Government Subsidies for Research Programs Facing "If" and "When" Uncertainty*

David Besanko† Jianjun Wu‡

Northwestern University University of Arizona

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Abstract

This paper studies the impact of R&D subsidies in a setting in which there is uncertainty not only about the timing of a breakthrough ("when" uncertainty), but also about whether a breakthrough is even possible ("if" uncertainty). This setting seems particularly applicable to firms engaged in fundamental scientific research. Our paper makes two broad contributions. First, we show that the *way* in which R&D is subsidized matters. Certain types of subsidies may crowd out private investment while other types of subsidies may stimulate private investment. Second, we show that simple subsidy mechanisms can be very effective in dealing with market failures, and under important conditions, it may be possible to implement the first-best outcome with a minimum of information. This demonstrates that even in complex dynamic settings R&D subsidies have the potential to improve social welfare. More specifically, the paper utilizes a two-armed bandit framework to model the impact of government subsidies for private R&D investment. We focus on two cases: monopoly and competitive R&D. We derive an individual firm's optimal R&D investment decision and noncooperative firms' symmetric Markov Perfect equilibrium investment strategies. If there is no shadow cost of public funding, we show that for a monopoly the first-best welfare can be attained through a pure matching subsidy that does not rely on firm beliefs about project viability. Under competition, the first-best can be attained using a combination of a (belief-free) matching subsidy and a (belief-free) unrestricted subsidy policy. The unrestricted component is needed because of the free-rider problem that arises under R&D competition. By contrast, if there is shadow cost of public funding, we show that earmark and unrestricted subsidies are never optimal for the monopoly case for a large set of parameters. In fact, numerical examples demonstrate that a pure matching policy is optimal for all cases under monopoly and when spillovers are sufficiently small under R&D competition.

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†Department of Management and Strategy, Kellogg School of Management, Northwestern University, Evanston, IL 60208. Email: d-besanko@kellogg.northwestern.edu

‡Department of Economics, University of Arizona, Tucson, AZ 85721-0108. Email: jwu@eller.arizona.edu.

1 Introduction

Governments have long played a role in subsidizing private investments in R&D. The principal economic justification for R&D subsidies is the presence of market imperfections (e.g., limits on appropriability or free-rider problems) that result in a less than socially optimal provision of private R&D (Arrow, 1962). But from the existing literature it is not clear how successful subsidies have been, or can be, in addressing these market failures. The empirical literature on R&D subsidies presents decidedly mixed results, with some studies concluding that R&D subsidies do indeed stimulate private investment in R&D (e.g., Lach, 2002 and Almus and Czarnitzki, 2003), while others find that subsidies crowd out private R&D investments (e.g., Irwin and Klenow, 1996 and Wallsten, 2000) or leave it unchanged (e.g., Klette and Moen, 1998, 1999). The theory literature (discussed in more detail below) shows that subsidies can, in principle, stimulate private R&D investment and increase social welfare, but the models in the literature are typically cast within static and/or deterministic settings that seem far removed from the dynamic and uncertain environments in which much modern research (especially basic research) takes place. It is fair to say that from both a theoretical and empirical perspective, the effectiveness of government R&D subsidies is an unsettled question, and as Hall (2005) suggests in her review of the economics of innovation financing, it deserves further research.

The purpose of this paper is to advance the theory of R&D subsidies by studying their impact in a setting in which there is uncertainty not only about the timing of a scientific or engineering breakthrough ("when" uncertainty), but also about whether a breakthrough is even possible ("if" uncertainty). Our paper makes two broad contributions. First, we show that the *way* in which R&D is subsidized matters. Certain types of subsidies (i.e., earmarked funding) may crowd out private investment, while other types of subsidies (i.e., matching subsidies) may stimulate private investment. This suggests that empirical studies of the impact of R&D subsidies on private R&D investment need to be cognizant of the subsidy mechanism. Second, we show that simple subsidy mechanisms can be very effective in dealing with market failures, and under important conditions, it may be possible to implement the first-best outcome with a minimum of information. This demonstrates that even in complex dynamic settings R&D subsidies have the potential to improve social welfare.

More specifically, our paper uses a two-armed bandit model of R&D competition in which firms seek to achieve a significant scientific breakthrough.[1] As time passes and the breakthrough is not achieved, firms become more pessimistic about the likelihood that this path of inquiry will ever pay off, and if they become sufficiently pessimistic, they will eventually terminate the project. Conditional on the project being viable, the likelihood and timing of a breakthrough depends on how persistent the firms are, i.e., how willing they are to continue to fund the project over time.

The government implements a subsidy mechanism in which a firm's R&D subsidy is a function of its actual R&D effort. The mechanism subsumes three specific funding schemes commonly used in practice: a *pure matching subsidy*, an *earmarked subsidy*, and a *pure unrestricted subsidy*. Under a matching subsidy, the government reimburses a fraction of the firm's actual R&D expenses. Under an earmarked subsidy, the government commits to a certain level of R&D funding, subject to the requirement that the firm provides a minimum mandated level of R&D effort on the focal project. Under an unrestricted subsidy, the government makes an open-ended commitment to fund the project until a breakthrough occurs. Unlike an earmarked subsidy there is no formal requirement that the firm actually spend the money on the focal R&D project.

We study the impact of this subsidy mechanism under both monopoly (by which we mean a firm that faces no R&D competition) and R&D competition. A monopolist's optimal R&D investment decision is a "bang-bang" rule: depending on its posterior beliefs about the project's viability, it either invests "flat out" in R&D at each instant in time or not at all. Relative to the case of no subsidies, the matching component of the subsidy expands the range of posterior beliefs over which the monopolist invests "flat out," while the earmarked and unrestricted components of the subsidy shrinks that range. Thus, increases in the matching rate stimulate private spending on R&D, while increases in unrestricted funding or the minimum mandated R&D effort crowd out private spending. However, with minimum mandated R&D effort, under an earmarked subsidy, R&D effort on behalf of the project continues below the point at which the firm stops investing, and thus an earmarked subsidy could potentially stimulate overall investment in R&D.

We then derive the optimal subsidy policy that maximizes *ex ante* social welfare. Perhaps unexpectedly given the ambiguous effect of different types of R&D subsidies, when a subsidy is a pure transfer

[1] Using two-armed bandit model to analyze economic problems dates back to Rothschild (1974). Recently, a number of papers focus on the strategic interaction among agents in a bandit framework (e.g., Keller, Rady, and Cripps, 2005, and Klein and Rady 2010). Our paper is closest to Keller, Rady, and Cripps (2005), as our second stage R&D competition is based on their Poisson bandit framework. In Besanko and Wu (2008), we explore R&D competition and cooperation in a model inspired by Keller, Rady, and Cripps (2005).

between taxpayers and firms, the optimal subsidy mechanism is a pure matching subsidy that induces the firm to follow the first-best R&D policy irrespective of prior beliefs about the viability of the project. This implies even if the government has no knowledge of the firm's prior belief on the feasibility of the project, it can still use the matching policy to induce the firm to conduct R&D along a socially optimal path. The information needed is minimal: the government only needs to estimate the total social benefit and the percentage of which that can be appropriated by the firm. By contrast, when there is a positive shadow cost of public funds,[2] the first-best investment policy cannot, in general, be achieved. However, through a combination of analytical sufficient conditions and calculations based on empirically plausible parameter values, we establish that there is a wide range of circumstances under which the optimal subsidy policy is a pure matching policy. Nevertheless, the matching rate is generally less than the matching rate that implements the first-best solution in the absence of a shadow costs. In some cases, depending on prior beliefs and the appropriability of the social returns from R&D, the optimal matching rate may be 0. That is, there are circumstances under which it is best not to subsidize the firm.

Under competition, an additional complication arises that does not exist under monopoly: the possibility that firms may free ride on the R&D efforts of other firms. As we show below, the free-rider problem implies that in a symmetric equilibrium, investment by firms is no longer "bang-bang." Instead, it may involve a range of posterior beliefs over which each firm invests a positive amount in R&D which is less than the technological maximum. When the free-rider problem can arise, an unrestricted component to the subsidy and a minimum mandated level of R&D — which had unambiguously adverse incentive effects under monopoly — can actually eliminate the free-rider problem. Indeed, without the unrestricted component or a minimum mandate, the free-rider problem always arises. Perhaps surprisingly, when there is a zero shadow costs of public funds, there exists a subsidy policy that induces firms in equilibrium to follow the first-best investment policy. However, unlike monopoly, that policy is not a pure matching policy. Instead, it involves a combination of a matching rate and an unrestricted component. The unrestricted component of the subsidy eliminates the free-rider problem, and given this, the matching rate is then set to mimic the social planner's optimal investment policy. As in the monopoly case, this subsidy policy is also belief independent and requires only minimal information. For

[2] The shadow cost would typically arise because the government must increase distortionary taxes to fund the subsidy. Romano (1989) uses "social cost" of subsidy to denote the sum of each subsidy-dollar and its shadow cost.

a small positive shadow costs of public funds, numerical computations show that the optimal subsidy policy continues to involve only matching and unrestricted components, but for larger (and probably more empirically plausible) shadow costs of public funds, the optimal subsidy policy reverts to a pure matching policy when the spillover is not very large.

Our paper fits within the theoretical literature on R&D subsidy policy. Papers in this literature have focused a number of broad issues. Some, such as Spencer and Brander (1983) and Qiu and Tao (1998), study the use of R&D subsidies to enhance national competitiveness. Other papers consider the role of subsidies to help overcome informational problems. For example, Socorro (2007) explores optimal patent subsidies for R&D in the context of a mechanism design problem in which firms have private information about the value of an uncertain R&D project, while Takalo and Tanayama (2010) examine whether R&D subsidies can alleviate financing constraints due to adverse selection. Most closely related to this paper are papers by Hinloopen (1997, 2000), Stenbacka and Tomback (1998), and Romano (1989) that explore the impact of subsidies on the level of R&D and social welfare. Hinloopen (1997) analyzes a model similar to the framework of d'Aspremont and Jacquemin (1988) and Kamien et. al. (1992) in which firms' investments in cost-reducing effort deterministically reduces their costs, and possibly the costs of other firms as well due to spillovers. R&D subsidies are shown to increase the level of investment activity and social welfare and are more effective at increasing R&D investment than allowing firms to cooperate through research joint ventures or R&D cartels. Stenbacka and Tomback (1998) analyze the best way to organize R&D (e.g., competitive research joint venture, cartelized research joint venture, R&D competition) given that the government chooses an optimal subsidy rate for the mode of organization being considered. With optimal subsidies, research joint ventures are shown to be superior to competition provided that the social cost of subsidies is not too large. Romano (1989) analyzes subsidies for research projects aimed at achieving process innovations in the presence of "when" uncertainty. He shows that it is always socially optimal to subsidize a monopolist, but under certain circumstances (e.g., sufficiently long patent life) it is not optimal to subsidize competitive firms.

Our paper differs from the existing theoretical literature in several important respects. First, unlike the existing literature that analyze R&D subsidies in reduced-form static or two-stage models, our model is explicitly dynamic. By employing the two-armed bandit framework, we can analyze how alternative subsidy policies affect belief updating and the abandonment of R&D projects, issues that cannot be studied in static or two-stage models. Second, we consider more general subsidy policies than those considered in the existing literature. Hinloopen (1997, 2000) and Stenbacka and Tomback (1998) consider pure matching subsidies, while Romano considers unrestricted subsidies. Our paper analyzes a more general subsidy mechanism that embraces both matching and unrestricted subsidies as special cases as well. Finally, in contrast to many of the papers cited above, a key focus of our paper is on the properties of an optimal subsidy policy and how that policy is affected by underlying economic fundamentals.

Our paper is also related to several papers in the broader literature on the financing of innovative activity, in particular Bergemann and Hege (1998, 2005) and Hörner and Samuelson (2009). These papers, like ours, study R&D projects that are characterized by both "if" and "when" uncertainty. The main focus of these papers is to explore the hidden action and/or hidden information problems in a context of venture capital financing. Although it is critical to understand the problem caused by asymmetric information in R&D experimentation, it is equally important to understand the economics of R&D subsidies. By assuming away the asymmetric information, our paper complements the current literature on funding experimentation by including a number of new important features. First, unlike a venture capitalist who provides exclusive funding, we allow firms to use their own funding to pursue R&D while receiving financial support from the government. Second, we address the appropriability and free-riding problems simultaneouly with a funding scheme that includes matching, earmarked, and unrestricted subsidies as special cases, while the current literature largely restricts the venture capitalist's financial support to unrestricted funding only. Third, we study the funding policy for multiple firms while the Bergemann and Hege and Hörner and Samuelson papers focus on the one-firm case. Finally, their models assume no friction in venture captial funding, while we consider cases with and without frictions in terms of a shadow cost. We are able to show that the optimal R&D subsidy policy is simple and belief-free if there is no friction of public funding and is still simple (but may be belief-dependent) if shadow cost of public funding is positive.

The paper is divided into five sections. Section 2 lays out the model. Section 3 considers the case of monopoly, while Section 4 consider the N-firm case. Section 5 explores the possibility that firms can form a research joint venture. Section 6 summarizes and concludes.

2 The Model

We present a model of R&D investment based on the exponential bandit framework of Keller, Rady, and Cripps (2005). We state the model with N firms, with the analysis of monopoly corresponding to the special case of $N = 1$.

Each of the N firms faces an opportunity to invest in an R&D program aimed at achieving a significant breakthrough. *Ex ante* the firms do not know if a breakthrough is possible. Let p_0 denote the firms' common prior that the project is viable, i.e., that the breakthrough will eventually be achieved by some firm. Conditional on the project being viable, the time the breakthrough occurs is random. Higher R&D investment increases the likelihood that the breakthrough occurs sooner. Specifically, let k_t^i denote firm i's R&D effort at time t. Conditional on the project being viable, the hazard rate of success on the R&D project is $\lambda k_t^i dt$, where $\lambda > 0$ is a parameter. We assume that each firm faces a technological constraint that limits its investment in R&D to at most 1 unit of effort at any point in time t. Thus, $k_t^i \in [0, 1]$. One can interpret this constraint as an extreme form of diminishing marginal returns to R&D. If the project was indeed viable, and a firm exerted the maximum feasible level of R&D effort ($k = 1$), then $\frac{1}{\lambda}$ would be the expected time until a breakthrough occurs.

R&D effort is costly, and the total cost $C(k_t^i)$ of R&D effort is assumed to be an identical linear function for each firm, $C\left(k_t^i\right) = \alpha k_t^i$, where $\alpha > 0$ denotes the marginal cost of R&D effort.[3] This cost could either be a direct cost of R&D effort, or an opportunity cost of redeploying scarce internal resources to support the focal R&D effort.

The achievement of a breakthrough is assumed to be "big news" and visible to all firms competing in the R&D race. As time passes and a breakthrough has not occurred, firms become more pessimistic about the viability of the project. Let $p(t)$ denote firms' posterior belief about the project's viability at date t. If no breakthrough occurs, $p(t)$ adjusts downward according to Bayes rule:

$$p(t + dt) = \frac{p(t)\left(1 - \lambda \sum_{i=1}^{N} k_t^i dt\right)}{1 - p(t) + p(t)\left(1 - \lambda \sum_{i=1}^{N} k_t^i dt\right)}. \tag{1}$$

[3]The linearity of the cost function is needed to solve for the equilibrium investment level in closed form. The basic intuition underlying the results does not depend on the linearity of the cost function.

It follows that

$$\frac{dp}{dt} = \lim_{dt \to 0} \frac{p(t+dt) - p(t)}{dt} = -\lambda \sum_{i=1}^{N} k_t^i p(t)(1 - p(t)). \tag{2}$$

This rate of belief updating is independent of its starting state, so we may rewrite it as

$$dp = -\lambda \sum_{i=1}^{N} k_t^i p (1-p) \, dt. \tag{3}$$

The solution concept is Markov Perfect Equilibrium, with each firm's common posterior belief p being the payoff-relevant state variable and equation (3) representing the law of motion for the state variable. Investment behavior and firm value functions are thus conditioned on p.

We assume the firm that wins the R&D race earns a payoff $\Pi > 0$. This payoff is the present value of the winning firm's profits, which are assumed to be discounted at a rate r. Each of the $N-1$ non-winning firms is assumed to receive a payoff $\theta\Pi$, where $\theta \in [0,1]$. If $\theta = 0$, the R&D race is winner-take-all; if $\theta > 0$, the breakthrough has positive spillovers. The discounted present value of the social benefit from the new technology is given by $CS + \Pi + (N-1)\theta\Pi$, where CS is the benefit of the breakthrough to consumers that the discovering and non-discovering firms cannot capture.[4] For later use, let $\rho \equiv \frac{\Pi + (N-1)\theta\Pi}{CS + \Pi + (N-1)\theta\Pi} \in (0,1]$ be the appropriability ratio, i.e., the share of the total benefit captured by firms and thus $1-\rho$ is the share of the total benefit that accrues to consumers. There are thus two potential justifications for government intervention in the market for R&D: the presence of R&D spillovers across firms (when $\theta > 0$) and the imperfect appropriability of the benefit from the breakthrough (when $\rho < 1$). Throughout the analysis, we assume that $\frac{\lambda[CS + \Pi + (N-1)\theta\Pi]}{\alpha} > 1$, which implies that the social benefit-cost ratio of a viable R&D project exceeds 1.[5]

The government's R&D subsidy policy is represented by an array of three instruments (z, s, ϕ) that form a schedule $S(k_t^i | z, s, \phi)$ that determines the funding flow a firm receives at each instant in time

[4] Throughout the analysis, we assume that Π, CS, and θ do not depend on N. In other words, we assume that the structural conditions that determine post-breakthrough profit, consumer surplus, and spillovers are independent of the number of firms engaged in competition to achieve the breakthrough itself.

[5] Let $\widetilde{T}$ be the random time to discovery for a project that is certain to be viable. With hazard rate λ and flat-out investment at any point in time, $\widetilde{T}$ is an exponential random variable with parameter λ. The *ex ante* expected social benefit of a viable R&D project would be $[CS + \Pi + (N-1)\theta\Pi] \, E(e^{-r\widetilde{T}})$, which equals $\frac{\lambda[CS+\Pi+(N-1)\theta\Pi]}{\lambda + r}$. The *ex ante* expected cost of a viable R&D project would be $\int_0^\infty \alpha \left(\int_0^t e^{-r\tau} d\tau \right) \lambda e^{-\lambda t} dt$ which can be shown to equal $\frac{\alpha}{r+\lambda}$. The *ex ante* benefit cost ratio is thus $\frac{\lambda[CS+\Pi+(N-1)\theta\Pi]}{\alpha}$.

prior to a breakthrough:

$$S(k_t^i|z, s, \phi) = \begin{cases} s + \phi\alpha(k_t^i - z) & \text{if } k_t^i \geq z, \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

In this schedule:

- $z \in [0, 1]$ is the minimum R&D effort mandated by the government at each instant in time in order for the firm to be eligible for any funding, and thus αz is the minimum mandated spending on R&D.

- $s \in [\alpha z, \infty)$ is the baseline amount of funding the firm receives, provided it satisfies the mandate. We require that $s \geq \alpha z$ so that at any point in time the firm would prefer to adhere to the mandate and accept the associated funding, rather than rejecting it. If $s = \alpha z$, the government exactly reimburses the firm its mandated R&D spending, while if $s > \alpha z$, the firm receives a subsidy in excess of its minimum mandated R&D expenditure. In this latter case, the firm could (in principle) spend some of its government funds on other activities besides the focal R&D project (e.g., it could fund other R&D projects). We thus refer to $s - \alpha z$ as the *unrestricted component* of its R&D subsidy.

- $\phi \in [0, 1]$ is the matching rate: the additional funding the firm receives for every additional dollar of R&D spending undertaken above the mandated level.

We assume throughout that the government does not know the firms' prior belief p_0 and thus cannot infer the posterior belief $p(t)$. This rules out the possibility that the government can write a fine-tuned "forcing contract" in which it ties the parameters of the subsidy function to the posterior belief p in such a way that it replicates the first-best investment policy.

The policy instruments (z, s, ϕ) embrace three interesting special cases:

- If $z = 0$, $s = 0$, and $\phi \in (0, 1]$, then a firm receives a *pure matching subsidy*: for every αk dollars of R&D investment, the government "matches" the firm's R&D spending by providing a subsidy of $\phi\alpha k$.

- If $z > 0$, $s = \alpha z$, and $\phi = 0$, then a firm receives an *earmarked subsidy*: it receives a subsidy of αz dollars, provided that its R&D effort satisfies the mandate z.

- If $z = 0$, $s > 0$, and $\phi = 0$, then a firm receives a *pure unrestricted subsidy*: it receives a no-strings-attached grant of s.

Note that we focus exclusively on *ex ante* subsidies that are paid during the research phase of the project. We do not consider *ex post* subsidies that are contingent on the success of the project (sometimes called patent subsidies; see Socorro, 2007).[6] Further, we restrict attention to subsidy policies that are both time invariant (i.e., (z, s, ϕ) are independent of t) and common to all firms in the industry. Focusing on such policies is not only useful for building intuition about the impact of alternative subsidy instruments on firms' incentives, but as we show below, restricting attention to such policies may entail no loss of generality since, under important circumstances, they are powerful enough to attain the first-best outcome.[7]

3 Monopoly R&D

This section considers the case of a single firm engaged in the search to achieve the R&D breakthrough, i.e., $N = 1$. Faced with a subsidy policy (z, s, ϕ), the firm's Bellman equation is

$$V(p) = \max_{k \in [z,1]} \Big[(s - \alpha\phi z - \alpha(1-\phi)k)dt + \lambda kpdt\Pi + (1 - \lambda kpdt)e^{-rdt}V(p + dp) \Big]. \tag{5}$$

The firm's profit flow at an instant in time dt is $s - \alpha(1-\phi)k$. With probability $\lambda kpdt$, a breakthrough will take place within the interval $[t, t + dt)$, which gives the firm the prize Π from discovering the new technology. With probability $1 - \lambda kpdt$, no breakthrough occurs within the interval $[t, t + dt)$, and the

[6] We focus on *ex ante* subsidies because they are the most common way that governments support private R&D activity. In our model, patent subsidies would enable the government to achieve the first-best solution under monoply when the shadow cost of public funds is zero. Even in this case, though, they would require that the government pass along the entire consumer surplus to the winning firm. This may be possible if the government itself is the only consumer of the products created by the research, as in the case of defense-related R&D. But this could be difficult in the context of other crucial technologies such as stem cell research.

That said, *ex post* subsidies are not without economic interest. Patent subsidies would not necessarily achieve the first best under monopoly with non-zero shadow costs, nor would they attain the first best under competition, irrespective of the shadow cost. This raises the issue of whether *ex post* subsidies are generally a more effective substitute for *ex ante* subsidies or a complement to them. This question is beyond the scope of this paper, but it is one we intend to study in future work. For the purpose of the current analysis, one can interpret Π as embodying all feasible patent subsidies and CS as any residual social benefit that arises from a successful breakthrough. The special case of full patent subsidization corresponds to the special case in which $CS = 0$.

[7] A particular example of a time-varying subsidy policy would be one in which funding is cut off after a certain deadline. Bonatti and Hörner (forthcoming) analyze the role of deadlines in collaborative research, and they show how deadlines can overcome the moral hazard in teams. In the concluding section, we discuss the role that deadlines might play in our model and how our model relates to the insights developed by Bonatti and Hörner.

firm's value become $e^{-rdt}V(p+dp)$. Using standard arguments, the firm's value function can be shown to be implicitly defined by the following differential equation:[8]

$$(1+r)V(p) = s - \alpha z + \max_{k\in[z,1]} \left\{ \begin{array}{c} [\lambda kp\Pi + (1-\lambda kp)V(p)] \\ -\alpha(1-\phi)(k-z) - \lambda kp(1-p)V'(p) \end{array} \right\}. \tag{6}$$

The firm's value consists of four components:

1. $s - \alpha z$ is the flow benefit from the unrestricted component of the subsidy;

2. $\alpha(1-\phi)(k-z)$ is the firm's net-of-subsidy total cost of k units of R&D effort;

3. $[\lambda kp\Pi + (1-\lambda kp)V(p)]$ is the expected benefit from k units of R&D effort (with probability λkp the firm achieves the breakthrough and gets Π and with probability $(1-\lambda kp)$ the firm gets the continuation value $V(p)$);

4. $\lambda kp(1-p)V'(p)$ is the expectation, with k units of R&D effort, of the option value of waiting that is foregone if the firm achieves a breakthrough.

In the Appendix we prove:

Proposition 1 *Given a subsidy policy (z, s, ϕ), a monopolist's optimal R&D strategy is as follows:*[9]

If $\frac{\alpha(1-\phi)}{\lambda\left[\Pi - \frac{s-\alpha z}{r}\right]\frac{r}{(r+\lambda z)}} < 1$

$$k_1(p) = \left\{ \begin{array}{ll} 1 & \text{if } p \in (p_1, 1] \\ z & \text{if } p \in [0, p_1] \end{array} \right., \tag{7}$$

where p_1 solves

$$\lambda p_1 \left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda z}\right) - (1-\phi)\alpha = 0, \tag{8}$$

which implies

$$p_1 = \frac{\alpha(1-\phi)}{\lambda\left[\Pi - \left(\frac{s-\alpha z}{r}\right)\right]\left(\frac{r}{r+\lambda z}\right)}. \tag{9}$$

[8] The approach is to form a Taylor expansion of $V(p)$ (ignoring the higher order terms which disappear as dt goes to 0), which gives

$$V(p+dp) = V(p) + V'(p)\,dp = V(p) - \lambda kp(1-p)V'(p)\,dt.$$

Substituting this into (5), taking limits as $dt \to 0$, and simplifying yields (6).

[9] We use the subscript "1" to denote the monopoly case, and the subscript "N" to denote the N-firm case.

The firm's value function is

$$V_1(p) = \begin{cases} \frac{s-(1-\phi)\alpha-\alpha\phi z}{r} + \frac{\lambda}{r+\lambda}\left(\Pi - \left[\frac{s-(1-\phi)\alpha-\alpha\phi z}{r}\right]\right)p + B_1 p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & \text{if } p \in (p_1, 1] \\ \frac{s-\alpha z}{r} + \frac{\lambda z}{r+\lambda z}\frac{r\Pi-(s-\alpha z)}{r}p & \text{if } p \in [0, p_1] \end{cases},$$

where

$$B_1 = \frac{\lambda\alpha(1-z)(1-\phi)}{r(r+\lambda)}\left(\frac{1-p_1}{p_1}\right)^{-\frac{r}{\lambda}}.$$

If $\frac{\alpha(1-\phi)}{\lambda\left[\Pi-\frac{s-\alpha z}{r}\right]\frac{r}{(r+\lambda z)}} \geq 1,$

$$k_1(p) = z \text{ for all } p \in [0,1], \tag{10}$$

which implies $p_1 = 1$*, and the firm's value function is*

$$V_1(p) = \frac{s-\alpha z}{r} + \frac{\lambda z}{r+\lambda z}\frac{r\Pi-(s-\alpha z)}{r}p, \text{ for all } p \in [0,1].$$

According to Proposition 1, the monopolist's optimal investment decision $k_1(p)$ is a "bang-bang" rule: k either equals the minimum required level z, or the maximum feasible level 1, depending on whether the posterior belief p is greater or less than p_1. We refer to p_1 as the *abandonment threshold* because it is the point at which the firm abandons a "flat out" commitment to the R&D program. A monopolist following the optimal policy would thus behave in one of two ways. If the firm is sufficiently optimistic about the project's viability *ex ante* so that $p_0 > p_1$, the firm would begin by exerting R&D maximum effort (i.e., $k = 1$). As time passes without a breakthrough, the firm becomes more pessimistic about the viability of the project, and once its posterior falls to p_1, the firm would switch from "flat out" investment (i.e., $k = 1$), to the minimum level mandated by the government (i.e., $k = z$). As long as a breakthrough does not occur, the firm would persist with the minimum mandated level. By contrast, if the firm is sufficiently pessimistic about the viability of the project *ex ante* so that $p_0 \leq p_1$, it would simply exert the minimum mandated level z.

The abandonment threshold p_1 comes from a "marginal cost equals marginal benefit condition" (8). The marginal cost of an additional unit of R&D effort above the minimum threshold is $(1-\phi)\alpha$. The matching component of the subsidy thus reduces the firm's marginal cost. The firm's marginal benefit equals $\lambda p\left[\Pi - \frac{s-\alpha z}{r}\right]\left(\frac{r}{r+\lambda z}\right)$. The marginal benefit consists of two components: (a) the incremental increase in the likelihood of a breakthrough, λp and (b) the net prize to the firm if a breakthrough occurs, $\left[\Pi - \frac{s-\alpha z}{r}\right]\left(\frac{r}{r+\lambda z}\right)$. The firm's net prize is the present value of profits from a breakthrough, Π, minus the present value of the fungible portion of the subsidy, $\frac{s-\alpha z}{r}$, which the firm foregoes if it achieves a breakthrough. The net prize is further "deflated" by the term $\frac{r}{r+\lambda z}$ which is less than 1 when $z > 0$; as z increases the extent of this deflation increases, and the firm's marginal benefit falls.[10]

Given (9), we can immediately determine the incentive properties of the various policy instruments.

Proposition 2 *(a) Holding s and z fixed, an increase in the matching rate ϕ decreases the monopolist's abandonment threshold, thus expanding the range over which firm invests "flat out" in excess of the mandated minimum; (b) Holding ϕ and z fixed, an increase in the baseline subsidy s (which thus increases the unrestricted portion of the subsidy $s - \alpha z$) increases the monopolist's abandonment threshold, thus contracting the range over which the firm invests "flat out" in excess of the mandated minimum; (c) Holding ϕ fixed and the unrestricted portion of the subsidy $s - \alpha z$ fixed, an increase in the mandated minimum z increases the monopolist's abandonment threshold, thus contracting the range over which the firm invests "flat out" in excess of the mandated minimum.*

The unrestricted component of the subsidy is a drag on R&D incentives because by investing more heavily and accelerating the expected time to a breakthrough, the firm brings to an end more quickly the flow $s-\alpha z$ of fungible benefits. Increases in unrestricted funding magnify this negative consequence. The minimum mandate z is also a drag on R&D incentives, but for a different, reason. When $k = z$, the firm, in effect, receives a fully-funded option from the government: the government is paying for the R&D investment z, but the firm receives the benefit Π if a discovery is made. A firm faces a trade-off between accelerating the time to breakthrough at its own cost or retaining the free option from the government. A larger z increases the option value and thus reduces a firm's incentives to use its own resources for R&D.

[10] Formally, it can be shown that $\frac{r}{r+\lambda z}$ is the arc elasticity of the expected present value of dollar with respect to z. Specifically:

$$\frac{r}{r+\lambda z} = \frac{\frac{E(e^{-r\tilde{T}}|k=1)-E(e^{-r\tilde{T}}|k=z)}{E(e^{-r\tilde{T}}|k=1)}}{\frac{1-z}{1}},$$

where $\widetilde{T}$ denotes the time to discovery; $E(e^{-r\widetilde{T}}|k=1) = \int_0^\infty e^{-rt}\lambda e^{-\lambda t}dt = \frac{\lambda}{\lambda+r}$ is the expected present value of \$1 when the firm invests flat out; and $E(e^{-r\widetilde{T}}|k=z) = \int_0^\infty e^{-rt}\lambda z e^{-\lambda z t}dt = \frac{\lambda z}{\lambda z+r}$ is the expected present value of \$1 when the firm invests at level $z \in [0,1]$. These expressions arise because, conditional on the project being viable, if the investment effort is a constant k, then discovery time is an exponetial random variable with parameter λk. Thus, $\frac{r}{r+\lambda z}$ represents (approximately) the percentage change in the expected value of \$1 worth of a prize per one pecent change in R&D effort above the minimum level. When $z = 0$, this elasticity equals 1, As the mandated minimum increases, this elasticity decreases.

The insight from Proposition 2 can be reinforced by considering the special cases of a pure matching subsidy, an earmarked subsidy, and a pure unrestricted subsidy and comparing the outcome in those cases to the case in which the firm does not receive a subsidy. Using (9), when the firm does not receive a subsidy, its abandonment threshold is

$$p_1^{NO} = \frac{\alpha}{\lambda \Pi}. \tag{11}$$

The abandonment thresholds p_1^M, p_1^E, and p_1^U for a pure matching subsidy, earmarked subsidy, and unrestricted subsidy are respectively

$$p_1^M = \frac{(1-\phi)\,\alpha}{\lambda \Pi}, \tag{12}$$

$$p_1^E = \frac{\alpha}{\lambda \Pi \left(\frac{r}{r+\lambda z}\right)}.$$

$$p_1^U = \frac{\alpha}{\lambda \left[\Pi - \frac{s}{r}\right]}.$$

Proposition 3 *(a) $p_1^M < p_1^{NO}$, i.e., with the pure matching subsidy, the firm persists with the R&D project for a longer duration than it would have in the absence of a subsidy. If $p_0 \in [p_1^M, p_1^{NO})$, a pure matching subsidy induces the firm to invest in a project that a non-subsidized firm would not have. (b) $p_1^E > p_1^{NO}$, i.e., with an earmarked subsidy, the firm stops funding R&D from its own resources earlier than it would have without a subsidy. If $p_0 \in [p_1^{NO}, p_1^E]$ then net-of-subsidy R&D spending by a firm receiving an earmarked subsidy would be zero, while R&D spending by a non-subsidized firm would be positive. (c) $p_1^E > p_1^{NO}$, i.e., with a pure unrestricted subsidy, the firm persists with the R&D project for a shorter duration than it would have in the absence of a subsidy. If $p_0 \in [p_1^{NO}, p_1^U]$ a pure unrestricted subsidy induces the firm to shut down investment in a project that a non-subsidized firm would have continued to fund.*

We can now summarize the incentive properties of subsidies on R&D investment behavior. Increases in the matching rate ϕ will stimulate private R&D activity, while increases in the unrestricted portion of the subsidy $s - \alpha z$ (holding z fixed) suppresses private R&D activity. An increase in the minimum mandate z (holding $s - \alpha z$ fixed) also suppresses private R&D activity, but at the same time, it ensures that a minimum level of R&D activity will be undertaken.

3.1 Socially Optimal R&D Subsidy Policies: $N = 1$

We now consider the subsidy policy that would be chosen by a government planner seeking to maximize expected social welfare. As a benchmark, the first-best R&D investment policy is[11]

$$k^*(p) = \begin{cases} 1 & \text{if } p \in [p^*, 1] \\ 0 & \text{if } p \in [0, p^*] \end{cases},$$

where

$$p^* = \frac{\alpha}{\lambda[CS + \Pi]} < 1.$$

The first-best abandonment threshold p^* is the reciprocal of the social benefit-cost ratio $\frac{\lambda[CS+\Pi]}{\alpha}$.

However, as discussed above, we assume that the government cannot direct the firm to follow this policy by fiat or a forcing contract and must instead rely on subsidies to provide incentives to the firm. The subsidy policies considered here are thus inherently second best. Throughout the analysis, we assume that the subsidy is funded from revenues from broad-based taxes that do not materially affect the firm's incentive to invest in R&D. However, we do allow for the possibility of a shadow cost of public funds $\gamma \geq 0$. That is, a subsidy S to the firm entails a transfer of S from taxpayers plus a social cost γS, where $\gamma \geq 0$.

We begin by deriving the expression for expected social welfare induced by the firm's optimal investment rule $k_1(p)$ derived in Proposition 1. To do so, we note that if the firm invests $k_1(p)$ units in the R&D project for the time interval $[t, t + dt)$ when the belief about the project's viability is p, then the social welfare schedule $W_1(p)$ is given recursively as follows:

$$\begin{aligned} W_1(p) &= -\alpha k_1(p)dt - \gamma[s + \phi\alpha(k_1(p) - z)]\,dt \\ &\quad + \lambda k_1(p)pdt\,[CS + \Pi] + (1 - \lambda k_1(p)pdt)e^{-rdt}W_1(p + dp). \end{aligned}$$

This recursion can be transformed into the following differential equation:

$$0 = -\alpha k_1(p) - \gamma[s + \phi\alpha(k_1(p) - z)] + \lambda k_1(p)p\,(CS + \Pi) - (r + \lambda k_1(p)p)\,W(p) - \lambda k_1(p)p\,(1 - p)\,W'(p). \tag{13}$$

[11] This policy is derived as part of the proof of Proposition 4.

In the Appendix, we show that the solution to this differential equation is

$$W_1(p) = \begin{cases} -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right) + \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p + B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & p \in [p_1, 1] \\ -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right]p & p \in [0, p_1] \end{cases}, \tag{14}$$

where p_1 is given by Proposition 1, and B_W makes the welfare schedule continuous at $p = p_1$.[12] Note that if there is no shadow cost of public funding ($\gamma = 0$) and no minimum R&D mandate ($z = 0$), then for posterior beliefs less than or equal to the abandonment threshold p_1, social welfare is 0.

Ex ante social welfare, denoted by $EW_1(z, s, \phi)$, is found by evaluating $W_1(p)$ at the prior belief p_0. We express it as

$$EW_1(z, s, \phi) \equiv W(p_0) = \begin{cases} \Psi(B_W(p_1(z, s, \phi), z, s, \phi), z, s, \phi) & p_0 \in [p_1(z, s, \phi), 1] \\ -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right]p_0 & p_0 \in [0, p_1(z, s, \phi)] \end{cases} \tag{15}$$

where

$$\Psi(B_W, z, s, \phi) \equiv \left\{ \begin{array}{c} -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right) \\ +\frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p_0 \\ +B_W p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}} \end{array} \right\}. \tag{16}$$

$$p_1(z, s, \phi) \equiv \min\left\{\frac{\alpha(1-\phi)}{\lambda\left[\Pi - \left(\frac{s-\alpha z}{r}\right)\right]\left(\frac{r}{r+\lambda z}\right)}, 1\right\}. \tag{17}$$

$$B_W(p_1, z, s, \phi) \equiv \frac{\left\{ \begin{array}{c} \frac{\alpha(1-z)}{r} + \frac{\gamma\phi\alpha(1-z)}{r} - \\ \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p_1 \\ +\frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right]p_1 \end{array} \right\}}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}.$$

The social planner's problem in choosing a subsidy policy is

$$\max_{z\in[0,1], s\geq\alpha z, \phi\in[0,1]} EW_1(z, s, \phi) \tag{18}$$

Throughout, we let $(z^{**}, s^{**}, \phi^{**})$ denote the solution to this problem.

[12] Continuity of the welfare schedule follows because at $p = p_1$, the firm is indifferent between $k = z$ and $k = 1$.

When $\gamma = 0$, a subsidy is a pure transfer between taxpayers and the firm. In this case, as the following proposition shows, there is a simple way to achieve the first-best outcome: use a pure matching subsidy whose matching rate is 1 minus the appropriability ratio. No mandated minimum level of R&D is necessary, and the policy is robust to all priors.

Proposition 4 *Under monopoly, if a subsidy is a pure transfer between consumers and firms, i.e.,* $\gamma = 0$, *then the optimal subsidy policy is a pure matching subsidy, i.e.,* $z^{**} = 0$, $s^{**} = 0$, *with a matching rate* $\phi^{**} = 1 - \rho$ *(where recall* $\rho = \frac{\Pi}{\Pi + CS}$*). This policy, which is independent of the prior belief* p_0, *induces the firm to choose an investment policy* $k_1(p) = k^*(p)$ *and thus achieves the first-best level of ex ante welfare for any prior belief* p_0.

Proof. See Appendix ■

When there is a positive shadow cost of public funds, the first-best policy cannot be implemented through subsidies. This is because a positive shadow cost of public funds creates a trade-off between inducing more R&D and incurring higher social costs due to the subsidy. Still, as we show in the following sequence of propositions and numerical computations, for a wide range of circumstances, the optimal subsidy policy is a pure matching subsidy. To begin, we show that at an optimal policy, the unrestricted component of the subsidy is set to 0, i.e., $s = \alpha z$.

Proposition 5 *Suppose a subsidy entails a positive shadow cost of public funds, i.e.,* $\gamma > 0$. *If the optimal subsidy policy involves a matching rate* $\phi^{**} \in (0, 1)$, *then* $s^{**} = \alpha z^{**}$,*i.e., the subsidy policy does not involve an unrestricted component.*

Proof. See Appendix ■

Proposition 5 follows from the poor incentive properties of the unrestricted component $s - \alpha z$ of the subsidy: as we saw above, increases in $s - \alpha z$ crowd out private incentives for R&D.

Could the optimal policy involve $z > 0$? A mandate of positive z ensures that a minimum level of R&D effort is expended. While the government could induce voluntary provision of R&D effort with a large enough matching rate, that approach might be quite expensive when there is a shadow cost of public funding. Thus, a potential advantage of a minimum mandate is that it may be a less expensive way of inducing R&D effort than a matching subsidy. But the mandate is also socially costly, since $s = \alpha z$. *A priori*, then, neither subsidy tool appears to have an inherent advantage over the other.

However, our analysis suggests that a minimum mandate is never optimal. We begin by providing sufficient conditions under which the second-best optimal policy is a pure matching policy, and using those condition, we characterize the nature of that policy and the behavior that is induced.

Proposition 6 *Suppose a subsidy entails a positive shadow cost of public funds, i.e.,* $\gamma > 0$*. Suppose, further, that the optimal subsidy policy involves a matching rate* $\phi^{**} \in (0,1)$ *and is such that* $p_1^{**} \equiv p_1(z^{**}, s^{**}, \phi^{**}) < p_0$*. Then the following are sufficient conditions for* $z^{**} = 0$*: (a)* γ *sufficiently close to* 0*; (b)* $\gamma \geq \frac{\lambda(CS+\Pi)}{\alpha} - 1$*; (c)* $\gamma < \frac{\lambda(CS+\Pi)}{\alpha} - 1$ *and* $r > \lambda$*. Furthermore, under conditions (a), (b), or (c)* $p_1^{**} > p^* = \frac{\alpha}{\lambda(CS+\Pi)}$ *and* $\phi^{**} < 1 - \rho$*, i.e., the abandonment threshold is greater than the first-best abandonment threshold, and the subsidy rate is less than the subsidy rate* $1 - \rho$ *that induces the first-best outcome in the absence of a shadow cost.*

Proof. See Appendix. ■

The sufficient conditions in this proposition establish a wide set of circumstances under which the optimal subsidy policy is a pure matching policy, i.e., $z = 0$. Moreover, under these conditions, there is less subsidization than there would be if there was no shadow cost of public funds, and the R&D effort that is induced by the optimal subsidy scheme is less than first-best level. However, the proposition leaves open the question of whether there are other parameter conditions under which it is optimal for $z > 0$. To explore this question, we have done numerical calculations using empirically plausible parameter values that do not satisfy the sufficient conditions in Proposition 6 . The results are summarized in Table 1 in the Appendix. The table shows the optimal subsidy policy $(z^{**}, \phi^{**}))$ for various values of p_0 (the prior belief) and $\rho = \frac{\Pi}{CS+\Pi}$ (the appropriability ratio).[13]

As shown in Table 1, a positive value of z is never optimal, even in those cases in which $\phi = 0$.[14] Thus, the calculations provide additional circumstances under which a pure matching policy is optimal

[13] The parameter values for these calculations are: $r = 0.03$ (a real interest rate of 3 percent annually), $\lambda = 0.1$ (the expected time to discovery, conditional on the project being viable and maximum R&D effort, is 10 years.), $\alpha = 0.2$, $CS + \Pi = 15$ (upper table), or $CS + \Pi = 45$ (lower table), so that the social benefit-cost ratio $\frac{\lambda[CS+\Pi]}{\alpha}$ is 7.5 (upper table) or 22.5 (lower table) and $\gamma = 0.2$. (i.e., a \$1 subsidy entails a net shadow cost of public funds of \$0.20.)

These parameter values are empirically plausible. Sandmo (1998) suggests that a plausible estimate of the shadow cost of public funds is in the range of 0.1 to 0.2. In addition, when $\lambda = 0.1$, $\alpha = 0.2$, and $CS + \Pi = 45$, the realized social rate of return would be 46.6 percent. Jones and Williams (1998), in reviewing the empirical literature on the social rate of return to R&D, write: "Estimates of the social return average about 27 percent when only R&D from one's own industry is included and average nearly 100 percent when the broadest concept of return ... is employed" (p. 1129).

[14] The optimal policies were determined using a grid search over values of (z, ϕ) in $[0,1] \times [0,1]$. The table reports the results of calculations with a grid of 0.1. Making the grid finer does not change the result and does not uncover cases in which $z > 0$.

under monopoly, and thus strongly suggest that in the case of monopoly, a matching rate strictly dominates a minimum mandate as a policy tool for eliciting higher levels of R&D. The calculations reported here are only a subset of the calculations we have conducted, and we have yet to find an example in which the optimal $z > 0$.[15]

It may seem surprising that a minimum R&D mandate is not optimal. After all, a mandate is the most direct mechanism for achieving a minimum level of R&D effort that might not otherwise be provided. Indeed, when $z > 0$, the probability of a breakthrough is 1, conditional on the project being viable. That is, with a mandate, the social commitment to the project is absolute and will inevitably lead to a breakthrough if a breakthrough is possible. Since the monopolist firm underinvests in R&D relative to the social optimum ($p_1^{NO} > p^*$), a minimum mandate might appear to be a potentially useful policy tool to counteract this underinvestment.

However, the matching rate ϕ dominates the mandate z for two reasons. First, ϕ has superior incentive properties. As we saw above, increasing the minimum mandate z results in a crowding out of private R&D effort, while, by contrast, increasing the matching rate ϕ stimulates private R&D effort. Second, in those circumstances in which it is socially desirable to prolong the R&D effort (i.e., when $p^* \approx 0$), which are the circumstances in which a minimum R&D mandate z might potentially be most desirable, the planner can nearly replicate the outcome with a minimum R&D mandate by setting ϕ sufficiently close to 1. Thus, the advantages of earmarking can be nearly replicated by a matching subsidy.

The calculations suggest two other features of the optimal subsidy policy. First, consistent with Proposition 6, the optimal matching rate in all of the calculations is less than $1-\rho$, the matching rate that implements the first-best solution when the shadow cost of public funds is zero. For example, when $CS+\Pi = 15$ and $\rho = 0.6$, the optimal matching rates ϕ^* are 0, 0.3, 0.2, 0, and 0 for $p_0 = 0.1, 0.3, 0.5, 0.7,$ and 0.9, respectively. Second, there are circumstances under which no subsidy is optimal, i.e., $z^* = 0, \phi^* = 0$. Broadly this occurs under two sets of circumstances: (a) if the social benefit-cost ratio is relatively low ($CS + \Pi = 15$) and the prior belief p_0 about the viability of the project is low, it will not

[15] This suggests that it may be possible to prove analytically that $z > 0$ is never optimal. However, we have been unable to prove such a result. As one will see from the proof of Proposition 6, in the Apendix, the expressions for the derivatives of EW_1 with respect to z and ϕ are very complex. Ruling out the optimality of $z > 0$ using first-order conditions would require analytical characterization of a higher-order polynomial. Perhaps there is non-calculus based proof, but we have been unable to find it. The basic problem, intuitively, is that the minimum mandate z is not inherently dominated by ϕ since, as noted, by increasing z above 0, the government can always ensure some R&D and this can be socially valuable.

be worthwhile to subsidize the project; (b) if the appropriability ratio ρ is sufficiently high, it will not be worthwhile subsidizing the project. In this case, given the shadow cost of public funds, it is better to rely solely on the firm's private incentives to engage in R&D effort, which are rather strong when it can appropriate a significant share of the surplus.

4 N-Firm Oligopoly R&D

We now consider the case in which N firms compete to achieve the R&D breakthrough. As we will see, the N-firm case is not a simple multiple of one-firm case, because an individual firm will behave strategically in choosing its R&D investment strategy. Strategic behavior among firms introduces the possibility of free-riding, which in turn influences how subsidy policy shapes incentives.

4.1 Equilibrium Investment Strategy

We begin by considering a firm's optimal investment strategy given the government's R&D subsidy policy. The value function of firm i, $i = 1, .., N$, is given by the recursion

$$V^i(p) = \max_{k^i \in [z,1]} \left[\begin{array}{c} \left(s - \alpha\phi z - \alpha(1-\phi)k^i\right) dt + \lambda p k^i dt \Pi + \lambda p K^{-i} dt \theta \Pi \\ + \left(1 - \lambda p \left(k^i + K^{-i}\right) dt\right) e^{-rdt} V^i(p + dp) \end{array} \right], \tag{19}$$

where $K^{-i} = \sum_{j \neq i} k^j$ is the sum of the R&D investments by firm i's rivals. We can rewrite the value function in (19) as a differential equation:

$$(1+r)V^i(p) = s - \alpha z + \max_{k^i \in [z,1]} \left\{ \begin{array}{c} \left[\lambda k^i p \Pi + \lambda p K^{-i} \theta \Pi + (1 - \lambda\left(k^i + K^{-i}\right) p) V^i(p)\right] \\ -\alpha(1-\phi)(k-z) - \lambda\left(k^i + K^{-i}\right) p (1-p) V^{i\prime}(p) \end{array} \right\} \tag{20}$$

As in the case of $N = 1$, a firm's value has four components. Two of these components are identical to the monopoly case; the other two differ due to the presence of competitors who can also achieve the breakthrough:

1. $s - \alpha z$ is the flow benefit from the unrestricted component of its subsidy;

2. $\alpha(1-\phi)(k-z)$ is the firm's net-of-subsidy total cost of k^i units of R&D effort;

3. $\left[\lambda k^i p\Pi + \lambda p K^{-i}\theta\Pi + (1-\lambda\left(k^i+K^{-i}\right)p)V^i(p)\right]$ is the expected benefit to a firm from its own k^i units of R&D effort and its rivals' collective R&D effort K^{-i}.

4. $\lambda\left(k^i+K^{-i}\right)p(1-p)V^{i\prime}(p)$ is the expectation, with k^i+K^{-i} units of R&D effort in total, of the option value of waiting that is foregone if some firm achieves a breakthrough.

Throughout, we focus on the symmetric equilibrium, i.e., $k^1 = \ldots = k^N = k$. In the Appendix we prove:

Proposition 7 *Let*

$$p_N = \frac{\alpha(1-\phi)}{\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda N z}\right) + (N-1)(1-\theta)\Pi\left(\frac{\lambda z}{r+\lambda N z}\right)\right)}; \tag{21}$$

define

$$\overline{\theta}(z,s) \equiv 1 - \left[\frac{\Pi - \left(\frac{s-\alpha z}{r}\right)}{\Pi}\right]\left[\frac{r}{r+\lambda z}\right] \in [0,1]. \tag{22}$$

(i) If the spillover θ exceeds the critical level $\overline{\theta}(z,s)$, there exists a unique symmetric Markov perfect equilibrium with the following strategy:[16]

$$k_N(p) = \begin{cases} 1 & \text{if } p \geq q_N \\ \frac{rV_N^M(p)-s+\alpha\phi z}{(N-1)(\alpha(1-\phi)-\lambda p(1-\theta)\Pi)} & \text{if } p_N > p > q_N \\ z & \text{if } p \leq p_N \end{cases}$$

and an individual firm's value function is

$$V_N(p) = \begin{cases} \frac{s-\alpha(1-\phi)-\alpha\phi z}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha(1-\phi)+\alpha\phi z\right)}{r(r+\lambda N)}p + B_H p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}} & \text{if } p \geq q_N & (V_{N1}) \\ \frac{\lambda\Pi-(1-\phi)\alpha}{\lambda} - B_M(1-p) + \frac{(1-\phi)\alpha(1-p)}{\lambda}\ln\frac{1-p}{p} & \text{if } p_N < p < q_N & (V_{N2}) \\ \frac{s-\alpha z}{r} + \frac{\lambda N z\left[\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha z\right]}{r(r+\lambda N z)}p & \text{if } p \leq p_N & (V_{N3}) \end{cases}$$

where B_M equates (V_{N2}) and (V_{N3}) at $p = p_N$, q_N satisfies

$$\frac{r\left[\frac{\lambda\Pi-(1-\phi)\alpha}{\lambda} - B_M(1-q_N) + \frac{(1-\phi)\alpha(1-q_N)}{\lambda}\ln\frac{1-q_N}{q_N}\right] - s + \alpha\phi z}{(N-1)(\alpha(1-\phi)-\lambda q_N(1-\theta)\Pi)} = 1,$$

[16] Recall that the subscript N denotes the N-firm case.

and B_H equates (V_{N1}) and (V_{N2}) at $p = q_N$.

(ii) If the spillover θ is below the critical level $\overline{\theta}(z,s)$, there exists a unique symmetric Markov perfect equilibrium with the following strategy:

$$k_N(p) = \begin{cases} 1 & \text{if } p \geq p_N \\ z & \text{if } p \leq p_N \end{cases}$$

with the value function

$$V_N(p) = \begin{cases} \frac{s-\alpha(1-\phi)}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha(1-\phi)\right)}{r(r+\lambda N)}p + B_H^* p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}} & \text{if } p > p_N \quad (V_{N4}) \\ \frac{s-\alpha z(1-\phi)}{r} + \frac{\lambda N z\left[\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha z(1-\phi)\right]}{r(r+\lambda N z)}p & \text{if } p \leq p_N. \quad (V_{N5}) \end{cases},$$

where B_H^* equates (V_{N4}) and (V_{N5}) at $p = p_N$.

Proof. See Appendix. ■

Panel (a) of Figure 1 illustrates the equilibrium investment policy when the spillover parameter θ exceeds the critical level $\overline{\theta}(z,s) \equiv 1 - \left[\frac{\Pi-\left(\frac{s-\alpha z}{r}\right)}{\Pi}\right]\left[\frac{r}{r+\lambda z}\right]$. In contrast to monopoly, in which the equilibrium investment policy is "bang-bang," there is a range of beliefs (p_N, q_N) over which $k_N(p) \in (z,1)$. If the posterior belief falls to the slowdown threshold q_N, firms start to taper off their research efforts by reducing k below 1. If the posterior reaches the abandonment threshold p_N, a firm chooses the minimum required R&D effort, $k_N(p) = z$.

The equilibrium involves $k_N(p) \in (z,1)$ because of a free-rider problem. The free-rider problem arises because the firm can achieve a positive payoff $\theta\Pi$ from spillover even if it loses the R&D competition, a phenomenon that does not arise under monopoly. In particular, when $\theta > \overline{\theta}(z,s)$ and $p \in (p_N, q_N)$, given that all other firms invest "flat out," it will be optimal for a firm to reduce its R&D investment below the maximum level. On the other hand, though, given that all other firms invest at the minimum level, it will be optimal for a firm to invest "flat out" in R&D. The "concession" to the free-rider problem that is made in equilibrium is that for $p \in (p_N, q_N)$, all firms reduce k to a positive number less than 1. The equilibrium value of $k_N(p)$ is such that when a firm's $N-1$ competitors invest $k_N(p)$, it is indifferent in investing among all $k \in (z,1)$ and thus chooses $k_N(p)$ in a symmetric equilibrium.

By contrast, when the spillover parameter is less than $\overline{\theta}(z,s)$, the free-rider problem does not arise, and as shown in panel (b) of Figure 1, the equilibrium investment policy is "bang-bang," as in the case of a monopoly. However, the abandonment threshold p_N does not correspond to the monopoly threshold. In fact, when the free-rider problem is absent, for any given subsidy policy (z, s, ϕ) there is unambiguously more investment with N firms than with a monopolist. This is because from (9) and (21),

$$\frac{p_1}{p_N} = \frac{\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda N z}\right) + (N-1)(1-\theta)\,\Pi\left(\frac{\lambda z}{r+\lambda N z}\right)\right)}{\lambda\left[\Pi - \left(\frac{s-\alpha z}{r}\right)\right]\left(\frac{r}{r+\lambda z}\right)} > 1.$$

Thus R&D competition with no free riding increases the provision of R&D effort relative to that in a monopoly.

Because $\overline{\theta}$ depends on z and s, whether or not free riding arises in equilibrium depends on the subsidy policy. As the next proposition shows, if there is no unrestricted funding or minimum investment mandate, free riding will always arise irrespective of ϕ.

Proposition 8 *If $N > 1$, $\theta > 0$, and $z = s = 0$, then $\overline{\theta}(z, s) = 0$ and the free-rider problem always arises.*

This result tells us that a necessary condition for avoiding the free-rider problem is to establish a mandated minimum level of R&D ($z > 0$) or provide positive baseline funding ($s > 0$), or both. To understand why, recall from the discussion of monopoly that a subsidy with an unrestricted component $s - \alpha z$ creates an implicit loss for the firm when it achieves a breakthrough. By the same token, an unrestricted component creates an implicit loss when *another firm* achieves a breakthrough. Thus, the unrestricted component of the subsidy offsets part of the gain the firm receives when another firm makes the discovery, thereby reducing the temptation to free ride. A subsidy policy with a minimum mandate also creates an implicit loss for the firm when another firm achieves the breakthrough, but for a different reason. To see why note that when $k = z$, the firm is receiving an option (the possibility that it, or another firm, achieves a breakthrough) that is fully paid for by the government. When another firm achieves a breakthrough, this government-funded option goes away, creating an implicit loss that can offset some of the gains from free riding. Thus, in contrast to the monopoly case, in which increases in s and z had unambiguously adverse effects on the provision of R&D, Proposition 8 suggests that s and/or z may have potentially beneficial incentive effects by mitigating the extent of free riding behavior by firms.

Each of the policy choices affects investment incentives through the entire equilibrium strategy $k_N(p)$. These effects cannot be determined analytically, but we can determine the impact of z, s, and ϕ on the abandonment threshold p_N:

Proposition 9 *(a) Holding s and z fixed, an increase in the matching rate ϕ decreases the N-firm equilibrium abandonment threshold, thus expanding the range over which firm invests in excess of the mandated minimum; (b) Holding ϕ and z fixed, an increase in the baseline subsidy s (which thus increases the unrestricted component of the subsidy $s-\alpha z$) increases the N-firm equilibrium abandonment threshold, thus contracting the range over which the firm invests in excess of the mandated minimum.*

Proof. The result follows immediately from the expression for the abandonment threshold (21). ■

Propositions 8 and 9 hint at an interesting tension involving the baseline subsidy s. On the one hand, it can crowd out private R&D investment, by contracting the range over which the firm invests in excess of the mandated minimum. On the other hand, it can counteract the free-rider problem. In the next section, we will see that in designing an optimal subsidy scheme, a social planner can exploit this tension to balance the firm's incentives so that the first-best outcome can be attained.

To summarize the impact of subsidies in the N-firm case, if s and z are sufficiently large, the free-rider problem will not arise in equilibrium. However, changes in ϕ have no impact on free riding. As in the monopoly case, increases in ϕ decrease the abandonment threshold p_N (thus expanding the range of private funding of R&D), while decreases in s increase the abandonment threshold. Unlike the monopoly case, changes in z have an ambiguous impact on the abandonment threshold. The impact of z, s, and ϕ on the slowdown threshold q_N and the equilibrium investment policy $k_N(p)$ more generally cannot be determined analytically.

4.2 Socially Optimal R&D Subsidy Policies: $N > 1$

With N firms, the first-best R&D investment policy is a straightforward extension of the first-best solution for the monopoly case:[17]

$$k^*(p) = \begin{cases} 1 & \text{if } p \in [p^*, 1] \\ 0 & \text{if } p \in [0, p^*] \end{cases},$$

[17] It is straightforward to transform this problem to one that is identical to the social planner's problem under monopoly (discussed in the proof of Proposition 4 using the transformations $\alpha' = N\alpha$, $\lambda' = N\lambda$, and $\Pi' = \Pi + (N-1)\theta\Pi$.

where

$$p^* = \frac{\alpha}{\lambda[CS + \Pi + (N-1)\theta\Pi]} < 1.$$

As in the case of monopoly, we want to characterize the subsidy policy $(z^{**}, s^{**}, \phi^{**})$ that maximizes *ex ante* social welfare, $EW_N(z,s,\phi) = W_N(p_0)$, where $W_N(p)$ is the welfare schedule given by the recursion

$$\begin{aligned} W_N(p) &= -\alpha N k_N(p)dt - \gamma N\left[s + \phi\alpha(k_N(p) - z)\right]dt \\ &\quad + \lambda k_N(p)pdt\left[CS + \Pi + (N-1)\theta\Pi\right] + (1 - \lambda k_N(p)pdt)e^{-rdt}W_N(p + dp). \end{aligned}$$

The welfare schedule can be transformed into the following differential equation:

$$\begin{aligned} 0 &= -\alpha N k_N(p) - \gamma N\left[s + \phi\alpha(k_N(p) - z)\right] + \lambda N k_N(p)p\left[CS + (1 + (N-1)\theta)\Pi\right] \\ &\quad - (r + \lambda N k_N(p)p)W_N(p) - \lambda N k_N(p)p(1-p)W_N'(p). \end{aligned} \tag{23}$$

When $k_N(p) = 1$ (i.e., $p > q_N$), the solution to this differential equation is

$$\begin{aligned} W_N(p) &= -\frac{\alpha N}{r} - \frac{N\gamma\left[s + \alpha\phi(1-z)\right]}{r} \\ &\quad + \frac{\lambda N\left(r\left[CS + (1 + (N-1)\theta)\Pi\right] + N\alpha + N\gamma\left[s + \alpha\phi(1-z)\right]\right)}{r(r + \lambda N)}p + B_W^H p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}}, \end{aligned}$$

where B_W^H is a constant. When $k(p) = z$ (i.e., $p < p_N$), the solution to this differential equation is

$$W_N(p) = -\frac{N\alpha z}{r} - \frac{N\gamma s}{r} + \frac{\lambda N z\left(r\left[CS + (1 + (N-1)\theta)\Pi\right] + \alpha N z + N s\gamma\right)}{r(r + \lambda N z)}p.$$

When $k_N(p) \in (z, 1)$, the differential equation (23) does not have a closed form solution. However, we will be able to numerically compute the welfare schedule.

We now show that when there is no shadow cost of public funds ($\gamma = 0$), the first-best outcome can be attained by combining a matching policy with an unrestricted subsidy.

Proposition 10 *With N firms, if a subsidy is a pure transfer between consumers and firms, i.e., $\gamma = 0$, then the optimal subsidy policy is $z^{**} = 0$; $\phi^{**} = \frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$; and $s^{**} = r\theta\Pi$. This policy, which is independent of the prior belief p_0, induces each firm to choose an investment policy $k_N(p) = k^*(p)$ and thus achieves the first-best level of ex ante welfare for any prior belief p_0. If there are no spillovers*

($\theta = 0$), the optimal subsidy policy for N firms is identical to that for a monopoly.

Proof. See Appendix ■

In the absence of a shadow cost of public funds, the subsidy policy that implements the first-best solution has an intuitively appealing form. The firm receives an unrestricted subsidy s that equals the flow equivalent of the spillover benefits $\theta\Pi$ that it would have received had another firm won the R&D competition. This ensures that the only way that a firm can improve its payoff is by winning the R&D competition, thus eliminating the free-rider problem and making the firm's goal to win the competition. Though a positive value of s eliminates the free-rider problem, it does not fully align the private marginal benefit of R&D with the social marginal benefit of R&D. By choosing the matching rate ϕ to equal the fraction of social surplus $\frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$ that is not internalized, private and social marginal benefits are aligned.

As in the case of monopoly, the first-best solution cannot be implemented when there is a positive shadow cost of public funds. In this case, analytical results (even sufficient conditions) cannot be obtained, and so we illustrate the optimal subsidy policy using numerical calculations. The parameters used in those calculations are the same as those used in the monopoly calculations above, but with $N = 4$. Table 2 reports the optimal R&D subsidy policies for the case of $CS + \Pi = 15$. As in the monopoly case, there is a set of parameter values for which no subsidy is optimal. This generally occurs when p_0 is small or ρ is large. In addition, given the chosen parameters, for θ less than 0.5, it is not optimal to have either a minimum mandate or an unrestricted component to the subsidy for most cases. As a result, the optimal subsidy induces free riding.[18] For θ greater or equal to 0.5, there are a few cases involving very small minimum mandate and an unrestricted subsidy component. They are not significant enough to allow us to conclude that optimal subsidy policy requires a minimum mandate, although they suggest that when spillover is large, it may be necessary to introduce a small mandate to induce more investment.

[18] In other calculations not reported here, we find that the shadow cost of public funds must be sufficiently close to 0 before an unrestricted component of the subsidy becomes optimal. In these cases, $z = 0$ continues to be optimal.

5 R&D Consortium

An alternative arrangement to non-cooperative R&D with N competing firms is an N-firm research consortium. Faced with a subsidy policy (z, s, ϕ) the research consortium solves the following problem:

$$V(p) = \max_{k \in [z,1]} \left[(s - \alpha\phi z - \alpha(1-\phi)k)dt + \lambda N kpdt \left(\frac{\Pi + (N-1)\theta\Pi}{N} \right) + (1 - \lambda N kpdt) e^{-rdt} V(p + dp) \right].$$

This can be transformed into the following differential equation

$$rV(p) = s - \alpha\phi z + \max_{k \in [z,1]} \left\{ -\alpha(1-\phi)k + \lambda N kp \left[\left(\frac{\Pi + (N-1)\theta\Pi}{N} \right) - V(p) - (1-p)V'(p) \right] \right\}.$$

Like a monopolist, the research consortium's optimal R&D policy $k_N^C(p)$ is bang-bang:

$$k_N^C(p) = \begin{cases} 1 & \text{if } p \geq p_N^C \\ z & \text{if } p \leq p_N^C \end{cases},$$

where the abandonment threshold p_N^C is given by

$$p_N^C = \frac{\alpha(1-\phi)}{\lambda \left[\Pi + (N-1)\theta\Pi - \frac{s - \alpha z}{r} \right] \left(\frac{r}{r + \lambda z} \right)}.$$

The research consortium's optimal investment plan induces the following expected social welfare:

$$W_N^c(p) = \begin{cases} -\frac{\alpha N}{r} - \frac{N\gamma[s+\alpha\phi(1-z)]}{r} + \frac{\lambda N(r[CS+(1+(N-1)\theta)\Pi]+N\alpha+N\gamma[s+\alpha\phi(1-z)])}{r(r+\lambda N)} p + B_W^c p \left(\frac{1-p}{p} \right)^{\frac{r+\lambda N}{\lambda N}} & \text{if } p \geq p_N^C \\ -\frac{N\alpha z}{r} - \frac{N\gamma s}{r} + \frac{\lambda N z(r[CS+(1+(N-1)\theta)\Pi]+\alpha N z+N s\gamma)}{r(r+\lambda N z)} p & \text{if } p < p_N^C \end{cases}.$$

As with a monopoly, in an N-firm research consortium, the first-best level of welfare can be attained when there is no shadow cost of public funds.

Proposition 11 *Under a research consortium,*

(i)if a subsidy is a pure transfer between consumers and firms, i.e., $\gamma = 0$*, then the optimal subsidy policy is a pure matching subsidy, i.e.,* $z^{**} = 0$*,* $s^{**} = 0$*, with a matching rate* $\phi^{**} = 1 - \rho$ *(where recall* $\rho = \frac{\Pi+(N-1)\theta\Pi}{\Pi+(N-1)\theta\Pi+CS}$*). This policy, which is independent of the prior belief* p_0*, induces the the firm to choose an investment policy* $k_N^C(p) = k^*(p)$ *and thus achieves the first-best level of ex ante welfare for*

any prior belief p_0*;*

(ii) if a subsidy entails a positive shadow cost of public funds, and further, the optimal subsidy policy involves a matching rate $\phi^{**} \in (0,1)$ *and is such that* $p_N^{C*} \equiv p_N^C(z^{**}, s^{**}, \phi^{**}) < p_0$*, then the following are sufficient conditions for* $z^{**} = 0$*: (a)* γ *sufficiently close to* 0*. (b)* $\gamma \geq \frac{\lambda(CS+\Pi+(N-1)\theta\Pi)}{\alpha} - 1$*; (c)* $\gamma < \frac{\lambda(CS+\Pi+(N-1)\theta\Pi)}{\alpha} - 1$ *and* $r > \lambda$*. Furthermore, under conditions (a), (b), or (c)* $p_N^{C*} > p^* = \frac{\alpha}{\lambda(CS+\Pi+(N-1)\theta\Pi)}$ *and* $\phi^{**} < 1 - \rho$*, i.e., the abandonment threshold is greater than the first-best abandonment threshold, and the subsidy rate is less than the subsidy rate* $1-\rho$ *that induces the first-best outcome in the absence of a shadow cost.*

Proof. The proof is directly analogous to the proof of Proposition 4 in the Appendix and is thus omitted. ■

When there is a zero shadow cost of public funds, the government is indifferent between subsidizing N non-cooperative firms or N firms organized into a research consortium, provided that the matching rates and unrestricted subsidies are appropriately chosen as indicated in Propositions 10 and 11.

If there is a positive shadow cost of public funds, part (ii) of Proposition 11 implies that for a large range of parameter values, the optimal policy is a pure matching policy. The intuition is very similar to the monopoly case. This is because a research consortium coordinate its R&D efforts to achieve the maximum joint profit $\Pi + (N-1)\,\theta\Pi$, which replaces the monopoly profit Π in the monopoly case.[19]

We note from part (i) of Proposition 11 that when $\theta > 0$, the matching rate $\frac{CS}{CS+\Pi+(N-1)\theta\Pi}$ needed to attain the first-best outcome under a research consortium is less than the matching rate $\frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$ needed to attain the first-best outcome under non-cooperative research. Further, with a research consortium attaining the first-best outcome entails no unrestricted subsidy, while an unrestricted subsidy is required under non-cooperative research. Thus, when there is no shadow cost of public funds, attaining the first-best outcome with a research consortium involves a smaller overall subsidy than attaining the first-best outcome with N non-cooperative firms. This suggests that the research consortium may have an advantage over non-cooperative research when the shadow cost of public funds is positive. To determine whether *ex ante* welfare is higher under an optimally subsidized research consortium or an optimally subsidized non-cooperative firms, we turn to numerical calculations. We use the same set of parameter values we used in the N-firm calculations in the previous section. Table 3 shows two

[19] Here we assume the research corsortium can only jointly choose investment strategy and internal transfer rule. In other words, they cannot monopolize the market once a member firm succeeds.

ratios under the corresponding optimal subsidy policies: the numbers in column C are the ratios of the maximum level of expected welfare with an N-firm research consortium with respect to the first-best social welfare, and the numbers in column N are the ratios of maximum level of expected welfare with N-firm non-cooperative research with respect to the first-best social welfare. In all cases, the R&D consortium leads to higher expected social welfare.

6 Conclusions

In this paper, we employ a two-armed bandit model to study the optimal subsidy policy for research programs bearing both "if" and "when" uncertainty. Although the government has the option to design a complex subsidy scheme by combining a number of commonly used instruments—unrestricted funding, earmarked and matching subsidies, the optimal subsidy policy is surprisingly simple. If there is no shadow cost of public funding, the only information that the government needs is the social value of the project and the proportion of the value that can be appropriated by private firms. With these two pieces of information, a government with no knowledge of project viability can still devise a simple subsidy policy to achieve the first-best welfare outcome. In the case of monopoly, the optimal subsidy scheme is a pure matching policy with the matching rate equal to the ratio of the portion of social welfare that is not appropriable by the firm to total social welfare. In the case of competition, the optimal policy is a combination of two instruments: a matching subsidy and an unrestricted subsidy. The unrestricted subsidy eliminates firms' incentives to free ride and the matching subsidy ensures firms to follow the social planner's optimal R&D investment path.

The first-best welfare cannot be achieved when the shadow cost of public funding is positive. Nevertheless, the optimal policies are still simple. For the one-firm case, a pure matching policy is optimal under a wide range of conditions. For the N-firm case, numerical examples show that pure matching policy is optimal for empirically plausible cases. Overall, our results suggest that a matching policy stands out in terms of welfare maximization in a wide range of cases. The matching subsidy also has an advantage over the other pure subsidy policies in that it will not lead to unlimited funding for projects that are not viable, an outcome which is achieved by delegating the stopping decision to private firms.

There are a number of interesting issues not addressed in this paper that warrant further attention. First, the belief updating structure in our paper is rather simple. As more time passes without a discovery, the updated likelihood that the project is viable falls. The simplicity of this updating rule allows us to derive a closed-form solution to our problem. However, this rule does not allow upward revision of the viability probability. To model this, we need to allow for the possibility that firms acquire new information as the research program progresses. This would be a useful extension of our model.

Second, as noted above, our paper restricts attention to time-invariant subsidy policies. In particular, this rules out the use of funding deadlines and terminations as a way of motivating R&D investment. Bonatti and Hörner (2009) study the use of deadlines in addressing the free-riding problem in an R&D collaboration. They show that a finite deadline T can be chosen to induce the agents to contribute maximum effort throughout the process. The intuition is that because agents cannot continue the project on their own once the deadline is reached, as the deadline approaches agents will increase their effort level to race against time. They show that there is an optimal time $T' < T$ after which the agents will exert maximum effort levels. Under certain conditions, T can be chosen so that $T' \leq 0$, which implies that the agents will exert full effort throughout the collaboration process.

Our paper differs from Bonatti and Hörner in that private incentives in our model are not only affected by the free-rider problem, but also by the appropriability problem. While Bonatti and Hörner show that a deadline can neutralize the free-rider problem, it is less clear that a deadline would fully neutralize the appropriability problem. Given the potential consumer benefit from the project that cannot be appropriated by firms, a social planner may not want to set a deadline such that firms are forbidden to conduct research after the deadline expires. Further, in the context of our model, government could not completely forbid firms to conduct self-funded research, and so a strict deadline on research activity would not be feasible. Still, deadlines on subsidies may be useful, and we intend to study them in future research. Removing governmental support after certain point in time would, as in Bonatti and Hörner, generate additional incentives. However, we conjecture that in contrast to the Bonatti and Hörner model, a subsidy deadline by itself would not achieve the first-best outcome when the appropriability problem is present. More generally, though, allowing for the possibility of a time-varying subsidy mechanism may move the social planner closer to the first-best solution in those cases in which time-invariant policies cannot attain the first-best solution (i.e., when there is a shadow-cost of public funds). In such cases, it would useful to explore the interaction of deadlines and other subsidy instruments and in particular, whether a deadline is a complement to the matching rate, or a substitute for it, in generating incentives.

7 Appendix A: Figures and Tables

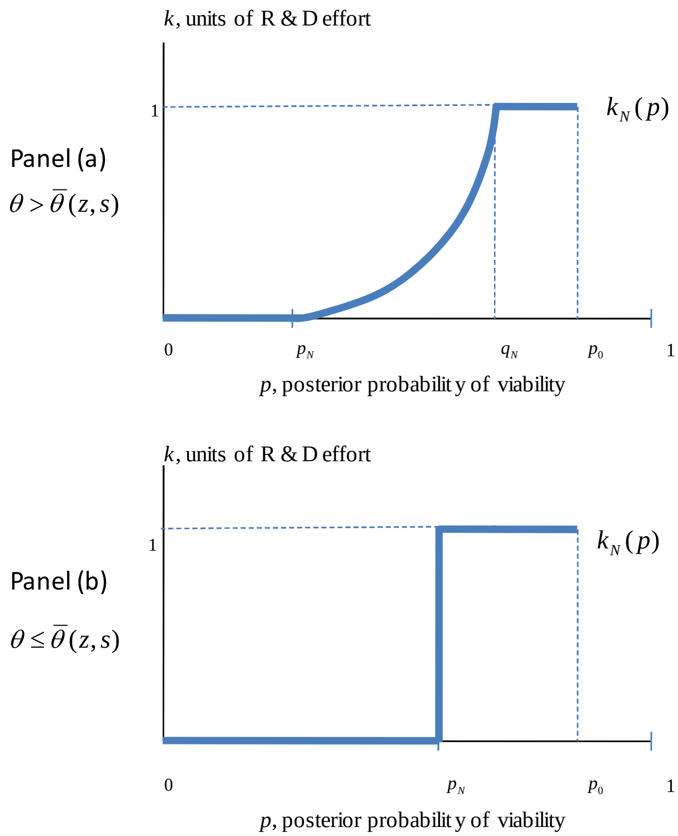

Figure 1: Equilibrium investment policy

Table 1

$CS + \Pi = 15$

ρ \ p_0	0.1	0.3	0.5	0.7	0.9
0.1	$(0, 0)$	$(0, 0.9)$	$(0, 0.9)$	$(0, 0.9)$	$(0, 0.8)$
0.2	$(0, 0)$	$(0, 0.8)$	$(0, 0.7)$	$(0, 0.7)$	$(0, 0.6)$
0.3	$(0, 0)$	$(0, 0.6)$	$(0, 0.6)$	$(0, 0.6)$	$(0, 0.3)$
0.4	$(0, 0)$	$(0, 0.5)$	$(0, 0.5)$	$(0, 0.4)$	$(0, 0)$
0.5	$(0, 0)$	$(0, 0.4)$	$(0, 0.3)$	$(0, 0.2)$	$(0, 0)$
0.6	$(0, 0)$	$(0, 0.3)$	$(0, 0.2)$	$(0, 0)$	$(0, 0)$
0.7	$(0, 0)$	$(0, 0.2)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
0.8	$(0, 0)$	$(0, 0.1)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
0.9	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$

$CS + \Pi = 45$

ρ \ p_0	0.1	0.3	0.5	0.7	0.9
0.1	$(0, 0.9)$	$(0, 0.9)$	$(0, 0.9)$	$(0, 0.9)$	$(0, 0.8)$
0.2	$(0, 0.8)$	$(0, 0.7)$	$(0, 0.7)$	$(0, 0.6)$	$(0, 0.2)$
0.3	$(0, 0.6)$	$(0, 0.6)$	$(0, 0.5)$	$(0, 0.4)$	$(0, 0)$
0.4	$(0, 0.5)$	$(0, 0.4)$	$(0, 0.3)$	$(0, 0)$	$(0, 0)$
0.5	$(0, 0.4)$	$(0, 0.3)$	$(0, 0.1)$	$(0, 0)$	$(0, 0)$
0.6	$(0, 0.3)$	$(0, 0.1)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
0.7	$(0, 0.2)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
0.8	$(0, 0.1)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
0.9	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$

Table 2: Optimal R&D Subsidy Policy for N Noncooperative Firms (N=4)

θ=0.1

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.91)	(0,0,0.91)	(0,0,0.91)	(0,0,0.90)
0.2	(0,0,0)	(0,0,0.83)	(0,0,0.82)	(0,0,0.81)	(0,0,0.77)
0.3	(0,0,0)	(0,0,0.74)	(0,0,0.73)	(0,0,0.70)	(0,0,0.61)
0.4	(0,0,0)	(0,0,0.65)	(0,0,0.63)	(0,0,0.59)	(0,0,0.43)
0.5	(0,0,0)	(0,0,0.57)	(0,0,0.54)	(0,0,0.48)	(0,0,0.22)
0.6	(0,0,0)	(0,0,0.48)	(0,0,0.44)	(0,0,0.36)	(0,0,0)
0.7	(0,0,0)	(0,0,0.40)	(0,0,0.34)	(0,0,0.23)	(0,0,0)
0.8	(0,0,0)	(0,0,0.32)	(0,0,0.24)	(0,0,0.09)	(0,0,0)
0.9	(0,0,0)	(0,0,0.23)	(0,0,0.13)	(0,0,0)	(0,0,0)

θ=0.2

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.93)	(0,0,0.93)	(0,0,0.93)	(0,0,0.92)
0.2	(0,0,0)	(0,0,0.87)	(0,0,0.86)	(0,0,0.86)	(0,0,0.83)
0.3	(0,0,0)	(0,0,0.80)	(0,0,0.79)	(0,0,0.78)	(0,0,0.73)
0.4	(0,0,0)	(0,0,0.73)	(0,0,0.72)	(0,0,0.70)	(0,0,0.60)
0.5	(0,0,0)	(0,0,0.67)	(0,0,0.65)	(0,0,0.62)	(0,0,0.46)
0.6	(0,0,0)	(0,0,0.60)	(0,0,0.57)	(0,0,0.53)	(0,0,0.30)
0.7	(0,0,0)	(0,0,0.53)	(0,0,0.51)	(0,0,0.43)	(0,0,0.13)
0.8	(0,0,0)	(0,0,0.47)	(0,0,0.42)	(0,0,0.34)	(0,0,0)
0.9	(0,0,0)	(0,0,0.41)	(0,0,0.35)	(0,0,0.23)	(0,0,0)

θ=0.3

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.95)	(0,0,0.95)	(0,0,0.94)	(0,0,0.94)
0.2	(0,0,0)	(0,0,0.89)	(0,0,0.89)	(0,0,0.89)	(0,0,0.87)
0.3	(0,0,0)	(0,0,0.84)	(0,0,0.84)	(0,0,0.83)	(0,0,0.80)
0.4	(0,0,0)	(0,0,0.79)	(0,0,0.78)	(0,0,0.76)	(0,0,0.70)
0.5	(0,0,0)	(0,0,0.74)	(0,0,0.72)	(0,0,0.70)	(0,0,0.60)
0.6	(0,0,0)	(0,0,0.68)	(0,0,0.66)	(0,0,0.63)	(0,0,0.50)
0.7	(0,0,0)	(0,0,0.63)	(0,0,0.61)	(0,0,0.57)	(0,0,0.37)
0.8	(0,0,0)	(0,0,0.58)	(0,0,0.55)	(0,0,0.50)	(0,0,0.23)
0.9	(0,0,0)	(0,0,0.52)	(0,0,0.49)	(0,0,0.42)	(0,0,0.11)

θ=0.4

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.96)	(0,0,0.96)	(0,0,0.96)	(0,0,0.95)
0.2	(0,0,0)	(0,0,0.91)	(0,0,0.91)	(0,0,0.91)	(0,0,0.90)
0.3	(0,0,0)	(0,0,0.87)	(0,0,0.87)	(0,0,0.86)	(0,0,0.84)
0.4	(0,0,0)	(0,0,0.83)	(0,0,0.82)	(0,0,0.81)	(0,0,0.78)
0.5	(0,0,0)	(0,0,0.79)	(0,0,0.78)	(0,0,0.77)	(0,0,0.70)
0.6	(0,0,0)	(0,0,0.74)	(0,0,0.73)	(0,0,0.71)	(0,0,0.62)
0.7	(0,0,0)	(0,0,0.70)	(0,0,0.69)	(0,0,0.66)	(0,0,0.54)
0.8	(0,0,0)	(0,0,0.65)	(0,0,0.64)	(0,0,0.60)	(0,0,0.45)
0.9	(0,0,0)	(0,0,0.61)	(0,0,0.58)	(0,0,0.55)	(0,0,0.34)

θ=0.5

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.97)	(0,0,0.96)	(0,0,0.96)	(0,0,0.95)
0.2	(0,0,0)	(0,0,0.93)	(0,0,0.93)	(0,0,0.93)	(0,0,0.92)
0.3	(0,0,0)	(0,0,0.89)	(0,0,0.89)	(0,0,0.89)	(0,0,0.88)
0.4	(0,0,0)	(0,0,0.86)	(0,0,0.86)	(0,0,0.85)	(0,0,0.83)
0.5	(0,0,0)	(0,0,0.83)	(0,0,0.82)	(0,0,0.81)	(0,0,0.77)
0.6	(0,0,0)	(0,0,0.79)	(0.02,0.004,0.77)	(0,0,0.76)	(0,0,0.71)
0.7	(0,0,0)	(0,0,0.74)	(0.02,0.004,0.73)	(0,0,0.72)	(0,0,0.65)
0.8	(0,0,0)	(0,0,0.71)	(0.02,0.004,0.69)	(0,0,0.69)	(0,0,0.59)
0.9	(0,0,0)	(0,0,0.69)	(0.01,0.003,0.66)	(0,0,0.64)	(0,0,0.51)

θ=0.6

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.97)	(0,0,0.97)	(0,0,0.97)	(0,0,0.97)
0.2	(0,0,0)	(0,0,0.94)	(0,0,0.94)	(0,0,0.94)	(0,0,0.94)
0.3	(0,0,0)	(0,0,0.91)	(0,0,0.91)	(0,0,0.91)	(0,0,0.90)
0.4	(0,0,0)	(0,0,0.89)	(0,0.0.88)	(0,0,0.88)	(0,0,0.87)
0.5	(0,0,0)	(0.02,0.003,0.85)	(0.02,0.004,0.85)	(0.02,0.004,0.84)	(0.02,0.004,0.82)
0.6	(0,0,0)	(0,0,0.82)	(0.02,0.004,0.82)	(0.02,0.004,0.81)	(0,0,0.78)
0.7	(0,0,0)	(0,0,0.80)	(0.02,0.003,0.79)	(0,0,0.78)	(0,0,0.73)
0.8	(0,0,0)	(0.02,0.004,0.77)	(0.02,0.004,0.75)	(0.03,0.006,0.73)	(0,0,0.68)
0.9	(0,0,0)	(0.02,0.004,0.73)	(0.02,0.004,0.73)	(0,0,0.72)	(0,0,0.64)

θ=0.7

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.97)	(0,0,0.98)	(0,0,0.98)	(0,0,0.98)
0.2	(0,0,0)	(0,0,0.96)	(0,0,0.95)	(0,0,0.95)	(0,0,0.95)
0.3	(0,0,0)	(0,0,0.93)	(0,0,0.93)	(0,0,0.93)	(0,0,0.93)
0.4	(0,0,0)	(0,0,0.91)	(0,0,0.91)	(0,0,0.90)	(0,0,0.90)
0.5	(0,0,0)	(0.02,0.004,0.88)	(0.02,0.004,0.88)	(0.02,0.004,0.88)	(0,0,0.86)
0.6	(0,0,0)	(0,0,0.84)	(0.02,0.004,0.86)	(0.02,0.004,0.85)	(0,0,0.83)
0.7	(0,0,0)	(0,0,0.82)	(0,0,0.84)	(0.02,0.005,0.82)	(0.02,0.005,0.79)
0.8	(0,0,0)	(0,0,0.82)	(0.02,0.004,0.81)	(0.02,0.005,0.80)	(0.02,0.004,0.76)
0.9	(0,0,0)	(0,0,0.77)	(0,0,0.79)	(0.02,0.004,0.77)	(0.01,0.002,0.73)

θ=0.8

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0.02,0.004,0.98)	(0,0,0.98)	(0,0,0.98)	(0,0,0.98)
0.2	(0,0,0)	(0,0,0.96)	(0,0,0.97)	(0,0,0.97)	(0,0,0.96)
0.3	(0,0,0)	(0,0,0.93)	(0.02,0.004,0.95)	(0.02,0.004,0.95)	(0,0,0.95)
0.4	(0,0,0)	(0.02,0.004,0.91)	(0.02,0.004,0.93)	(0,0,0.93)	(0.02,0.004,0.92)
0.5	(0,0,0)	(0.03,0.006,0.88)	(0.02,0.004,0.91)	(0.02,0.004,0.91)	(0,0,0.9)
0.6	(0,0,0)	(0.02,0.004,0.87)	(0.02,0.004,0.89)	(0,0,0.89)	(0,0,0.88)
0.7	(0,0,0)	(0.02,0.005,0.83)	(0.02,0.003,0.88)	(0.02,0.04,0.87)	(0,0,0.86)
0.8	(0,0,0)	(0.04,0.009,0.77)	(0.03,0.006,0.85)	(0.02,0.004,0.85)	(0.04,0.008,0.82)
0.9	(0,0,0)	(0.02,0.004,0.78)	(0,0,0.84)	(0,0,0.84)	(0,0,0.81)

θ=0.9

ρ \ p_o	0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.98)	(0,0,0.99)	(0.02,0.004,0.99)	(0,0,0.99)
0.2	(0,0,0)	(0.02,0.005,0.96)	(0,0,0.98)	(0,0,0.98)	(0,0,0.98)
0.3	(0,0,0)	(0.02,0.004,0.93)	(0,0,0.96)	(0,0,0.96)	(0,0,0.96)
0.4	(0,0,0)	(0.02,0.005,0.91)	(0.02,0.004,0.95)	(0.02,0.004,0.95)	(0,0,0.95)
0.5	(0,0,0)	(0.02,0.005,0.89)	(0.03,0.005,0.94)	(0.04,0.008,0.94)	(0.03,0.006,0.93)
0.6	(0,0,0)	(0,0,0.87)	(0.03,0.005,0.93)	(0.03,0.006,0.92)	(0,0,0.92)
0.7	(0,0,0)	(0.02,0.005,0.84)	(0,0,0.92)	(0.03,0.006,0.91)	(0,0,0.91)
0.8	(0,0,0)	(0,0,0.82)	(0,0,0.90)	(0,0,0.90)	(0,0,0.89)
0.9	(0,0,0)	(0.04,0.008,0.8)	(0,0,0.89)	(0.01,0.002,0.89)	(0,0,0.88)

Note: 1. Parameters: r=0.03, λ=0.1, α=0.2, and γ=0.2.
2.The optimal policy is presented as a vector (z,s,ϕ).

Table 3: Expected Welfare as a Percentage of First-best Social Welfare

θ=0.1

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.718	0.876	0.872	0.929	0.926	0.96	0.958
0.2	-	-	0.761	0.742	0.893	0.884	0.939	0.933	0.967	0.963
0.3	-	-	0.794	0.766	0.909	0.895	0.949	0.941	0.976	0.968
0.4	-	-	0.827	0.79	0.925	0.907	0.96	0.948	0.985	0.974
0.5	-	-	0.86	0.814	0.941	0.919	0.971	0.956	0.993	0.981
0.6	-	-	0.893	0.839	0.958	0.931	0.983	0.964	0.996	0.988
0.7	-	-	0.926	0.863	0.975	0.943	0.992	0.972	0.998	0.993
0.8	-	-	0.959	0.887	0.991	0.956	0.998	0.98	0.999	0.996
0.9	-	-	0.99	0.91	0.998	0.968	0.999	0.988	1	0.998

θ=0.2

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.707	0.876	0.868	0.929	0.924	0.96	0.957
0.2	-	-	0.761	0.725	0.893	0.877	0.939	0.93	0.967	0.96
0.3	-	-	0.794	0.744	0.909	0.886	0.949	0.935	0.976	0.964
0.4	-	-	0.827	0.762	0.925	0.895	0.96	0.941	0.985	0.969
0.5	-	-	0.86	0.781	0.941	0.904	0.971	0.947	0.993	0.973
0.6	-	-	0.893	0.8	0.958	0.913	0.983	0.953	0.996	0.978
0.7	-	-	0.926	0.817	0.975	0.923	0.992	0.959	0.998	0.983
0.8	-	-	0.959	0.837	0.991	0.931	0.998	0.945	0.999	0.988
0.9	-	-	0.99	0.855	0.998	0.941	0.999	0.971	1	0.993

θ=0.3

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.693	0.876	0.864	0.929	0.922	0.96	0.956
0.2	-	-	0.761	0.705	0.893	0.871	0.939	0.927	0.967	0.957
0.3	-	-	0.794	0.722	0.909	0.878	0.949	0.931	0.976	0.962
0.4	-	-	0.827	0.736	0.925	0.886	0.96	0.935	0.985	0.965
0.5	-	-	0.86	0.752	0.941	0.893	0.971	0.94	0.993	0.968
0.6	-	-	0.893	0.767	0.958	0.9	0.983	0.945	0.996	0.972
0.7	-	-	0.926	0.78	0.975	0.907	0.992	0.949	0.998	0.976
0.8	-	-	0.959	0.791	0.991	0.915	0.998	0.954	0.999	0.98
0.9	-	-	0.99	0.804	0.998	0.922	0.999	0.959	1	0.984

θ=0.4

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.67	0.876	0.859	0.929	0.92	0.96	0.955
0.2	-	-	0.761	0.685	0.893	0.865	0.939	0.924	0.967	0.957
0.3	-	-	0.794	0.696	0.909	0.871	0.949	0.927	0.976	0.96
0.4	-	-	0.827	0.707	0.925	0.876	0.96	0.931	0.985	0.962
0.5	-	-	0.86	0.72	0.941	0.882	0.971	0.935	0.993	0.965
0.6	-	-	0.893	0.732	0.958	0.888	0.983	0.938	0.996	0.967
0.7	-	-	0.926	0.742	0.975	0.894	0.992	0.942	0.998	0.97
0.8	-	-	0.959	0.741	0.991	0.899	0.998	0.946	0.999	0.973
0.9	-	-	0.99	0.753	0.998	0.903	0.999	0.949	1	0.976

θ=0.5

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.642	0.876	0.843	0.929	0.915	0.96	0.954
0.2	-	-	0.761	0.651	0.893	0.842	0.939	0.917	0.967	0.956
0.3	-	-	0.794	0.655	0.909	0.848	0.949	0.919	0.976	0.958
0.4	-	-	0.827	0.67	0.925	0.854	0.96	0.922	0.985	0.96
0.5	-	-	0.86	0.679	0.941	0.854	0.971	0.923	0.993	0.962
0.6	-	-	0.893	0.679	0.958	0.859	0.983	0.925	0.996	0.964
0.7	-	-	0.926	0.662	0.975	0.863	0.992	0.929	0.998	0.966
0.8	-	-	0.959	0.696	0.991	0.863	0.998	0.928	0.999	0.968
0.9	-	-	0.99	0.721	0.998	0.87	0.999	0.934	1	0.971

θ=0.6

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.589	0.876	0.843	0.929	0.915	0.96	0.954
0.2	-	-	0.761	0.581	0.893	0.842	0.939	0.917	0.967	0.955
0.3	-	-	0.794	0.604	0.909	0.848	0.949	0.919	0.976	0.956
0.4	-	-	0.827	0.621	0.925	0.854	0.96	0.922	0.985	0.958
0.5	-	-	0.86	0.608	0.941	0.854	0.971	0.923	0.993	0.959
0.6	-	-	0.893	0.626	0.958	0.859	0.983	0.925	0.996	0.961
0.7	-	-	0.926	0.633	0.975	0.863	0.992	0.929	0.998	0.963
0.8	-	-	0.959	0.632	0.991	0.863	0.998	0.928	0.999	0.965
0.9	-	-	0.99	0.624	0.998	0.87	0.999	0.934	1	0.966

θ=0.7

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.529	0.876	0.829	0.929	0.91	0.96	0.953
0.2	-	-	0.761	0.535	0.893	0.831	0.939	0.91	0.967	0.954
0.3	-	-	0.794	0.541	0.909	0.834	0.949	0.914	0.976	0.955
0.4	-	-	0.827	0.54	0.925	0.836	0.96	0.916	0.985	0.956
0.5	-	-	0.86	0.545	0.941	0.837	0.971	0.917	0.993	0.957
0.6	-	-	0.893	0.556	0.958	0.841	0.983	0.918	0.996	0.958
0.7	-	-	0.926	0.564	0.975	0.847	0.992	0.92	0.998	0.959
0.8	-	-	0.959	0.573	0.991	0.847	0.998	0.921	0.999	0.961
0.9	-	-	0.99	0.573	0.998	0.852	0.999	0.924	1	0.962

θ=0.8

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.481	0.876	0.806	0.929	0.903	0.96	0.951
0.2	-	-	0.761	0.48	0.893	0.807	0.939	0.905	0.967	0.952
0.3	-	-	0.794	0.485	0.909	0.81	0.949	0.906	0.976	0.953
0.4	-	-	0.827	0.493	0.925	0.811	0.96	0.907	0.985	0.953
0.5	-	-	0.86	0.487	0.941	0.814	0.971	0.908	0.993	0.954
0.6	-	-	0.893	0.503	0.958	0.814	0.983	0.91	0.996	0.955
0.7	-	-	0.926	0.501	0.975	0.819	0.992	0.912	0.998	0.956
0.8	-	-	0.959	0.43	0.991	0.814	0.998	0.913	0.999	0.956
0.9	-	-	0.99	0.51	0.998	0.825	0.999	0.915	1	0.958

θ=0.9

ρ	p_0=0.1 C	0.1 N	0.3 C	0.3 N	0.5 C	0.5 N	0.7 C	0.7 N	0.9 C	0.9 N
0.1	-	-	0.728	0.426	0.876	0.766	0.929	0.892	0.96	0.949
0.2	-	-	0.761	0.473	0.893	0.769	0.939	0.892	0.967	0.949
0.3	-	-	0.794	0.464	0.909	0.771	0.949	0.894	0.976	0.949
0.4	-	-	0.827	0.477	0.925	0.772	0.96	0.894	0.985	0.95
0.5	-	-	0.86	0.479	0.941	0.771	0.971	0.892	0.993	0.951
0.6	-	-	0.893	0.449	0.958	0.772	0.983	0.895	0.996	0.951
0.7	-	-	0.926	0.482	0.975	0.776	0.992	0.896	0.998	0.952
0.8	-	-	0.959	0.467	0.991	0.779	0.998	0.899	0.999	0.953
0.9	-	-	0.99	0.465	0.998	0.781	0.999	0.9	1	0.953

N: N-Firm Noncooperative R&D

C: N-Firm R&D consortium

Parameters: r=0.03, λ=0.1, α=0.2, γ=0.2, and N=4.

8 Appendix B: Proofs

Proof of Proposition 1:

Because the objective function is linear in k, the firm's optimal investment decision $k_1(p)$ is a "bang-bang" rule. This implies that it either sets k equal to the minimum required level z, or its maximum feasible level 1, i.e.,

$$k_1(p) = \begin{cases} 1 & \text{if } p > p_1 \\ z & \text{if } p \leq p_1 \end{cases}.$$

If $k = z$, the general solution to the equation (6) is [20]

$$V_L(p) = \frac{s - \alpha z}{r} + \frac{\lambda z}{r + \lambda z} \frac{r\Pi - (s - \alpha z)}{r} p.$$

If $k = 1$, the general solution to the equation (6) is

$$V_H(p) = \frac{s - (1-\phi)\alpha - \alpha\phi z}{r} + \frac{\lambda}{r+\lambda}\left(\Pi - \left[\frac{s - (1-\phi)\alpha - \alpha\phi z}{r}\right]\right) p + B_1 p \left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}},$$

where B_1 is a constant. The abandonment threshold p_1 and the constant B_1 are determined by the value matching and smooth pasting conditions for $V_L(p)$ and $V_H(p)$:

$$V_L(p_1) = V_H(p_1). \tag{24}$$

$$V_L'(p_1) = V_H'(p_1). \tag{25}$$

These yield:

$$p_1 = \frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi - s + \alpha z)}. \tag{26}$$

$$B_1 = \frac{\lambda\alpha(1-z)(1-\phi)}{r(r+\lambda)}\left(\frac{1-p_1}{p_1}\right)^{-\frac{r}{\lambda}}. \tag{27}$$

[20] Note that there is a nonlinear term $B_0 p \left(\frac{1-p}{p}\right)^{\frac{r+\lambda z}{\lambda z}}$ to the general soluction of this differential equation, but is dropped because $V(0)$ needs to be finite and thus implies $B_0 = 0$. In fact, $V(0)$ represents the value of the firm when the R&D project is destined to fail.

Algebraic analysis reveals that p_1 can be expressed as the solution to:

$$\frac{\lambda}{r+\lambda}\left[\Pi-\frac{(s-\alpha z)}{r}\right]p_1=\frac{r}{r+\lambda}\left[\frac{\alpha(1-\phi)(1-z)}{r}\right]+\frac{\lambda z}{r+\lambda z}\left[\Pi-\frac{(s-\alpha z)}{r}\right]p_1. \tag{28}$$

If $\alpha, \phi, r, \lambda, z, s$ and Π are such that $p_1=\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi-s+\alpha z)}<1$, then over the meaningful range of possible beliefs, $p\in[0,1]$, the value function has two pieces:

$$V_1(p)=\begin{cases} V_H(p)=\frac{s-(1-\phi)\alpha-\alpha\phi z}{r}+\frac{\lambda}{r+\lambda}\left(\Pi-\left[\frac{s-(1-\phi)\alpha-\alpha\phi z}{r}\right]\right)p+B_1 p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & \text{if } p\in[p_1,1] \\ V_L(p)=\frac{s-\alpha z}{r}+\frac{\lambda z}{r+\lambda z}\frac{r\Pi-(s-\alpha z)}{r}p & \text{if } p\in[0,p_1] \end{cases}.$$

If $\alpha, \phi, r, \lambda, z, s$ and Π are such that $p_1=\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi-s+\alpha z)}\geq 1$, then over the meaningful range of possible beliefs the value function consists only of the $V_L(p)$ piece:

$$V_1(p)=V_L(p)=\frac{s-\alpha z}{r}+\frac{\lambda z}{r+\lambda z}\frac{r\Pi-(s-\alpha z)}{r}p \text{ for all } p\in[0,1]. \tag{29}$$

■

Derivation of the Welfare Schedule $W_1(p)$:

As Proposition 1 shows, if $\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi-s+\alpha z)}<1$, the monopoly firm either invests $k=1$ or $k=z$ in the R&D project. If $k_1(p)=1$ (which occurs if $p>p_1$), the solution to the differential equation in (13) is:

$$W_1(p)=-\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)+\frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p+B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}.$$

If $k_1(p)=z$ (which occurs if $p<p_1$), the solution to the differential equation in (13) is:[21]

$$W_1(p)=-\left(\frac{\alpha z+\gamma s}{r}\right)+\frac{\lambda z}{r+\lambda z}\left[CS+\Pi+\left(\frac{\alpha z+\gamma s}{r}\right)\right]p.$$

Thus,

$$W_1(p)=\begin{cases} -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)+\frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p+B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & p\in[p_1,1] \\ -\left(\frac{\alpha z+\gamma s}{r}\right)+\frac{\lambda z}{r+\lambda z}\left[CS+\Pi+\left(\frac{\alpha z+\gamma s}{r}\right)\right]p & p\in[0,p_1] \end{cases}, \tag{30}$$

[21] As usual, we drop the nonlinear term by requiring the value function to be finite at $p=0$.

Because at p_1, the firm is indifferent between $k = 1$ and $k = z$, the welfare schedule is continuous at p_1. Thus, the constant B_W equates the upper piece of $W_1(p)$ and the lower piece. Straightforward algebra establishes

$$B_W = B_W(p_1, z, s, \phi) = \frac{\left\{ \begin{array}{c} \frac{\alpha(1-z)}{r} + \frac{\gamma\phi\alpha(1-z)}{r} + \\ \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right] p_1 \\ -\frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right] p_1 \end{array} \right\}}{p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{r}}}$$

If $\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi - s+\alpha z)} \geq 1$, the monopoly firm invests $k = z$ in the R&D project for all p, and thus

$$W_1(p) = -\left(\frac{\alpha z + \gamma s}{r}\right) + \frac{\lambda z}{r + \lambda z}\left[CS + \Pi + \left(\frac{\alpha z + \gamma s}{r}\right)\right] p.$$

Proof of Proposition 4:

To prove the proposition, we will show that the first-best level of welfare can be attained by setting $s = z = 0$ and choosing $\phi = 1 - \rho$. The first-best R&D policy $k^*(p)$ solves the social planner's problem:

$$W^*(p) = \max_{k\in[0,1]} \left[-\alpha k dt + \lambda k p dt\,[CS + \Pi] + (1 - \lambda k p dt)e^{-rdt}W^*(p + dp)\right]. \tag{31}$$

Using an approach similar to that used to prove Proposition 1, the first-best R&D policy is given by:

$$k^*(p) = \begin{cases} 1 & \text{if } p \in [p^*, 1] \\ 0 & \text{if } p \in [0, p^*] \end{cases}, \tag{32}$$

where

$$p^* = \frac{\alpha}{\lambda(\Pi + CS)}.$$ [22]

is the first-best abandonment threshold.

Now, when $\gamma = 0$,

$$W_1(p) = -\alpha k_1(p)dt + \lambda k_1(p)pdt\,[CS + \Pi] + (1 - \lambda k_1(p)pdt)e^{-rdt}W_1(p + dp),$$

[22] Because we have assumed that the social benefit-cost ratio for a viable project exceeds 1, we have $\frac{\lambda[CS+\Pi]}{\alpha} > 1$, and thus $p^* < 1$. This implies that there must some exist some set of prior beliefs for which flat-out R&D investment would occur under the socially optimal policy.

and because of (31), for any arbitrary subsidy policy, it must be the case that $W_1(p) \leq W^*(p)$, and in particular $EW_1(z,s,\phi) = W_1(p_0) \leq W^*(p_0)$. However, if we can find a subsidy policy such that $k_1(p) = k^*(p)$,then $W_1(p) = W^*(p)$ and in particular, $EW_1(z,s,\phi) = W_1(p_0) = W^*(p_0)$ for any prior belief p_0. A subsidy policy that implements the first-best investment policy must therefore maximize expected *ex ante* welfare when $\gamma = 0$.

Note that if $s = z = 0$,

$$p_1(z,s,\phi) = \frac{\alpha(1-\phi)}{\lambda(\Pi + CS)}$$

A matching rate given by $\phi = 1 - \frac{\Pi}{\Pi + CS} = 1 - \rho$, along with $s = z = 0$, ensures that $p_1(z,s,\phi) = p^*$, and thus implements the maximum level of expected welfare for any prior belief p_0. Therefore, $z = 0, s = 0$, $\phi = 1 - \frac{\Pi}{\Pi + CS}$ is the optimal policy.■

Proof of Proposition 5:

Suppose, to the contrary, that the optimal subsidy policy entails $s > \alpha z$. If the subsidy policy occurs in the range where $p_0 \in [0, p_1(z,s,\phi))$, then $EW_1(z,s,\phi) = -\left(\frac{\alpha z + \gamma s}{r}\right) + \frac{\lambda z}{r + \lambda z}\left[CS + \Pi + \left(\frac{\alpha z + \gamma s}{r}\right)\right] p$, which is strictly decreasing in s, and expected welfare can be increased by decreasing s, contradicting the presumed optimality of policy. Suppose, then, the optimal subsidy policy occurs in the range where $p_0 \in [p_1(z,s,\phi), 1]$. Then $EW_1(z,s,\phi) = \Psi(B_W(p_1(z,s,\phi),z,s,\phi),z,s,\phi)$, and the first-order conditions for an optimal $s > \alpha z$ and a $\phi > 0$ are

$$\frac{\partial EW_1}{\partial \phi} = \frac{\partial \Psi}{\partial B_W}\left[\frac{\partial B_W}{\partial p_1}\frac{\partial p_1}{\partial \phi} + \frac{\partial B_W}{\partial \phi}\right] + \frac{\partial \Psi}{\partial \phi} = 0 \quad (33)$$

$$\frac{\partial EW_1}{\partial s} = \frac{\partial \Psi}{\partial B_W}\left[\frac{\partial B_W}{\partial p_1}\frac{\partial p_1}{\partial s} + \frac{\partial B_W}{\partial s}\right] + \frac{\partial \Psi}{\partial s} = 0 \quad (34)$$

or equivalently

$$\frac{-\left[\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial \phi} + \frac{\partial \Psi}{\partial \phi}\right]}{\frac{\partial p_1}{\partial \phi}} = \frac{-\left[\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial s} + \frac{\partial \Psi}{\partial s}\right]}{\frac{\partial p_1}{\partial s}} \quad (35)$$

We know from Proposition 2 that $\frac{\partial p_1}{\partial \phi} < 0$, while $\frac{\partial p_1}{\partial s} > 0$. Thus, $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial \phi} + \frac{\partial \Psi}{\partial \phi}$ and $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial s} + \frac{\partial \Psi}{\partial s}$

have opposite signs. Now,

$$\begin{aligned}
\frac{\partial \Psi}{\partial B_W} &= p_0 \left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}} > 0 \\
\frac{\partial B_W}{\partial s} &= \frac{\frac{\gamma p_1}{r}\left[\frac{\lambda z}{r+\lambda z} - \frac{\lambda}{r+\lambda}\right]}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} < 0 \\
\frac{\partial \Psi}{\partial s} &= -\frac{\gamma}{r}\left[1 - \frac{\lambda}{r+\lambda} p_0\right] < 0.
\end{aligned}$$

Hence, $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial s} + \frac{\partial \Psi}{\partial s} < 0$, and we must have $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial \phi} + \frac{\partial \Psi}{\partial \phi} > 0$. Now,

$$\begin{aligned}
\frac{\partial B_W}{\partial \phi} &= \frac{\frac{\gamma\alpha(1-z)}{r}\left[1 - \frac{\lambda p_1}{r+\lambda}\right]}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} \\
\frac{\partial \Psi}{\partial \phi} &= -\frac{\gamma\alpha(1-z)}{r}\left[1 - \frac{\lambda p_0}{r+\lambda}\right].
\end{aligned}$$

Thus, straightforward algebra establishes $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial \phi} + \frac{\partial \Psi}{\partial \phi} > 0 \Longleftrightarrow$

$$\frac{\gamma\alpha(1-z)}{r}\left\{\frac{p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}}{1-\frac{\lambda p_0}{r+\lambda}} - \frac{p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}{1-\frac{\lambda p_1}{r+\lambda}}\right\} > 0.$$

The function $F(p) \equiv \frac{p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}}{1-\frac{\lambda p}{r+\lambda}}$ can be shown to be a strictly decreasing function of p.[23] However, $p_1 < p_0$, which implies that the above expression is negative, not positive, and we have a contradiction. This establishes that *if* that the optimal ϕ is interior and that the optimal policy occurs within the range where $p_1(z, s, \phi) < p_0$, then we must have $s = \alpha z$.

Proof of Proposition 6:

Preliminary Steps

With $\phi^{**} \in (0,1)$, Proposition 5 implies that $s^{**} = \alpha z^{**}$. Thus, we redefine the function $\Psi(\cdot)$ to

[23] Differentiating F with respect to p yields

$$F'(p) = \left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}\left\{\frac{1}{1-p} - \frac{\frac{r+\lambda}{\lambda}}{1-p}\right\} < 0.$$

depend on B_W, z and ϕ; we redefine $B_W(\cdot)$ to depend on p_1, z, and ϕ, and we redefine $p_1(\cdot)$ to depend on z and ϕ. This gives us:

$$\Psi(B_W,z,\phi) \equiv \left\{ \begin{array}{c} -\left(\frac{\alpha+\gamma\alpha z+\gamma\phi\alpha(1-z)}{r}\right) \\ +\frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma\alpha z+\gamma\phi\alpha(1-z)}{r}\right)\right]p \\ +B_W p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}} \end{array} \right\}$$

$$B_W(p_1,z,\phi) \equiv \frac{\left\{ \begin{array}{c} \frac{\alpha(1-z)}{r}+\frac{\gamma\phi\alpha(1-z)}{r} \\ -\frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma\alpha z+\gamma\phi\alpha(1-z)}{r}\right)\right]p_1 \\ +\frac{\lambda z}{r+\lambda z}\left[CS+\Pi+\left(\frac{\alpha z+\gamma\alpha z}{r}\right)\right]p_1 \end{array} \right\}}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}.$$

$$p_1(z,\phi) \equiv \frac{\alpha(1-\phi)}{\lambda\Pi\left(\frac{r}{r+\lambda z}\right)}.$$

With $\phi \in (0,1)$ and $p_1(z,s,\phi) < p_0$, ϕ satisfies the first-order condition[24]

$$\frac{\partial EW_1}{\partial\phi} = \frac{\partial\Psi}{\partial B_W}\left[\frac{\partial B_W}{\partial p_1}\frac{\partial p_1}{\partial\phi}+\frac{\partial B_W}{\partial\phi}\right]+\frac{\partial\Psi}{\partial\phi}=0,$$

where, as noted above, $\frac{\partial\Psi}{\partial B_W} = p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}$ and

$$\frac{\partial B_W}{\partial p_1} = \frac{\alpha(1-z)\left([CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]\right)}{\lambda\Pi p_1^2(1-p_1)\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}$$

$$\frac{\partial B_W}{\partial\phi} = \frac{\frac{\alpha\gamma(1-z)}{r}\left(1-\frac{\lambda}{r+\lambda}p_1\right)}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}$$

$$\begin{aligned} \frac{\partial\Psi}{\partial\phi} &= -\frac{\alpha\gamma(1-z)}{r}\left(1-\frac{\lambda}{r+\lambda}p_0\right) \\ \frac{\partial p_1}{\partial\phi} &= -\frac{\alpha}{\lambda\Pi}\frac{r+\lambda z}{r}. \end{aligned}$$

[24]To simplify notation, hereafter we drop the ** on the optimal subsidy policy.

We can thus rewrite the first-order condition for ϕ as

$$\begin{aligned}\frac{\partial EW_1}{\partial \phi} &= p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}\left[\begin{array}{c}\frac{\alpha(1-z)\left([CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]\right)}{\lambda\Pi(1-p_1)p_1^2\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}\left(-\frac{\alpha}{\lambda\Pi}\frac{r+\lambda z}{r}\right)\\+\frac{\frac{\alpha\gamma(1-z)}{r}\left(1-\frac{\lambda}{r+\lambda}p_1\right)}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}\end{array}\right]\\&\quad-\frac{\alpha\gamma\,(1-z)}{r}\left(1-\frac{\lambda}{r+\lambda}p_0\right)\\&= 0.\end{aligned}$$

Because $p_1(z,\phi)\equiv\frac{\alpha(1-\phi)}{\lambda\Pi}\frac{r+\lambda z}{r}$, it follows that $-\frac{\alpha}{\lambda\Pi}\frac{r+\lambda z}{r}=-\frac{p_1}{1-\phi}$. Substituting this into the previous expression and rearranging terms, we can rewrite the first-order condition for ϕ as

$$\frac{\partial EW_1}{\partial \phi}=p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}\alpha(1-z)\left\{\frac{\frac{-1}{1-\phi}\left([CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]\right)}{\lambda\Pi(1-p_1)p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}+G(p_1)-G(p_0)\right\}=0, \tag{36}$$

where

$$G(p)\equiv\frac{\frac{\gamma}{r}\left(1-\frac{\lambda}{r+\lambda}p\right)}{p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}}.$$

When $\gamma>0$, the function $G(p)$ can be shown to be a strictly increasing function of p. Since $p_1<p_0$, it follows that when $\gamma>0$, $G(p_1)-G(p_0)<0$. Thus, (36) implies that when $\gamma>0$,

$$[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]<0, \tag{37}$$

or

$$1-\phi>\frac{\Pi(1+\gamma)}{CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}}. \tag{38}$$

Now, consider the expression for $\frac{\partial EW_1}{\partial z}$:

$$\frac{\partial EW_1}{\partial z}=\frac{\partial \Psi}{\partial B_W}\left[\frac{\partial B_W}{\partial p_1}\frac{\partial p_1}{\partial z}+\frac{\partial B_W}{\partial z}\right]+\frac{\partial \Psi}{\partial z}=0. \tag{39}$$

It can be shown that

$$\begin{aligned}\frac{\partial p_1}{\partial z} &= \frac{\alpha(1-\phi)}{r\Pi} \\ \frac{\partial \Psi}{\partial z} &= -\frac{\alpha\gamma(1-\phi)}{r}\left(1-\frac{\lambda}{r+\lambda}p_0\right) \\ \frac{\partial B_W}{\partial z} &= \frac{\frac{\alpha\gamma(1-\phi)}{r}\left(1-\frac{\lambda}{r+\lambda}p_1\right)}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} + \frac{\left[CS+\Pi+\frac{\alpha(1+\gamma)z}{r}\right]\frac{\alpha(1-\phi)}{\Pi(r+\lambda z)}}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} - \frac{\frac{\alpha(1+\gamma)}{r}\left[1-\frac{(1-\phi)\alpha z}{r\Pi}\right]}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}\end{aligned}$$

Noting that $\frac{\partial p_1}{\partial z} = \frac{\alpha(1-\phi)}{r\Pi} = \frac{\lambda p_1}{r+\lambda z}$, and factoring out $\alpha(1-\phi)$ and $p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}$ from the components of $\frac{\partial EW_1}{\partial z}$, we can write $\frac{\partial EW_1}{\partial z}$ as

$$\frac{\partial EW_1}{\partial z} = p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}\alpha(1-\phi)\left\{\begin{array}{c}\frac{\frac{1}{1-\phi}\frac{\lambda(1-z)}{r+\lambda z}\left([CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]\right)}{\lambda\Pi(1-p_1)p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} + G(p_1)-G(p_0) \\ +\frac{\left[CS+\Pi+\frac{\alpha(1+\gamma)z}{r}\right]\frac{1}{\Pi(r+\lambda z)}}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} \\ -\frac{\frac{\alpha(1+\gamma)}{r}\frac{1}{\alpha(1-\phi)}\left[1-\frac{\alpha z(1-\phi)}{r\Pi}\right]}{p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}\end{array}\right\}. \tag{40}$$

Now, given that $\frac{\partial EW_1}{\partial \phi} = 0$, we can substitute (36) into (40) to get

$$\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0} = \frac{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}{p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}}\alpha(1-\phi)\left\{\begin{array}{c}\frac{r+\lambda}{r+\lambda z}\frac{1}{1-\phi}\left(\frac{[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]}{\lambda\Pi(1-p_1)}\right) \\ +\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{\Pi(r+\lambda z)} \\ -\frac{(1+\gamma)}{r}\frac{1}{(1-\phi)}\left[1-\frac{\frac{\alpha z}{r}(1-\phi)}{\Pi}\right]\end{array}\right\}.$$

In deriving this expression, we used the fact that $\frac{\lambda(1-z)}{r+\lambda z} - 1 = \frac{r+\lambda}{r+\lambda z}$; we factored out the term $p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}$; and we noted that $-\frac{(1+\gamma)}{r}\frac{1}{(1-\phi)}\left[1-\frac{\frac{\alpha z}{r}(1-\phi)}{\Pi}\right] = -\frac{1}{r}\left[\frac{(1+\gamma)}{1-\phi} - \frac{\frac{\alpha(1+\gamma)z}{r}}{\Pi}\right]$. Thus, the sign of $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0}$ is the sign of the expression

$$\begin{aligned} H(z,\phi) \equiv & \left(\frac{r+\lambda}{r+\lambda z}\right)\left(\frac{1}{1-\phi}\right)\left(\frac{[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]}{\lambda\Pi(1-p_1)}\right) \\ & +\left(\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{\Pi(r+\lambda z)}\right) \\ & -\frac{1}{r}\left[\frac{1+\gamma}{1-\phi}-\frac{\frac{\alpha(1+\gamma)z}{r}}{\Pi}\right]. \end{aligned} \tag{41}$$

Note that if we can establish that $H(z,\phi)<0$ for all $z\in[0,1]$, it follows that $z=0$.

Proof of (a):

We note that from (37) that $[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]<0$, so the first term in the expression for $H(z,\phi)$ in (41) is negative. Thus, to show that $z=0$, it suffices to show that the last two terms of $H(z,\phi)$ are negative or

$$\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{\Pi(r+\lambda z)}-\frac{(1+\gamma)}{r}\frac{1}{(1-\phi)}\left[1-\frac{\frac{\alpha z}{r}(1-\phi)}{\Pi}\right]<0.$$

Multiplying through by $\frac{\Pi(r+\lambda z)}{CS+\Pi}$ and combining terms, we can rewrite this condition as

$$\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}-\left(\frac{r+\lambda z}{r}\right)\left[\frac{(1+\gamma)\,\Pi}{(1-\phi)\,(CS+\Pi)}-\frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}\right]<0. \tag{42}$$

Now, since we know from (38) that $1-\phi>\frac{\Pi(1+\gamma)}{CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}}$, it follows that

$$\frac{1}{1-\phi}=\frac{CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}}{\Pi(1+\gamma)}-\varepsilon, \tag{43}$$

where ε is a positive number. Substituting (43) into (42), and rearranging terms algebraically allows us to rewrite (42) as

$$\left[\frac{\alpha}{\lambda\,[CS+\Pi]}-1\right]\frac{\lambda z}{r}+\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}-\left(\frac{r+\lambda z}{r}\right)\frac{[\gamma\Pi-(1+\gamma)\varepsilon\Pi]}{CS+\Pi}<0. \tag{44}$$

Now, note that since the social benefit-cost ratio $\frac{\lambda[CS+\Pi]}{\alpha}$ is assumed to exceed 1, it follows that $\left[\frac{\alpha}{\lambda[CS+\Pi]}-1\right]\frac{\lambda z}{r}<0$ for $z\in(0,1]$. Now, as $\gamma\to 0$,$\frac{r}{r+\lambda z}\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}\to 0$, and $\varepsilon\to 0$ (as implied by the limiting case of Proposition 4) so that $\left(\frac{r+\lambda z}{r}\right)\frac{[\gamma\Pi-(1+\gamma)\varepsilon\Pi]}{CS+\Pi}\rightarrow 0$. Thus, the inequality in (44) does indeed hold, which shows that as $\gamma\rightarrow 0$, $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial\phi}=0}<0$, establishing part (a) of the Proposition.

Proof of part (b):

To prove part (b), we return to condition (42) which, if it can be established, shows that $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial\phi}=0}<0$ for all z. Noting that $\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}=1+\frac{\frac{\alpha z}{r}}{CS+\Pi}+\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}$, adding and subtracting the term $\left(\frac{r+\lambda z}{r}\right)$, and rearranging terms gives us:

$$\left[\frac{\alpha}{\lambda\left[CS+\Pi\right]}-1\right]\frac{\lambda z}{r}+\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}+\left(\frac{r+\lambda z}{r}\right)\left[1+\frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}-\frac{1}{1-\phi}\frac{(1+\gamma)\,\Pi}{(CS+\Pi)}\right]<0. \tag{45}$$

Now, we know that $p_1=\frac{\alpha(1-\phi)}{\lambda\Pi}\left(\frac{r+\lambda z}{r}\right)\leq 1$, so $1-\phi\leq\frac{\lambda\Pi}{\alpha\left(\frac{r+\lambda z}{r}\right)}$, or $\frac{1}{1-\phi}\geq\left(\frac{r+\lambda z}{r}\right)\frac{\alpha}{\lambda\Pi}$. Thus, $-\frac{1}{1-\phi}\frac{(1+\gamma)\Pi}{CS+\Pi}\leq-\left(\frac{r+\lambda z}{r}\right)\frac{\alpha(1+\gamma)}{\lambda(CS+\Pi)}$. This latter condition implies that

$$\begin{aligned}&\left[\frac{\alpha}{\lambda\left[CS+\Pi\right]}-1\right]\frac{\lambda z}{r}+\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}+\left(\frac{r+\lambda z}{r}\right)\left[1+\frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}-\frac{1}{1-\phi}\frac{(1+\gamma)\,\Pi}{(CS+\Pi)}\right]\\ <\;&\left[\frac{\alpha}{\lambda\left[CS+\Pi\right]}-1\right]\frac{\lambda z}{r}+\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}+\left(\frac{r+\lambda z}{r}\right)\left[1+\frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}-\left(\frac{r+\lambda z}{r}\right)\frac{\alpha(1+\gamma)}{\lambda\left(CS+\Pi\right)}\right].\end{aligned} \tag{46}$$

With a number of steps of algebra to rearrange the right-hand side of the above inequality, we can write the inequality as

$$\begin{aligned}&\left[\frac{\alpha}{\lambda\left[CS+\Pi\right]}-1\right]\frac{\lambda z}{r}+\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}+\left(\frac{r+\lambda z}{r}\right)\left[1+\frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}-\frac{1}{1-\phi}\frac{(1+\gamma)\,\Pi}{(CS+\Pi)}\right]\\ <\;&1-\frac{\alpha(1+\gamma)}{\lambda(CS+\Pi)}\\ \leq\;&0,\end{aligned}$$

where the final inequality follows from the assumption in this part of the proposition that $\gamma\geq\frac{\lambda(CS+\Pi)}{\alpha}-1$. This then suffices to establish (42) which, in turn, shows that $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial\phi}=0}<0$.

Proof of (c):

To prove (c), we return to the expression for $H(z,\phi)$ in (41) , whose sign is the sign of $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial\phi}=0}<0$. Multiplying through by $\frac{\Pi}{CS+\Pi}(r+\lambda z)$ and rearranging terms yields the expression $I(z,\phi)$ whose sign is the same as $H(z,\phi)$:

$$I(z,\phi) \equiv \frac{r+\lambda}{1-\phi}\left(\frac{[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]}{\lambda\left(CS+\Pi\right)\left(1-p_1\right)}\right)$$
$$+\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}$$
$$-\left(\frac{r+\lambda z}{r}\right)\left[\frac{\left(1+\gamma\right)\Pi}{\left(1-\phi\right)\left(CS+\Pi\right)}-\frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}\right].$$

Algebraically rearranging terms in $I(z,\phi)$ yields

$$I(z,\phi) \equiv \frac{1}{1-\phi}\left[\frac{r+\lambda}{\lambda(1-p_1)}-\frac{r+\lambda z}{r}\right]\left(\frac{[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]}{CS+\Pi}\right)$$
$$+\left[\frac{\alpha(1+\gamma)}{\lambda\left(CS+\Pi\right)}-1\right]\frac{\lambda z}{r}-\frac{\gamma\Pi}{\left(CS+\Pi\right)}\left(\frac{r+\lambda z}{r}\right). \tag{47}$$

The last term in $I(z,\phi)$ is negative and by assumption in this part of the proposition, $\gamma < \frac{\lambda(CS+\Pi)}{\alpha}-1$, which implies $\left[\frac{\alpha(1+\gamma)}{\lambda(CS+\Pi)}-1\right]\frac{\lambda z}{r}$ is negative for $z \in (0,1]$. In addition, we have the following chain of inequalities:

$$\frac{r+\lambda}{\lambda(1-p_1)} > \frac{r+\lambda}{\lambda} > \frac{r+\lambda}{r} > \frac{r+\lambda z}{r},$$

where the second inequality in the chain follows from the assumption in this part of the proposition that $r > \lambda$. Since we had established earlier that $[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}] < 0$, it follows that $I(z,\phi) < 0$ for all z and hence $H(z,\phi) < 0$, thus establishing that $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0} < 0$.

Proof that $p_1^{**} > p^* = \frac{\alpha}{\lambda(CS+\Pi)}$ *and* $\phi^{**} < 1-\rho$*:*

Under conditions (a), (b), or (c), the optimal subsidy policy entails $z = s = 0$. Now, *given* the optimal matching rate ϕ^{**} and the positive shadow cost of public funds γ, consider the social planner's most preferred investment policy i.e., the policy it would want the firm follow given that it had committed to the subsidy. That policy would solve the following problem:

$$W_1(p) = \max_{k\in[0,1]} -\alpha k dt - \gamma\phi\alpha k dt$$
$$+\lambda k p dt\left[CS+\Pi\right]+(1-\lambda k p dt)e^{-rdt}W_1(p+dp).$$

That policy entails an abandonment threshold given by:

$$p_1^W = \frac{\alpha\left(1+\gamma\phi^{**}\right)}{\lambda\left(CS+\Pi\right)}.$$

By contrast, the abandonment threshold p_1^{**} induced by the optimal subsidy policy is

$$p_1^{**} = \frac{\alpha\left(1-\phi^{**}\right)}{\lambda\Pi}$$

Thus,

$$\begin{aligned}
\frac{p_1}{p_1^W} &= \frac{\frac{\alpha(1-\phi^{**})}{\lambda\Pi}}{\frac{\alpha(1+\gamma\phi^{**})}{\lambda(CS+\Pi)}} \\
&= \frac{\left(1-\phi^{**}\right)}{\left(1+\gamma\phi^{**}\right)}\frac{CS+\Pi}{\Pi} \\
&> \frac{1-\frac{CS}{CS+\Pi(1+\gamma)}}{1+\gamma\frac{CS}{CS+\Pi(1+\gamma)}}\frac{CS+\Pi}{\Pi} = 1.
\end{aligned}$$

where the inequality follows from the result that $\phi^{**} < \frac{CS+\frac{\alpha z(1+\gamma)}{r}}{CS+\Pi(1+\gamma)+\frac{\alpha z(1+\gamma)}{r}}$ proved in (38) above. Thus, we have: $p_1^{**} > p_1^W = \frac{\alpha(1+\gamma\phi^{**})}{\lambda(CS+\Pi)} > \frac{\alpha}{\lambda(CS+\Pi)} = p^*$. To complete the proof, note that $\phi^{**} < \frac{CS}{CS+\Pi(1+\gamma)} < \frac{CS}{CS+\Pi} = 1-\rho$.■

Proof of Proposition 7:[25]

We begin by defining

$$\begin{aligned}
b\left(p, V^i\right) &= \lambda p\left[\theta\Pi - V^i - (1-p)\,V^{i\prime}\right] \\
c\,(p) &= \alpha\,(1-\phi) - \lambda p\,(1-\theta)\,\Pi.
\end{aligned}$$

The term $b\left(p, V^i\right)$ is the marginal benefit to the firm from an additional unit of investment effort by a rival firm, while $c(p)$ is the *net* marginal cost to the firm from an additional unit of R&D. Specifically, it is the net- of-subsidy marginal cost of effort $\alpha\,(1-\phi)$ minus the net marginal benefit $\lambda p\,(1-\theta)\,\Pi$ of achieving the breakthrough rather than free riding on a competitor's discovery. We can now write the

[25] Part (i) of the proof follows the method employed by Keller Rady and Cripps (2005).

Bellman equation (20) as

$$rV^i(p) - (s - \alpha\phi z) = K^{-i} b\left(p, V^i\right) + \max_{k^i \in [z,1]} \left\{k^i \left[b\left(p, V^i\right) - c(p)\right]\right\},$$

The firm's optimal investment decision is given by the following "reaction function":

$$k^i(p) = \begin{cases} 1 & \text{if } V^i(p) > \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}K^{-i} \\ \in [z,1] & \text{if } V^i(p) = \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}K^{-i} \\ z & \text{if } V^i(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}K^{-i} \end{cases}. \tag{48}$$

The linearity of the value function implies that we only need to consider three cases: $k = z$, $k = 1$ and $k = \kappa \in (z, 1)$. As noted, we seek to characterize a symmetric equilibrium with investment policies $k^i(p) = k_N(p)$ and value function $V^i(p) = V_N(p)$, for all $i = 1, ..., N$. When $k_N(p) = z$, the differential equation (20) is:

$$\begin{aligned} rV_N(p) &= s - \alpha\phi z + \lambda p(N-1)z\left[\theta\Pi - V_N(p) - (1-p)V_N'(p)\right] \\ &\quad + z\left[\lambda p\left[\Pi - V_N(p) - (1-p)V_N'(p)\right] - \alpha(1-\phi)\right]. \end{aligned}$$

The solution to this equation, denoted by $V_N^L(p)$, is:[26]

$$V_N^L(p) = \frac{s - \alpha z}{r} + \frac{\lambda N z\left[\frac{r(1-\theta)\Pi}{N} + r\theta\Pi - s + \alpha z\right]}{r(r + \lambda N z)} p.$$

When $k_N(p) = 1$, the differential equation (20) is

$$\begin{aligned} rV_N(p) &= s - \alpha\phi z + \lambda p(N-1)\left[\theta\Pi - V_N(p) - (1-p)V_N'(p)\right] \\ &\quad + \left[\lambda p\left[\Pi - V_N(p) - (1-p)V_N(p)\right] - \alpha(1-\phi)\right]. \end{aligned}$$

The solution to this equation, denoted by $V_N^H(p)$, is

$$V_N^H(p) = \frac{s - \alpha(1-\phi) - \alpha\phi z}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N} + r\theta\Pi - s + \alpha(1-\phi) + \alpha\phi z\right)}{r(r + \lambda N)} p + B_H p \left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}},$$

[26] The coefficient to the nonlinear part is zero as we require finiteness for the value function at $p = 0$.

where B_H is a constant to be determined. Finally, if $k_N(p) \in (z,1)$, then from (48), $b(V_N(p),p) = c(p)$, or

$$\lambda p \left(\theta \Pi - V_N(p) - (1-p) V_N'(p)\right) - \left(\alpha(1-\phi) - \lambda p (1-\theta)\Pi\right) = 0.$$

The solution to this equation, which is denoted by $V_N^M(p)$, is

$$V_N^M(p) \equiv \frac{\lambda\Pi - (1-\phi)\alpha}{\lambda} - B_M(1-p) + \frac{(1-\phi)\alpha(1-p)}{\lambda} \ln \frac{1-p}{p},$$

where B_M is a constant to be determined. To solve for the abandonment threshold p_N at which the firm reduces its R&D effort to the minimum mandated level. The value matching and smooth pasting conditions imply that

$$\begin{aligned} V_N^M(p_N) &= V_N^L(p_N) \\ V_N^{M\prime}(p_N) &= V_N^{L\prime}(p_N), \end{aligned}$$

which gives us

$$p_N = \frac{\alpha(1-\phi)}{\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda N z}\right) + (1-\theta)\Pi\left(\frac{(N-1)\lambda z}{r+\lambda N z}\right)\right)}.$$

Now, from (48), $k_N(p) \in (z,1)$ if and only if $V_N^M(p) = \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)k_N(p)$. This implies:

$$k_N(p) = \frac{rV_N^M(p) - s + \alpha\phi z}{(N-1)c(p)},$$

which makes each firm indifferent between choosing any investment level between z and 1. A necessary condition for this to be well defined is that $c(p_N)$ be positive; otherwise $k_N(p)$ will be negative. From the definition of $c(p)$ above, $c(p_N) > 0$ if and only if $p_N < \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$. Straightforward algebra establishes that this condition is equivalent to

$$\theta > \overline{\theta}(z,s) \equiv 1 - \left[\frac{\Pi - \left(\frac{s-\alpha z}{r}\right)}{\Pi}\right]\left[\frac{r}{r+\lambda z}\right].$$

Since $k_N(p) \to \infty$ as $p \to \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$, there exists q_N such that $k(q_N) = 1$. In this case, the value function is thus:

$$V_N(p) = \begin{cases} V_N^H(p) & p \in [q_N, 1] \\ V_N^M(p) & p \in (p_N, q_N) \\ V_N^L(p) & p \in [0.p_N] \end{cases},$$

with B_H satisfying $V_H^H(q_N) = V_N^M(q_N)$.

(ii) If $\theta \leq \overline{\theta}(z,s)$, we will show that each firm's optimal strategy is either to invest "flat out" in R&D effort by setting $k=1$ when $p > p_N$ and invest at the minimum level z, when $p \leq p_N$. To establish the latter, suppose all other firms besides i are investing the minimum level z, so that $K^{-i} = (N-1)z$. To show that firm i's best response is also to invest z when $p \leq p_N$, then from (48), we must establish that $V_N(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)K^{-i}$, or equivalently $V_N^L(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)z$, when $p \leq p_N$. Using the expressions for $V_N^L(p)$ and $c(p)$, and some straightforward algebraic manipulations, $V_N^L(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)z$ can be shown to be equivalent to $\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda Nz}\right) + (1-\theta)\Pi\left(\frac{(N-1)\lambda z}{r+\lambda Nz}\right)\right)p \leq \alpha(1-\phi)$, and given the expression for p_N above, this indeed is true for $p \leq p_N$. Now assume $p > p_N$ and every other firm invests $k=1$. We need to show that firm i's best response is to invest $k=1$. Suppose not. Then, due to the linearity of the problem, the only case that the firm will invest $z < k < 1$ is when $V^i = \frac{s}{r} + \frac{c(p)}{r}K^{-i}$, but this implies $p_N < \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$. However, $\theta \leq \overline{\theta}(z,s)$ implies $z \geq \frac{r\theta\Pi - s}{\lambda(1-\theta)\Pi - \alpha}$, which, in turn, implies $p_N \geq \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$, a contradiction. The uniqueness also follows from the fact that the investment problem is linear in k so firms will not coordinate in investing in a lower level $\kappa \in (z,1)$ because if that is the case, it then implies $b(p, V_i^n) > c(p)$, then the firm will invest $k=1$ rather than $\kappa < 1$.[27] In this case the value function is:

$$V_N(p) = \begin{cases} V_N^H(p) & p \in [p_N, 1] \\ V_N^L(p) & p \in [0.p_N] \end{cases}.$$

Proof of Proposition 10:

We employ the same logic as in the proof of Proposition (4): when $\gamma = 0$ it suffices to show that if we can induce $k_N(p) = k^*(p)$ through an appropriate choice of (z, s, ϕ), then that subsidy policy must indeed maximize *ex ante* welfare $EW_N(z,s,\phi)$. Now, recall that under the equilibrium policy there is

[27] For a similar proof of this result, see Besanko and Wu (2008).

no free riding if and only if

$$\theta \leq \overline{\theta}(z,s) = 1 - \left[\frac{\Pi - \left(\frac{s-\alpha z}{r}\right)}{\Pi}\right]\left[\frac{r}{r+\lambda z}\right],.$$

When $z = 0$, $s = r\theta\Pi$, then it can be verified that $\overline{\theta}(z,s) = \theta$, which is just enough to eliminate the free-rider problem. The equilibrium investment policy in this case is

$$k_N(p) = \begin{cases} 1 & \text{if } p \geq p_N(0, r\theta\Pi, \phi) \\ 0 & \text{if } p \leq p_N(0, r\theta\Pi, \phi), \end{cases},$$

where

$$p_N(0, r\theta\Pi, \phi) = \frac{\alpha(1-\phi)}{\lambda(1-\theta)\Pi}.$$

By setting $\phi = \frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$, we can make $p_N(0, r\theta\Pi, \phi) = p_N^*$. Thus, the first-best investment policy can be induced by setting $z = 0$ and using a combination of unrestricted funding $s = r\theta\Pi$ and a matching rate $\phi = \frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$.$\blacksquare$

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