MATH499H Summary
Topic: Calculus of Variations
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A functional has a set of functions as its domain, and a set of real numbers as its range. For example, $G[y] = \int_0^1 y(x)dx$ is a functional. Note that the functionals we will examine here will be of the form$\int_a^b I\,dx$, where I is the integrand. Some of these have ranges that are bounded, and the ones examined here will typically have a minimum or maximum that we are interested in finding, such as $F[y] = \int_{-1}^2 y^2 - 2xy(x)dx$, and we will assume that they are continuous.

Next, we will examine the difference quotient for a functional. As one might expect, the difference quotient for a functional is:

$$\frac{F[y+\Delta y] - F[y]}{\Delta y}$$

Keeping in mind that y is a function of x, we evaluate F[y+Δy]:

$$\int_{-1}^{2} (y+\Delta y)^2 - 2x(y+\Delta y)dx = \int_{-1}^{2} y^2 - 2xy + \Delta y(2y-2x) + \Delta y^2\ dx$$

Now, this will be at an extremum when the coefficient of the first order of Δy is zero, giving the function y(x)=x. Since the second-order Δy term is always positive, y(x)=x gives a minimum.

We will call the first-order coefficient the 'functional derivative,' and it can be derived that for integral-form functionals whose integrand depends only on x, y(x), and y'(x), the functional derivative, where I is the integrand of the functional, is:

$$\frac{\delta F}{\delta y} = \frac{\partial I}{\partial y} - \frac{d}{dx}\frac{\partial I}{\partial y'}$$

When this is set equal to zero, we get the Euler-Lagrange equation:

$$\frac{\partial I}{\partial y} - \frac{d}{dx}\frac{\partial I}{\partial y'} = 0$$

However, sometimes these problems are unsolvable, such as:

$$K[y] = \int_{-3}^{4} y^2 - 2xy + x^5 y' + (y')^3$$

for which the Euler-Lagrange equation yields the nonlinear second-order ODE:

$$2y - 2x - 5x^4 - 6y'y'' = 0$$

There are several applications of the Euler-Lagrange equation. One simple example is the problem of finding the shortest path between two points. This starts with the arc length integral:

$$L[y] = \int_a^b \sqrt{1+(y')^2}\,dx$$

Evaluating the functional derivative of this integral provides the result of:

$$\frac{\delta L}{\delta y} = \frac{\partial I}{\partial y} - \frac{d}{dx}\frac{\partial I}{\partial y'} = 0 - \frac{d}{dx}\frac{\partial}{\partial y'}(1+(y')^2) = -\frac{d}{dx}[y'(1+y'^2)^{-\frac{1}{2}}] =$$

$$-[y'\frac{d}{dx}(1+y'^2)^{-\frac{1}{2}} + (1+y'^2)^{-\frac{1}{2}}\frac{d}{dx}(y')] =$$
$$-y' * -\frac{1}{2}\left(1+y'^2\right)^{-\frac{3}{2}} * 2y'y'' - \left(1+y'^2\right)^{-\frac{1}{2}} * y'' = y'^2y''\left(1+y'^2\right)^{-\frac{3}{2}} - y''\left(1+y'^2\right)^{-1/2} =$$
$$y''\left[\frac{y'^2}{\left(1+y'^2\right)^{\frac{3}{2}}} - \frac{1}{\left(1+y'^2\right)^{\frac{1}{2}}}\right] = y''\left[\frac{y'^2}{\left(1+y'^2\right)^{\frac{3}{2}}} - \frac{\left(1+y'^2\right)}{\left(1+y'^2\right)^{\frac{3}{2}}}\right] = -y''\left(1+y'^2\right)^{-\frac{3}{2}}$$

Now, using the Euler-Lagrange equation, the functional is at an extremum when this quantity is equal to zero (for all x). Note that this occurs when $y'' = 0$, or when y(x) is linear. Therefore, we have shown that the shortest distance between two points is a straight line, which does not come as a surprise but does give us some confidence in this method.

However, for more complex examples, it is often useful to use the energy function. In general, for integral-form functional dependent on y and y', but not x, there is an 'energy' for which the derivative with respect to x is zero at the extrema. As it turns out, this is the energy function:

$$E(x) = y'\frac{\partial I}{\partial y'} - I$$

We can use this to find the surface of revolution of smallest area between two rings (also known as the soap-bubble problem). First of all, the functional for this surface, from (x,r) points (-1,2) to (1,3) is the following:

$$S[r] = 2\pi\int_{-1}^{1} r\sqrt{(1+r'^2)}\ dx$$

Also, the functional derivative for this is:

$$\frac{\delta S}{\delta y} = \frac{\partial I}{\partial y} - \frac{d}{dx}\frac{\partial I}{\partial y'} = 2\pi\sqrt{\left(1+r'^2\right)} - \frac{d}{dx}\left[\frac{2\pi r r'}{\left(1+r'^2\right)^{\frac{1}{2}}}\right] =$$
$$2\pi\sqrt{(1+r'^2)} - \frac{2\pi r r'' + 2\pi(r')^2[1+(r')^2]}{[1+(r')^2]^{3/2}} = 2\pi\frac{(1+r'^2)^2 - rr'' - (r')^2[1+(r')^2]}{[1+(r')^2]^{3/2}}$$

Applying the Euler-Lagrange equation to this yields:

$$(1+r'^2)^2 - rr'' - (r')^2[1+(r')^2] = 0$$

and this reduces to the second-order nonlinear ODE $1 + r'^2 - rr'' = 0$.
On the other hand, considering the energy function above gives:

$$E(x) = r'\frac{\partial I}{\partial r'} - I = r'\left[\frac{rr'}{\sqrt{1+r'^2}}\right] - r\sqrt{1+r'^2} = \frac{r(r')^2 - r(1+r'^2)}{\sqrt{1+r'^2}}$$
$$= \frac{r(r')^2 - r(1+r'^2)}{\sqrt{1+r'^2}} = \frac{r(r')^2 - r - r(r')^2}{\sqrt{1+r'^2}} = \frac{-r}{\sqrt{1+r'^2}}$$

We then evaluate the derivative of this with respect to x:

$$\frac{dE}{dx} = -\frac{d}{dx}\left[r\left(1+r'^2\right)^{\frac{-1}{2}}\right] = -\left[r'\left(1+r'^2\right)^{\frac{-1}{2}} - \frac{1}{2}r\left(1+r'^2\right)^{-\frac{3}{2}} * 2r'r''\right] =$$

$$\frac{-r'}{\left(1+r'^2\right)^{\frac{1}{2}}}+\frac{rr'r''}{\left(1+r'^2\right)^{\frac{3}{2}}}=\frac{rr'r''-r'(1+r'^2)}{\left(1+r'^2\right)^{\frac{3}{2}}}=\frac{rr'r''-r'-(r')^3}{\left(1+r'^2\right)^{\frac{3}{2}}}$$

When this is set equal to zero, we get:

$$rr'r''-r'-(r')^3=0 => r'(rr''-1-r'^2)=0$$

Multiplying through by -1 and taking $r' \neq 0$, this yields the same thing as the Euler-Lagrange equation. Therefore, the energy function must be a constant:

$$E=\frac{-r}{\sqrt{1+r'^2}} => \frac{-r}{E}=\sqrt{1+r'^2} => \frac{r^2}{E^2}=1+r'^2 => r' = \pm\sqrt{\frac{r^2}{E^2}-1}$$

So, this forms a first-order separable ODE. Considering only the positive r', and using the substitution $\cosh u=\frac{r}{E}$, we can determine that the solution to this problem is a hyperbolic cosine curve.

There are also more complex problems that we can examine, such as geodesics. Geodesics are 2D surfaces in 3D space, such as the surface of a sphere. Recall the equations for transforming Cartesian coordinates to spherical coordinates:

$$x=Rsin\varphi cos\theta \quad y=Rsin\varphi sin\theta \quad z=Rcos\varphi$$

On a geodesic sphere, R is fixed. The change of the other two variables can be found as follows:

$$\Delta x=R[\sin(\varphi+\Delta\varphi)\cos(\theta+\Delta\theta)-\mathrm{sin\varphi cos\theta}]$$
$$\Delta y=R[\sin(\varphi+\Delta\varphi)\sin(\theta+\Delta\theta)-\mathrm{sin\varphi sin\theta}]$$
$$\Delta z=R[\cos(\varphi+\Delta\varphi)-\mathrm{sin\varphi}]$$

Since distance $d^2=(\Delta x)^2+(\Delta y)^2+(\Delta z)^2$, we can convert to φ and θ, and ultimately show:

$$d^2=R^2\Delta\varphi^2+R^2\Delta\theta^2 sin^2\varphi$$

This can be used to produce the functional for the distance between two points on a geodesic sphere using the arc length formula:

$$\int_a^b\sqrt{R^2(d\varphi)^2+R^2sin^2\varphi(d\theta)^2}=R\int_a^b\sqrt{\left(\frac{d\varphi}{d\theta}\right)^2+sin^2\varphi}\;d\theta$$

This gives a functional dependent of $\varphi(\theta)$ which can be examined through the methods developed earlier.

Additionally, with the use of the functional version of Lagrange multipliers, we can solve the isoperimetric problem (i.e. the shape that encloses the most area for a given perimeter, or has the least perimeter for a given area). To start, the functional for area and perimeter are:

$$A[y]=\int_a^b y\,dx \quad \text{and} \quad L[y]=\int_a^b\sqrt{1+(y')^2}dx$$

According to the functional version of Lagrange multipliers, the way to accomplish our goal is to extremize $A[y]-\lambda L[y]$ or $L[y]-\lambda A[y]$. The first will have a maximum and the second will have a minimum at the same point. We will maximize the first (call it F[y]) by the method of the energy function:

$$E(x)=y'\frac{\partial I}{\partial y'}-I=y'\frac{\partial}{\partial y'}(y-\lambda\sqrt{1+(y')^2})-(y-\lambda\sqrt{1+(y')^2})=$$

$$y'\left[-\lambda\frac{1}{2}(1+(y')^2)^{-1/2}*2y'\right]-y+\lambda\sqrt{1+(y')^2}=\frac{-\lambda(y')^2-y\sqrt{1+(y')^2}+\lambda(1+(y')^2)}{\sqrt{1+(y')^2}}$$

Finding the function y(x) for which this expression is equal to zero for all x should find this shape. Unfortunately, however, this marks the end of this independent study, and we did not have the time to do this.