ (a singleton) and $\left(\bar{\sigma}_{t_i}, \bar{\beta}_{t_i}\right) = (\sigma_i, \beta_i)$. For the leader $\iota(\varnothing)$, $T_{\iota(\varnothing)}(\bar{\lambda}_{\iota(\varnothing)}) = A_{\iota(\varnothing)}(\varnothing)$, and, for each type $a_{\iota(\varnothing)}$, $\bar{\sigma}_{a_{\iota(\varnothing)}}(a_{\iota(\varnothing)}|\varnothing) = 1$ and $\bar{\alpha}_{a_{\iota(\varnothing)}}(\cdot|a^1) = \prod_{i\in I} \sigma_i(\cdot|a^1)$ for all non-terminal a^1, where $\bar{\alpha}_{a_{\iota(\varnothing)}}$ is the first-order belief derived from $\bar{\beta}_{a_{\iota(\varnothing)}}$. By construction, each type of leader is indifferent, because the leader (who acts as-if selfish) is indifferent in the original assessment $(\sigma_i, \beta_i)_{i\in I}$. As for the followers, they have the same first-order beliefs, hence the same second-stage frustrations as in $(\sigma_i, \beta_i)_{i\in I}$. Under ABB/SA, blame always equals frustration in leader-followers games. As for ABI, Bayes' rule implies that, after observing $a^1 = a_{\iota(\varnothing)}$, each follower becomes certain that the leader indeed planned to choose $a_{\iota(\varnothing)}$ with probability one, and blame equals frustration in this case too. Therefore, the incentive conditions of the followers hold in $\bar{\lambda}$ as in $(\sigma_i, \beta_i)_{i\in I}$ for all kinds of decision utility (ABI, ABB, SA) under the same parameter profile $(\theta_i)_{i\in I}$. ■

2.4 Proof of Remark 4

Fix $h \in H$. We consider the following simple extension of could-have-been blame in multistage games under fast play:

$$\mathrm{B}_{ij}(h;\alpha_i) = \min\left\{\left[\max_{h' \prec h, a'_j \in A_j(h')} \mathbb{E}\left[\pi_i|(h', a'_j);\alpha_i\right] - \mathbb{E}[\pi_i|h;\alpha_i]\right]^+, \mathrm{F}_i(h;\alpha_i)\right\}. \tag{5}$$

We must show that $\mathrm{B}_{ij}(h;\alpha_i) = 0$ if j is not active at any $h' \prec h$, and $\mathrm{B}_{ij}(h;\alpha_i) = \mathrm{F}_i(h;\alpha_i)$ if j is the only active player at each $h' \prec h$.

First note that if j was never active before, then $A_j(h')$ is a singleton for each $h' \prec h$, hence the term in brackets of (5) is zero. Next suppose that i is frustrated at h and j was the only active player in the past. Then there must be some $\bar{h} \prec h$ such that j deviated from i's expectations $\alpha_i(\cdot|\bar{h})$ for the first time, that is, $\bar{h}$ is the shortest predecessor $h' \prec h$ such that $\alpha_j(a'_j|h') < 1$ for $(h', a'_j) \preceq h$. Such $\bar{h}$ must have probability one according to the initial belief $\alpha_i(\cdot|\varnothing)$, thus $\mathbb{E}[\pi_i|\bar{h};\alpha_i] = \mathbb{E}[\pi_i;\alpha_i]$. Since $\max_{a'_j \in A_j(\bar{h})} \mathbb{E}\left[\pi_i|(\bar{h}, a'_j);\alpha_i\right] \geq \mathbb{E}[\pi_i|\bar{h};\alpha_i]$, we have $\max_{a'_j \in A_j(\bar{h})} \mathbb{E}\left[\pi_i|(\bar{h}, a'_j);\alpha_i\right] \geq \mathbb{E}[\pi_i;\alpha_i]$. Therefore

$$\begin{aligned}
& \max_{h' \prec h, a'_j \in A_j(h')} \mathbb{E}\left[\pi_i|(h', a'_j);\alpha_i\right] - \mathbb{E}[\pi_i|h;\alpha_i] \\
\geq\ & \max_{a'_j \in A_j(\bar{h})} \mathbb{E}\left[\pi_i|(\bar{h}, a'_j);\alpha_i\right] - \mathbb{E}[\pi_i|h;\alpha_i] \\
\geq\ & \mathbb{E}[\pi_i;\alpha_i] - \mathbb{E}[\pi_i|h;\alpha_i] \\
\geq\ & \mathbb{E}[\pi_i;\alpha_i] - \max_{a_i \in A_i(h)} \mathbb{E}[\pi_i|(h, a_i);\alpha_i] = \mathrm{F}_i(h;\alpha_i),
\end{aligned}$$

which implies $\mathrm{B}_{ij}(h;\alpha_i) = \mathrm{F}_i(h;\alpha_i)$ according to (5). ■

References

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[2] Battigalli, P., and M. Siniscalchi (1999): "Hierarchies of Conditional Beliefs and Interactive Epistemology in Dynamic Games," *Journal of Economic Theory,* 88, 188-230.