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EXISTENCE AND UNIQUENESS OF MAXIMAL REDUCTIONS UNDER ITERATED STRICT DOMINANCE

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Iterated elimination of strictly dominated strategies is an order dependent procedure. It can also generate spurious Nash equilibria, fail to converge in countable steps, or converge to empty strategy sets. If best replies are well-defined, then spurious Nash equilibria cannot appear; if strategy spaces are compact and payoff functions are uppersemicontinuous in own strategies, then order does not matter; if strategy sets are compact and payoff functions are continuous in all strategies, then a unique and nonempty maximal reduction exists. These positive results extend neither to the better-reply secure games for which Reny has established the existence of a Nash equilibrium, nor to games in which (under iterated eliminations) any dominated strategy has an undominated dominator.

KEYWORDS: Game theory, strict dominance, iterated elimination, order independence, maximal reduction, existence.

1. INTRODUCTION

A BASIC RULE for predicting behavior in noncooperative games is that players should not adopt strictly dominated strategies. Eliminating such strategies from consideration may permit additional eliminations, and iterated elimination of strictly dominated strategies (IESDS) leads to a fundamental solution concept: the maximal reduction of a game. In some cases, the maximal reduction comprises a single strategy profile. For example, in a standard Cournot duopoly, after eliminating outputs that exceed a monopolist's output, small and large outputs may be eliminated sequentially until, in the limit, only the Cournot-Nash equilibrium remains (see Moulin (1984)).

Game theorists often assume, explicitly or implicitly, that play should be confined to the maximal reduction of a game. Nevertheless, little is known about this solution concept. This paper studies the conditions under which maximal reductions exist and are unique.

Because IESDS entails the shrinking of strategy sets, and shrinking sets always reach a limit, the question of existence reduces to: is the limit nonempty and does it own only undominated strategies? The answers are clearly yes for finite games, but we show by example that infinite games need not have maximal reductions. Uniqueness, on the other hand, concerns the speed and order of

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reductions: do all paths lead, in the limit, to the same maximal reduction? The answer is known to be yes for finite games, but we show, again by example, that order may matter for infinite games. Some of the examples are simple, but the finding may nevertheless appear surprising, as it seems to be widely believed that IESDS is an order independent procedure.

Reny (1999) has recently proved the existence of Nash equilibrium for a large class of games: those with finite player sets, compact and convex strategy sets, and payoff functions that are bounded, quasi-concave in the player's own strategy, and satisfy a condition of better-reply security. We show that this class includes games where order matters for IESDS.

More surprising, we also show that requiring that strategies be eliminated only by undominated strategies (a variation on Jackson's (1992) idea of "boundedness") does not solve the problem of order dependence. Specifically, we define a game to be closed under dominance (CD) if any dominated strategy has an undominated dominator, in any IESDS sequence. We prove that, in CD games, several elimination procedures are equivalent and can exhibit order dependence or other perverse properties.

To balance these negative findings, we establish several positive results concerning IESDS. We consider games having arbitrary numbers of players and strategy sets in arbitrary Hausdorff spaces, and we call such games compact and continuous if the strategy sets are compact and the payoff functions continuous. This class differs from Reny's because it does not require convex strategy sets, a finite (or even countable) player set, or quasi-concave payoffs; however, it does require continuous payoffs. We show that any compact and continuous game has a unique maximal reduction, which has nonempty strategy sets. For uniqueness (order independence), weaker assumptions suffice: if a compact game with payoffs uppersemicontinuous in own strategies has any maximal reduction with nonempty strategy sets, then its maximal reduction is unique. Outside of these classes of games, existence and uniqueness routinely fail.

We also identify a larger class of games, for which IESDS preserves the set of Nash equilibria. For this property, it is sufficient that each player have a well-defined best-response correspondence. The existence and uniqueness of maximally reduced games does not extend to this larger class.

Results concerning the order independence of IESDS in finite games have been derived by Gilboa, Kalai, and Zemel (1990), Stegeman (1990), Börgers (1993), and Osborne and Rubinstein (1994). We discuss their contributions and draw connections to our results.

We proceed as follows. Section 2 provides examples for which maximal reductions do not exist or are empty or for which order matters, or for which IESDS may generate spurious Nash equilibria. Section 3 states positive results concerning the existence and uniqueness of nonempty maximal reductions of compact and continuous games. Section 4 describes conditions under which IESDS does not affect the set of Nash equilibria. Section 5 defines games that are closed under dominance and shows that such games can exhibit all of the pathologies of Section 2. Section 6 discusses related literature, and Section 7 offers concluding remarks.

2. WHERE ORDER MATTERS OR EXISTENCE FAILS

In our first example, the order of elimination matters, and most elimination sequences introduce Nash equilibria that were not present in the original game.

EXAMPLE 1: Consider a game with player set $I=\{1,2\}$, strategy sets $G_1 = G_2=[0,1]$, and symmetric payoff functions u_i: $G_i \times G_j \to \mathbb{R}$ for $i, j \in I$ and $i \neq j$, defined by

$$\begin{aligned} &u_i(x, y) = x && \text{if } x < 1, \\ &u_i(1, y) = 0 && \text{if } y < 1, \\ &u_i(1, 1) = 1. \end{aligned}$$

Every strategy except 1 is strictly dominated, and the strategy profile (1, 1) is the game's unique Nash equilibrium. Eliminating $G_i \backslash \{1, x\}$ for some $x < 1$, for $i = 1, 2$, leaves the following 2×2 game, which cannot be further reduced:

	1	***x***
1	$1, 1$	$0, x$
x	$x, 0$	x, x

Since x is arbitrary, the example shows that IESDS is an order dependent procedure. The criterion of risk dominance selects the (x, x) equilibrium if $x > 1/2$, though (x, x) was not even a Nash equilibrium in the original game.

Example 1 belongs to a large class of games for which Reny (1999) has recently proved the existence of Nash equilibrium: games such that the player set is finite, strategy sets are compact and convex, payoffs are quasi-concave in the own player's strategy, and a condition of better-reply security holds. The last condition requires that for any nonequilibrium strategy profile $x^* = (x_j^*)_{j\in I}$ and every payoff vector limit $(u_j^*)_{j\in I}$ resulting from strategy profiles approaching x^*, some player i has a strategy guaranteeing himself a payoff strictly above u_i^* even if the others deviate slightly from x^*. The game in Example 1 is better-reply secure because given any nonequilibrium profile x^* we can pick a player $i \in \{1, 2\}$ such that $x_i^* < 1$ and an $\varepsilon > 0$ such that i can secure payoff $x_i^* + \varepsilon > u_i^* = x_i^*$. Moreover, $G_i = [0, 1]$ is compact and convex and $u_i(\cdot, x_j)$ is quasi-concave and bounded. Hence, Reny's conditions ensuring the existence of a Nash equilibrium are insufficient to ensure that IESDS is well-behaved.

Order dependence may have little practical significance in Example 1, because a natural sequence of reductions eliminates all strategies less than 1 in the first round, which immediately reduces the game to its original Nash equilibrium. In more complicated games, however, the burden of finding all strictly dominated strategies, in any given round of reduction, can be much greater. Order independence, when it applies, eliminates the problem of coordinating on a particular sequence of eliminations.

Discontinuous payoffs lead to order dependence and spurious Nash equilibria in Example 1, and Example 2 shows that they can also lead to maximal reductions with empty strategy sets.

EXAMPLE 2: Consider $I = \{1, 2\}$, strategy sets $G_1 = G_2 = [0, 1]$, and symmetric payoff functions u_i: $G_i \times G_j \to \mathbb{R}$ for $i, j \in I$ and $i \neq j$, defined by

$$\begin{aligned} u_i(x, y) &= 1 - x \quad \text{if } 0 < y/2 \leq x, \\ u_i(x, y) &= x \quad \text{else.} \end{aligned}$$

Any $x \in (0, 1)$ strictly dominates $x = 0$, and eliminating strategies as fast as possible produces this sequence of strategy sets: $(0, 1]$, $(0, 1/2]$, $(0, 1/4]$, $(0, 1/8]$, et cetera, with the limiting strategy sets being empty. Furthermore, the corollary to Theorem 1, in the sequel, implies that this is the unique maximal reduction of the game.

The next example shows that discontinuous payoffs can cause yet another problem for IESDS: the nonexistence of a maximal reduction.

EXAMPLE 3 (Cournot competition with outside wager): Consider $I = \{1, 2, 3\}$, $G_1 = G_2 = [0, 1]$, $G_3 = \{\alpha, \beta\}$, u_1: $G_1 \times G_2 \times G_3 \to \mathbb{R}$ defined by $u_1(x, y, z) = x(1 - x - y)$, $u_2 : G_1 \times G_2 \times G_3 \to \mathbb{R}$ defined by $u_2(x, y, z) = y(1 - x - y)$, and $u_3 : G_1 \times G_2 \times G_3 \to \mathbb{R}$ such that

$$\begin{aligned} &u_3(x, y, \alpha) < u_3(x, y, \beta) \quad \text{if } (x, y) \neq (1/3, 1/3), \\ &u_3(1/3, 1/3, \alpha) > u_3(1/3, 1/3, \beta). \end{aligned}$$

The interpretation is that 1 and 2 are competing firms, while 3 is a person who does not influence the firms' profits but who has an opportunity to wager on the event that each firm chooses strategy 1/3. Note that $(1/3, 1/3)$ is the Nash equilibrium of the corresponding Cournot game without player 3 present. IESDS reduces G_1 and G_2 to the common strategy set $\{1/3\}$, although this requires an infinite sequence of elimination rounds (e.g., with the surviving strategy sets described by the sequence $[0, 1]$, $[0, 1/2]$, $[1/4, 1/2]$, $[1/4, 3/8]$, $[5/16, 3/8], \ldots$). For each of these rounds, it is not possible to eliminate a strategy for player 3. The limit of this sequence is not a maximal reduction, because given the limiting strategy sets $\{1/3\}$, player 3's strategy β is strictly dominated. Therefore, there exists no maximal reduction (in countable steps).[2]

Examples 1 through 3 show that discontinuous payoffs can lead to nonuniqueness, or emptiness, or nonexistence of maximal reductions. The next example shows that unbounded strategy sets can create similar problems.[3] Here, as in Example 1, order matters and IESDS can introduce spurious Nash equilibria.

[2] Lipman (1994) discusses games that have similar properties. He shows that "common knowledge of rationality is not, in general, equivalent to the limit as $n \to \infty$ of order n mutual knowledge of rationality."

[3] Example 4, which appears in Stegeman (1990), is the first example of order dependence of IESDS of which we are aware.

EXAMPLE 4: Consider $I = \{1,2\}, G_1 = G_2 = \mathbb{R}_+$, and payoff function u_i: $G_1 \times G_2 \to \mathbb{R}$ for $i, j \in I$, defined by $u_i(s_1, s_2) = (\max\{s_1, 1-s_1-s_2\})/(1+s_1)$. The game is not symmetric, but the players have a common payoff function. The payoff function is continuous and has range $[0,1]$. If $s_2 > 0$, then player 1's optimal action is undefined, and it follows directly that the unique Nash equilibrium of the game is $(s_1, s_2) = (0, 0)$. One way to perform IESDS is as follows: eliminate every $s_1 > 0$ as it is strictly dominated by some $s_1' > s_1$. Given that $s_1 = 0$, every $s_2 > 0$ is then strictly dominated by $s_2 = 0$. IESDS thus eliminates all except Nash play. Another way to perform IESDS is: eliminate every $s_1 > 0$ except $s_1 = 1$, leaving the strategy sets $\{0, 1\}$ for player 1 and $\mathbb{R}_+$ for player 2. No more eliminations are possible and the residual game now has many Nash equilibria: $(s_1, s_2) = (0, 0)$ and $(s_1, s_2) = (1, y)$ for all $y \geq 1/2$.

To support our claim that the problems with IESDS are fundamental, consider the following, simplest possible, example of order dependence.

EXAMPLE 5: Consider a one-player game with strategy set $G_1 = (0, 1)$ and payoff function u_1: $G_1 \to \mathbb{R}$ defined by $u_1(x) = x$ for all $x \in G_1$. Every strategy is strictly dominated. For any $x \in G_1$, eliminate in round one all strategies in the set $G_1 \backslash \{x\}$, and only x survives IESDS.

These examples suggest that the perverse outcomes of IESDS are associated with the absence of a "best" dominating strategy. We show in Section 5, however, that the absence of a best dominator plays no special role; the same anomalies can appear in games having best dominators. In Section 3, we formulate and prove the weaker conjecture that IESDS is well-behaved in games having continuous payoff functions and compact strategy sets.

3. WHEN MAXIMAL REDUCTIONS EXIST AND ARE UNIQUE

This section describes classes of games for which IESDS works well, in the sense that maximal reductions exist or are unique. Preliminary definitions follow.

GAMES, PARINGS, AND STRICT DOMINANCE: A game is a triple $G = (I, (G_i)_{i\in I}, (u_i)_{i\in I})$, where I is the set of players. Assume $\#I \geq 2$, but the number of players is otherwise unrestricted (e.g., I may be uncountable). Player i's strategy set $G_i \neq \varnothing$ is a Hausdorff space (e.g., a metric space). The joint strategy space $\prod_i G_i$ is endowed with the product topology, the set of real numbers $\mathbb{R}$ is endowed with the usual topology, and u_i: $\prod_i G_i \to \mathbb{R}$ is the payoff of player i. We call the game G: compact if G_i is compact $\forall i \in I$; own-uppersemicontinuous if u_i is uppersemicontinuous in $s_i \,\forall i \in I$;[4] continuous if u_i is continuous in the product topology $\forall i \in I$. A paring of G is a triple $H = (I, (H_i)_{i\in I}, (u_i')_{i\in I})$, where $H_i \subseteq G_i$

[4] A function f: $S \to \mathbb{R}$ is uppersemicontinuous if $\{s \in S | f(s) \geq r\}$ is closed in S for all $r \in \mathbb{R}$; this is implied by f continuous.

and u_i' is the restriction of u_i to $\prod_i H_i$, $\forall i \in I$. As the notation suggests, we generally identify a game and its parings with the associated strategy sets. A paring is nonempty if $H_i \neq \varnothing \, \forall i \in I$. Hence, a nonempty paring is a game. For any paring H, let $H_{-i} \equiv \prod_{j \neq i} H_j$. Let $P(G)$ denote the set of all parings of G. Given a paring H of G, and $x, y \in G_i$: $y \succ_H x$ if $H_{-i} \neq \varnothing$ and $u_i(y, s_{-i}) > u_i(x, s_{-i}) \, \forall s_{-i} \in H_{-i}$. (The reordering of the arguments of u_i simplifies notation, where no confusion is possible.) The relation $\succ_H$ embodies the notion of strict dominance given rivals' options in paring H.

REDUCTION: Consider parings $H, H' \in P(G)$, such that $H_i' \subseteq H_i \, \forall i \in I$. $H \to H'$ if, for each $x \in H_i \backslash H_i'$, $\exists y \in H_i$ such that $y \succ_H x$. $H \to H'$ is defined to be fast if $y \succ_H x$ for some $x, y \in H_i$ implies $x \notin H_i'$. Clearly, $H \to H'$ is fast for some unique H'. We use the symbol $\to^*$ as follows: $H \to^* H'$ if there exists a (finite or countably infinite) sequence of parings, $A^t \in P(G), t = 0, 1, 2 \ldots$, such that $A^0 = H$, $A^t \to A^{t+1} \, \forall t \geq 0$, and $H_i' = \bigcap_t A_i^t \, \forall i \in I$. H is a maximal $(\to^*)$-reduction of G if $G \to^* H$ and $H \to H'$ only for $H' = H$.[5]

The following lemma is the key to our existence and uniqueness results.

LEMMA: *If $G \to^* H$ for some compact and own-uppersemicontinuous game G, and $y \succ_H x$ for some $x, y \in G_i$ and $i \in I$, then $\exists z^* \in H_i$ such that $z \not\succ_H z^* \succ_H x \, \forall z \in G_i$.*

PROOF: Given H as described, let $A^t \in P(G), t = 0, 1, 2 \ldots$, be the implied sequence of parings. Let $Z \equiv \{z \in G_i | u_i(z, s_{-i}) \geq u_i(y, s_{-i}) \, \forall s_{-i} \in H_{-i}\}$. The uppersemicontinuity of u_i in z implies that $\{z \in G_i | u_i(z, s_{-i}) \geq u_i(y, s_{-i})\}$ is closed for given s_{-i}, implying that the intersection Z is closed and therefore compact (because G_i is compact). Clearly $y \in Z$, and $y \succ_H x$ implies that $H_{-i} \neq \varnothing$. Define $f: Z \to \mathbb{R}$ by $f(z) = u_i(z, s_{-i}^*)$ for some fixed and arbitrary $s_{-i}^* \in H_{-i}$. The uppersemicontinuity of u_i in z implies that f is uppersemicontinuous, which with Z compact implies that f reaches a maximum at some $z^* \in Z$ (e.g., Bourbaki (1966, Theorem IV, 6, 2, 3)). $z^* \in Z$ and $y \succ_H x$ imply $z^* \succ_H x$. If $z \succ_H z^*$ for some $z \in G_i$, then $u_i(z, s_{-i}) > u_i(z^*, s_{-i}) \, \forall s_{-i} \in H_{-i}$, implying $z \in Z$ and $f(z) > f(z^*)$, a contradiction. Therefore, $z \not\succ_H z^* \, \forall z \in G_i$, implying $z \not\succ_{A^t} z^* \, \forall z \in G_i$, $\forall t \geq 0$ (because $H_{-i} \subseteq A_{-i}^t$), implying $z^* \in A_i^t \, \forall t \geq 0$, implying $z^* \in H_i$. *Q.E.D.*

THEOREM 1: (a) *If a game G is compact and own-uppersemicontinuous, then any nonempty maximal $(\to^*)$-reduction of G is the unique maximal $(\to^*)$-reduction of G.* (b) *If a game G is compact and continuous, then G has a unique maximal $(\to^*)$-reduction M; furthermore, M is nonempty, compact, and continuous.*

[5] An alternative definition of a maximal $(\to^*)$-reduction H would require, in addition, that H be nonempty. That language has an unnatural aspect: if a reduction to empty strategy sets is not maximal, then what is? More important, we want the claim that the "maximal reduction is unique" to imply that there does not exist an alternative reduction sequence that ends in an empty strategy set. Of course, there generally exist alternative reduction sequences that are so slow that they fail to converge to any maximal reduction.

PROOF: *Part (a).* Let M and M' be maximal $(\to^*)$-reductions of G, M nonempty. Given $G \to^* M'$, let $A^t \in P(G), t = 0, 1, 2 \ldots$, be the implied finite or infinite sequence of parings. Suppose that $M_i \nsubseteq M'_i$ for some i. Then $M_i \nsubseteq A^t_i$ $\forall\, t > T$, for some T such that A^{T+1}_i is well-defined. Let T take the largest value such that $M_i \subseteq A^T_i \,\forall\, i \in I$. Choose $i \in I$ and $x \in M_i \backslash A^{T+1}_i$. Then $x \in A^T_i \backslash A^{T+1}_i$, implying $\exists y \in A^T_i$ such that $y \succ_{A^T} x$, which with $\varnothing \neq M_i \subseteq A^T_i \,\forall\, i \in I$ implies $y \succ_M x$. The lemma implies $\exists z^* \in M_i$ such that $z^* \succ_M x$, contradicting that M is a maximal $(\to^*)$-reduction. Therefore, $M_i \subseteq M'_i \,\forall\, i \in I$, which implies that M' is nonempty. Similarly, $M'_i \subseteq M_i \,\forall\, i \in I$, implying $M = M'$.

Part (b). First, observe a trivial corollary of the lemma, for $H = G$: (i) If $y \succ_G x$ for some $x, y \in G_i$, then $\exists z^* \in G_i$ such that $z \not\succ_G z^* \succ_G x \,\forall\, z \in G_i$. Second, recall the standard fact for Hausdorff spaces: (ii) if $B_0 \supseteq B_1 \supseteq B_2 \ldots$ where $B_t \neq \varnothing$ is compact for all t, then $\bigcap_t B_t$ is nonempty and compact. We now establish (iii): if G is compact and continuous and $G \to H$ is fast, then H is compact and nonempty. Considering only the nontrivial case, choose i such that $H_i \neq G_i$. Then $y \succ_G x$ for some $x, y \in G_i$, and (i) implies $H_i \neq \varnothing$. It remains to show that H_i is compact. Choose $x \in H_i$, and let $Z \equiv \{s \in G | u_i(x, s_{-i}) \leq u_i(s)\}$.[6] Clearly $Z \neq \varnothing$. Define the temporary function f: $G \to G$ by $f(s) = (x, s_{-i})$. The elementary properties of the product topology imply that f is continuous, which with the continuity of u_i implies that $u_i(x, s_{-i}) = u_i(f(s))$ is continuous in s. Therefore, Z is closed. Since the projection function pr_i: $G \to G_i$ is continuous, $Z^x \equiv \text{pr}_i(Z)$ is a nonempty closed subset of G_i, with $x \in Z^x$. Consider arbitrary $w \in G_i$. For any $x \in H_i$, if $w \notin Z^x$, then $u_i(x, s_{-i}) > u_i(w, s_{-i})$ for all $s_{-i} \in G_{-i}$, implying $x \succ_G w$, implying $w \notin H_i$. Therefore, $H_i \subseteq \bigcap_x Z^x$, the intersection across $x \in H_i$. If $w \notin H_i$, then $x \succ_G w$ for some $x \in G_i$, and (i) then implies the existence of $x^* \in G_i$ such that $x^* \succ_G w$, implying $w \notin Z^{x^*}$, implying $w \notin \bigcap_x Z^x$. Therefore, $H_i \supseteq \bigcap_x Z^x$, implying $H_i = \bigcap_x Z^x$, but since Z^x is closed for all x, H_i is also closed and therefore compact.

Let $C(t), t = 0, 1, \ldots,$ denote the unique sequence of subgames of G such that $C(0) = G$ and $C(t) \to C(t+1)$ is fast, for all $t \geq 0$. Result (iii) implies by induction that $C(t)$ is compact and nonempty $\forall\, t$, and result (ii) then implies that paring M with strategy sets $M_i \equiv \bigcap_t C(t)_i$ is compact, continuous, and nonempty. To show that M is a maximal $(\to^*)$-reduction of G, it remains to show that nothing in M is strictly dominated. Consider any player i and any $x, y \in M_i$. Let $X \equiv \{s_{-i} \in G_{-i} | u_i(y, s_{-i}) \leq u_i(x, s_{-i})\}$. If $X \cap C(t)_{-i} = \varnothing$ for any t such that $C(t) \neq M$, then $y \succ_{C(t)} x$, contradicting $x \in M_i$. Therefore $X \cap C(t)_{-i} \neq \varnothing$, and the compactness of X and of $C(t)_{-i}$ (from (iii)) implies that $X \cap C(t)_{-i}$ is compact for all t such that $C(t) \neq M$, implying from (ii) that $X \cap M_{-i}$ is nonempty, implying that $y \not\succ_M x$. *Q.E.D.*

The following corollary provides a stronger and cleaner version of the order independence result, for two-player games.

[6] We owe set Z to Hannu Salonen.

COROLLARY 1: *If a two-player game G is compact and own-uppersemicontinuous, then G has at most one maximal reduction.*

PROOF: Let M and M' be distinct maximal $(\rightarrow^*)$-reductions of G. Then M_1, M_2, M'_1, and M'_2 cannot all be empty. Without loss of generality, assume $M_1 \neq \varnothing$. Then player 2's best replies to the elements of M_1 can never be eliminated, implying that M is nonempty, implying from Theorem 1(a) that M is the unique maximal $(\rightarrow^*)$-reduction of G, a contradiction. *Q.E.D.*

Theorem 1(b) says that a compact and continuous game can always be completely reduced by IESDS and that the procedure is order independent for such games. The theorem covers finite games, their mixed extensions, and important applications such as the Cournot game. IESDS sometimes requires a countably infinite sequence of elimination rounds, as in the Cournot game, but for compact and continuous games this always suffices.

If, instead, payoffs are merely uppersemicontinuous in own strategies, then Theorem 1(a) shows that order still does not matter for any game that has a nonempty maximal reduction, but Examples 2 and 3 show that maximal reductions may be empty or nonexistent.[7] If the assumptions are weakened further, then Examples 1, 4, and 5 show that order may matter.

We need the Hausdorff property only to establish fact (ii) in the proof of Theorem 1(b), which we use to show that any maximal reduction of a compact and continuous game has nonempty strategy sets. All other results in the paper (including the order independence result of Theorem 1(a)) hold for strategy sets in general topological spaces.

Theorem 1 admits an infinite number of players, and in this case the assumption that payoff functions are continuous may be stronger than intuition suggests. We close this section with an example that illustrates how easy it is to violate continuity with infinitely many players.

EXAMPLE 6: Consider a game with (finite or countably infinite) player set $I = \{0, 1, \dots\}$. Let $G_0 = \{\alpha, \beta, \gamma\}$ and $G_j = \{\alpha, \beta\}$ for all $j > 0$. Assume that player 0's payoffs satisfy

$$\begin{aligned}
u_0(\alpha, x_{-0}) &= 0 \quad \forall x_{-0} \in G_{-0},\\
u_0(\beta, x_{-0}) &= 1 \quad \forall x_{-0} \in G_{-0},\\
u_0(\gamma, x_{-0}) &= 2 \quad \text{if } \exists j > 0 \text{ with } x_j = \alpha,\\
u_0(\gamma, x_{-0}) &= 0 \quad \text{if } \forall j > 0\ x_j = \beta,
\end{aligned}$$

and that the payoffs of any player $j > 0$ satisfy

$$\begin{aligned}
u_j(\alpha, x_{-j}) &= 2 \quad \text{if } x_{j-1} = \alpha,\\
u_j(\alpha, x_{-j}) &= 0 \quad \text{if } x_{j-1} \neq \alpha,\\
u_j(\beta, x_{-j}) &= 1 \quad \forall x_{-j} \in G_{-j}.
\end{aligned}$$

[7] An earlier version of Theorem 1(a) assumed that the payoff function is continuous in own strategies. We are grateful to Pierpaolo Battigalli for suggesting the generalization to uppersemicontinuity.

Fast IESDS eliminates $x_0 = \alpha$ in round 1, and for each round $t > 1$ strategy α is eliminated for player $t-1$. If I is finite, then Theorem 1 applies and $\#I+1$ rounds of fast eliminations produce the maximally reduced game such that each player has the strategy set $\{\beta\}$. If, however, I is not finite, then a countably infinite number of elimination rounds converges to the strategy sets $H_0 = \{\beta, \gamma\}$ and $H_j = \{\beta\}$ for all $j > 0$. This is not a maximal reduction, because player 0's strategy γ is strictly dominated. Hence compactness or continuity must be violated. The problem is that player 0's payoff function is no longer continuous in the product topology.

4. NASH EQUILIBRIA

We now prove that IESDS preserves the set of Nash equilibria in any game such that each player possesses a best response to each opposing strategy profile. This class of games strictly contains the class of compact and continuous games, and existence and uniqueness of maximal reductions do not extend to this larger class. Examples 3 and 6 have well-defined best replies but no maximal reduction, and Example 7 (below) has well-defined best-replies but order matters.

Recall that a Nash equilibrium is a strategy profile $x = (x_i)_{i\in I}$ such that $\forall i \in I$, $\forall y_i \in G_i$: $u_i(x_i, x_{-i}) \geq u_i(y_i, x_{-i})$.

THEOREM 2: *Assume, for any $x \in G$, that there exists $z^* \in G$ such that $u_i(z_i^*, x_{-i}) \geq u_i(z_i, x_{-i})$ for all $z \in G$ and $i \in I$. If M is a $(\to^*)$-reduction of G, then games G and M have the same Nash equilibria.*

PROOF: Let $A^t \in P(G), t = 0, 1, 2, \ldots,$ be the implied sequence of parings (which need not be unique). Suppose that $x, x_i \in G_i$, is a Nash equilibrium in game G. Then, by induction on t, x_i is never eliminated in the sequence $A^t\,\forall i \in I$, implying $x \in M$. Since $M_i \subseteq G_i\,\forall i \in I, x$ is also a Nash equilibrium in game M. Going the other way, suppose that $x \in M$ is a Nash equilibrium in game M. Choose z^* as assumed. Since $x \in M, z_i^*$ is never eliminated in the sequence $A^t\,\forall i \in I$, implying $z^* \in M$. The choice of x and z^* imply $u_i(x_1, x_{-i}) \geq u_i(z_i^*, x_{-i}) \geq u_i(z_i, x_{-i})\,\forall z_i \in G_i\,\forall i \in I$, implying that x is a Nash equilibrium in game G. *Q.E.D.*

COROLLARY 2: *If G is compact and own-uppersemicontinuous, and it has a maximal $(\to^*)$-reduction M comprising singleton strategy sets, then M describes the unique Nash equilibrium of game G.*

PROOF: If G is compact and own-uppersemicontinuous, then best replies are well-defined (by the standard result cited in the proof of the lemma) and Theorem 2 implies that G and M have the same Nash equilibria. The unique strategy profile admitted by game M is also its unique Nash equilibrium. *Q.E.D.*

Theorem 2 shows that at least one player must have ill-defined best responses in games where IESDS introduces spurious Nash equilibria. For instance, in

Example 1 neither player has a best response if his opponent chooses a strategy from the set [0,1).

The following example shows that well-defined best replies are insufficient to ensure order independence.

EXAMPLE 7: Let $I = \{1, 2\}$, $G_1 = \{\alpha, \beta\} \cup [0, 1)$, $G_2 = \{\gamma, \delta\}$, and payoffs be as indicated in the following matrix where $x \in [0, 1)$:

	γ	δ
α	$1, 0$	$0, 0$
β	$0, 0$	$1, 0$
$\boldsymbol{x}$	$x, 0$	$x, 0$

Each player has well-defined best responses. All strategies in [0,1) are strictly dominated for player 1, but elimination of $[0,1)\backslash\{x\}$ for some arbitrary $x \in [0, 1)$ leaves exactly the above matrix, which cannot be further reduced.

5. GAMES CLOSED UNDER DOMINANCE

In the easiest examples of IESDS pathologies (including all that we have presented to this point), some strategies are eliminated only by strategies that are themselves dominated. One might conjecture, therefore, that IESDS performs well in games such that any dominated strategy has an undominated dominator, as IESDS is applied. In this section we show that this conjecture is false: all of the pathologies of IESDS that we have discussed can appear in such games.

To make this claim precise, we will say that game G is closed under dominance (CD) if $G \to^* H$ in finite steps and $y \succ_H x$ for some $x, y \in H_i$ and $i \in I$ imply: the existence of $z^* \in H_i$ such that $z \not\succ_H z^* \succ_H x$ for all $z \in H_i$. In words, at any point in any valid sequence of deletions, any strictly dominated strategy x has an undominated dominator z^*. More games satisfy this condition than one might expect: the lemma implies immediately that all compact and own-uppersemicontinuous games are CD. Therefore, Examples 2, 3, and 6 are all CD, and yet all of these games have empty or nonexistent maximal reductions. Because CD is a bit weaker than the implication of the lemma, order can also matter in CD games. The following example, which modifies the Cournot game of Example 3, shows that IESDS can be order-dependent and produce spurious Nash equilibria in CD games.

EXAMPLE 8: Consider $I = \{1, 2\}$, $G_1 = G_2 = [0, 1]\backslash\{1/3\}$, and u_i: $G_i \times G_j \to \mathbb{R}$ for $i, j, \in I$ and $i \neq j$, defined by

$$u_i(x, y) = x(1 - x - y) \qquad \text{if } y \in \mathbb{Q},$$
$$u_i(x, y) = x(1 - x - 1/3) \qquad \text{if } y \notin \mathbb{Q},$$

where $\mathbb{Q}$ denotes the set of rational numbers. This game differs from the benchmark Cournot game (with $u_i(x, y) = x(1 - x - y)$ for all $(x, y) \in [0, 1]^2$) in two respects: an opponent's irrational strategy (meaning, an irrational number) acts like the Nash strategy 1/3, but the strategy 1/3 is itself unavailable. Note that the game has neither compact strategy sets nor continuous payoffs. Nevertheless, the following proposition and proof, which follow closely the statement and proof of the lemma in Section 3, show that the game is CD.

PROPOSITION: *The game of Example* 8 *is closed under dominance*.

PROOF: Let G be the game in Example 8. We show that if $G \to^* H$ in finite steps and $y \succ_H x$ for some $x, y \in H_i$ and $i \in I$, then $\exists z^* \in H_i$ such that $z \not\succ_H z^* \succ_H x \, \forall z \in G_i$. Without loss of generality, let $i = 1$. Given H as described, let $A^t \in P(G), t = 0, 1, 2, \ldots, T$, be the implied sequence of parings, with $H = A^T$. For either player in $G = A^0$, the set of available best responses to the elements of $G_1 = G_2$ is $[0, 1/2] \cap \mathbb{Q} \backslash \{1/3\}$. Since (in any game) a strategy that is a best response to some vector of strategy choices by the other players cannot be strictly dominated, one infers in the case of G, in turn, that $[0, 1/2] \cap \mathbb{Q} \backslash \{1/3\} \subseteq A_j^1, [1/4, 1/2] \cap \mathbb{Q} \backslash \{1/3\} \subseteq A_j^2, [1/4, 3/8] \cap \mathbb{Q} \backslash \{1/3\} \subseteq A_j^3, \ldots,$ for $j = 1, 2$ (cf. Example 3). After T steps, it follows that $H_2 \cap \mathbb{Q} = A_2^T \cap \mathbb{Q} \neq \varnothing$. Let $Z \equiv \{z \in [0, 1] \mid u_1(z, w) \geq u_1(y, w) \, \forall w \in H_2\}$. Clearly $y \in Z$, and the continuity of $u_1(z, w)$ in z implies that Z is closed and therefore compact. Define $f \colon Z \to \mathbb{R}$ by $f(z) = u_1(z, w^*)$ for some fixed $w^* \in H_2 \cap \mathbb{Q}$. Because $w^* \neq 1/3$, the function f reaches a maximum at some $z^* \in G_1$. $z^* \in Z \cap G_1$ and $y \succ_H x$ imply $z^* \succ_H x$. If $z \succ_H z^*$ for some $z \in G_1$, then $u_1(z, w) > u_1(z^*, w) \, \forall w \in H_2$, implying $z \in Z$ and $f(z) > f(z^*)$, a contradiction. Therefore, $z \not\succ_H z^* \, \forall z \in G_1$, implying $z \not\succ_{A^t} z^* \, \forall z \in G_1, \forall t = 0, 1, \ldots, T$, implying $z^* \in A_1^t, \forall t = 0, 1, \ldots, T$, implying $z^* \in H_1$. *Q.E.D.*

We now show that applying IESDS to the game in Example 8 can produce a variety of maximal reductions, which can include empty strategy sets. The sequence of strategy sets surviving fast reduction is : $[0, 1] \backslash \{1/3\}, [0, 1/2] \backslash \{1/3\}, [1/4, 1/2] \backslash \{1/3\}, \ldots$ (cf. Example 3). In the limit no strategy survives IESDS. Alternatively, consider the same sequence of reduced games, except that for each player i some arbitrary irrational strategy x_i is never eliminated. This too is a valid sequence of eliminations (the presence of player i's strategy x_i interferes with none of player i's eliminations and merely replicates player i's other irrational strategies from the viewpoint of player j), and player i's maximally reduced strategy set is $\{x_i\}$. Since x_i is arbitrary, order matters, and IESDS supports any pair of irrational strategies as a spurious Nash equilibrium. We conclude that the problems of IESDS in infinite games are deeper than the possible nonexistence of "best" dominating strategies.

This finding sheds light on a proposal by Jackson (1992). Jackson observes that, in some games, playing a *weakly* dominated strategy is more sensible than it at first appears. Our examples show that Jackson's point extends to strict dominance: in Example 1, playing a strictly dominated strategy seems defensible; in Example 5, it is inevitable! To avoid such anomalies and make implementation in weakly undominated strategies more convincing, Jackson suggests that mechanism designers employ "bounded mechanisms," game forms in which any dominated strategy is dominated by a strategy that is itself undominated.[8] Similarly, one could restrict the use of IESDS to CD games, and this could make the maximally reduced game a more appealing solution concept. Our present point is merely that this restriction does not solve certain problems that are attached to IESDS.

Another way to implement Jackson's suggestion is to restrict the elimination procedure itself (rather than the class of games to which it is applied), allowing the elimination only of strategies that have undominated dominators. In CD games, this procedure is obviously equivalent to the standard procedure and consequently solves none of its shortcomings.

6. RELATED WORK

Stegeman (1990) proves Theorem 1 for the case of finite games. Other papers have proven order independence for finite games, using slightly different concepts of dominance. This section discusses these alternatives and draws connections to the present work.

6.1. *GKZ Reductions*

Gilboa, Kalai, and Zemel (1990) (GKZ) consider a variety of elimination procedures and provide sufficient conditions for order independence. Among the procedures is a form of IESDS, and GKZ prove that for finite games this procedure is order invariant.[9] GKZ base their result, however, on a notion of reduction that bounds the rate of elimination, unlike the standard ($\rightarrow$)-reduction we have considered so far. We shall use the symbol $\Rightarrow$ for GKZ's reduction. The difference between a ($\Rightarrow$)-reduction and a ($\rightarrow$)-reduction is that the former requires that for any strategy x that is eliminated there exists a strategy y that strictly dominates x and is not eliminated. This restriction on the set of allowable reductions may be viewed as an intermediate response to the problem of dominated dominators. Rather than require that the dominator be undominated (along the lines of Jackson (1992)), GKZ require merely that the dominator not be eliminated. This complicates the reduction procedure by requiring that eliminations be assessed as a set rather than strategy-by-strategy, and it proves to be an unnecessary loss of generality. In the finite games that GKZ study, GKZ's bound on the rate of elimination obviously cannot prevent convergence to a maximal reduction, and our Theorem 1 implies that any such maximal reduction must be the unique reduction that obtains under ordinary IESDS.

This finding can be generalized. GKZ consider IESDS only for a finite number of elimination rounds, but in games with infinite strategy spaces it is natural to

[8] Salonen (1996, Corollary 1) gives general conditions under which a game must have this property.

[9] Gilboa, Kalai, and Zemel (1993) study the computational complexity of this and some other elimination procedures for finite games.

allow an infinite sequence of elimination rounds. Given this modification, we prove that for all games closed under dominance, the outcome of IESDS is the same regardless of whether ($\Rightarrow$)-reductions or ($\rightarrow$)-reductions are used. This includes games in which IESDS performs well (e.g., Theorem 1) and games in which it performs badly (e.g., Example 8). Some new definitions are needed:

GKZ REDUCTION: Consider parings $H, H' \in P(G)$, such that $H_i' \subseteq H_i \, \forall i \in I$. $H \Rightarrow H'$ if, for each $x \in H_i \backslash H_i'$, $\exists y \in H_i'$ such that $y \succ_H x$. We use the symbol $\Rightarrow^*$ as follows: $H \Rightarrow {}^*H'$ if there exists a (finite or countably infinite) sequence of parings, $A^t \in P(G), t = 0, 1, 2, \ldots,$ such that $A^0 = H$, $A^t \Rightarrow A^{t+1} \, \forall t \geq 0$, and $H_i' = \bigcap_t A_i^t \, \forall i \in I$. H is a maximal ($\Rightarrow^*$)-reduction of G if $G \Rightarrow {}^*H$ and $H \Rightarrow H'$ only for $H' = H$.

$H \Rightarrow H'$ and $H \Rightarrow^* H'$ imply, respectively, $H \rightarrow H'$ and $H \rightarrow^* H'$. The following theorem shows that the converse holds for CD games, implying that ($\Rightarrow$)-reductions and ($\rightarrow$)-reductions produce identical maximal reductions in CD games.

THEOREM 3: *If a game G is closed under dominance, then $G \rightarrow^* H$ if and only if $G \Rightarrow^* H$.*

PROOF: $G \Rightarrow^* H$ immediately implies $G \rightarrow^* H$. Going the other way, suppose $G \rightarrow^* H$, and let $A^t \in P(G), t = 0, 1, 2 \ldots,$ be the implied sequence of parings. It is sufficient to show that $A' \Rightarrow A''$ for any two consecutive elements of this sequence. Consider such A' and A''. If $A' = A''$, then $A' \Rightarrow A''$ trivially. If not, then choose $i \in I$ and $x \in A_i' \backslash A_i''$. $A' \rightarrow A''$ implies $\exists y \in A_i'$ such that $y \succ_{A'} x$. Because G is closed under dominance, $\exists z^* \in A_i'$ such that $z \not\succ_{A'} z^* \succ_{A'} x \, \forall z \in A_i'$, and $A' \rightarrow A''$ then implies $z^* \in A_i''$. Hence, $x \in A_i' \backslash A_i''$, any $i \in I$, implies $\exists \, z^* \in A_i''$ such that $z^* \succ_{A'} x$. Therefore, $A' \Rightarrow A''$. *Q.E.D.*

COROLLARY 3: *If G is compact and continuous, then G has a unique maximal ($\Rightarrow^*$)-reduction, which coincides with its unique maximal ($\rightarrow^*$)-reduction.*

PROOF: The lemma implies that G is closed under dominance, and the claim follows from Theorems 1 and 3. *Q.E.D.*

Theorem 3 shows that GKZ's restriction, like requiring dominators to be undominated, has no impact on maximal reductions in CD games. Therefore, it solves none of the problems of IESDS in CD games. In other games, ($\rightarrow$)-reductions and ($\Rightarrow$)-reductions can produce different maximal reductions. For instance, in Example 5, the problematic ($\rightarrow$)-reduction would not be permitted as a ($\Rightarrow$)-reduction. Nevertheless, IESDS based on ($\Rightarrow$)-reductions does not escape the problem of order dependence. To see this, consider the following infinite sequence of ($\Rightarrow$)-reduced strategy sets in Example 5: $(0, 1), [x, 1], \{x\} \cup [1 - (1-x)/2, 1), \{x\} \cup [1 - (1-x)/3, 1), \{x\} \cup [1 - (1-x)/4, 1), \ldots$. For any choice of $x \in (0, 1)$, the limit of the intersections of these sets, $\{x\}$, is the strategy set corresponding to a maximal $(\Rightarrow^*)$-reduction. Order matters. Similarly one may show that, for IESDS based on $(\Rightarrow)$-reductions, order matters in the games of Examples 1 and 4.[10]

6.2. *Mixed Strategy Dominance*

In our framework a (pure) strategy is strictly dominated in a finite game if there exists a (pure) strategy that generates a higher payoff against any opponent strategy profile. In contrast, some textbooks define a (pure) strategy to be strictly dominated in a finite game if there exists a *mixed* strategy that generates a higher expected payoff against any opponent strategy profile. Osborne and Rubinstein (1994) (OR) prove that the corresponding formulation of IESDS is order independent in finite games; a strategy survives if and only if it is rationalizable.[11] They credit this result to David Pearce, because it builds on Pearce's (1984) Lemma 3.

Our formulation of IESDS is more conservative than OR's, as our last example indicates:

EXAMPLE 9:

	δ	ε
α	1, 1	0, 0
β	0, 1	1, 0
γ	0, 0	0, 0

In this finite game all strategies survive IESDS as defined here, while only α and δ survive by OR's definition. We can, however, generate OR's reduction by applying IESDS to the game's mixed extension. Our Theorem 1 shows, as a corollary, that IESDS is well-behaved when applied to the mixed extensions of finite games.

6.3. *Börgers Dominance*

Börgers (1993) defines another version of dominance, which, like OR's version, expands the set of what is dominated. A strategy x is B-dominated, meaning dominated in the sense of Börgers, if it is weakly dominated when opponents' play is restricted to arbitrary nonempty subsets of their strategy sets. If one requires

[10] If one returns to GKZ's original definition, which requires maximal $(\Rightarrow^*)$-reductions to end in a finite number of steps, then it is not possible to get multiple maximal $(\Rightarrow^*)$-reductions in our examples of order dependence, because none of those games has a maximal $(\Rightarrow^*)$-reduction in finite steps. For this reason, and examples such as the Cournot game, the restriction to finite steps seems too strong.

[11] See their Proposition 61.2 and the subsequent comment on p. 62. For this result, OR expand the set of rationalizable strategies to include best responses to opponents' correlated play, unlike in the original notion of rationalizability (Bernheim (1984), Pearce (1984)), which requires beliefs to have a product structure and leads to sharper predictions in games having more than two players.

dominating strategies to be pure, then B-dominance is sufficient for weak dominance and necessary for strict dominance; in this case, Börgers (1989) shows that iterated B-dominance is order-independent in finite games. Alternatively, if dominating strategies can be mixed, then Börgers (1989) shows that B-dominance is equivalent to strict dominance, that is, to dominance as defined by OR, and OR show that the procedure is order-independent in finite games.

The idea behind B-dominance is that if players' preferences over game outcomes are ordinal rather than cardinal, then it is appropriate to call a strategy "rational" if it maximizes expected utility given some subjective beliefs over opponents' play and some von Neumann-Morgenstern utility function that is consistent with those ordinal preferences. Börgers shows that a strategy is rational by this criterion if and only if it is not B-dominated, with dominating strategies required to be pure. It would be interesting (but beyond the scope of this paper) to discover whether this equivalence extends beyond finite games, and if so, what conditions are sufficient to ensure that B-dominance is order-independent. It is clear, at least, that B-dominance often produces different maximal reductions than does the standard strict dominance relationship studied in this paper, in both finite and infinite games.

7. CONCLUDING REMARKS

Many textbooks do not recommend iterated elimination of *weakly* dominated strategies (IEWDS) as a solution concept, and one reason is that order matters for that procedure in some games. We have shown that the same criticism applies to IESDS. In Examples 1, 4, 5, and 8, IESDS is order-dependent and introduces spurious Nash equilibria. Such perverse outcomes can arise from discontinuous payoffs or from strategy sets that fail to be closed or bounded, and they can occur in games that are closed under dominance.

For IEWDS, the possibility of order dependence has prompted researchers to search for classes of games for which order independence holds, on the presumption that IEWDS may be appropriate for such games (see, for example, Gretlein (1983) and Marx and Swinkels (1997)). Our positive results provide analogous consolation concerning the usefulness of IESDS. Our Theorem 2 shows that well-defined best replies are sufficient to eliminate the problem of spurious Nash equilibria seen in Examples 1, 4, 5, and 8. Example 7 shows that more is needed to ensure order independence; Theorem 1(a) shows that, if strategy sets are compact and payoffs are uppersemicontinuous in own strategies, then the maximal reduction is unique whenever a nonempty maximal reduction exists. To ensure the existence of such a reduction, several examples have shown that yet more is required. Examples 2, 3, and 6 satisfy the assumptions of Theorem 1(a); yet in Examples 3 and 6 no maximal reduction exists (in Example 3 because the strategy set is infinite, and in Example 6 because the player set is infinite) and in Example 2 the unique maximal reduction has empty strategy sets. To rule out all of these possibilities, Theorem 1(b) shows that it is enough to assume that payoffs are continuous in opponents' as well as own strategies. If strategy sets are compact and payoffs are continuous, then IESDS always produces a unique maximal reduction that is nonempty and introduces no spurious Nash equilibria.

The existence and uniqueness of maximal reductions is important even when players are unsophisticated. The standard justification for IESDS, in one-shot games, is that it is common knowledge that the players will perform the required computations. In a repeated setting, however, much less sophisticated players may learn (or evolve) not to play strictly dominated strategies. The sequential disappearance of such strategies may eventually confine play to the maximally reduced game, and this convergence may be more robust than convergence to a Nash equilibrium.

The proper definition and role of iterated strict dominance is unclear for games that are not compact and continuous. Example 5 shows that there are games for which the concept is intrinsically unsound. The identification of general classes of games for which IESDS is an attractive procedure, outside of the compact and continuous class, remains an open problem.

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