Irreversible Investment in Stochastically Cyclical Markets*

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Abstract

This paper studies entry and exit decisions in markets whose demand alternates between growth and decline phases at uncertain times. To capture these features of stochastic market evolution, we introduce a new stochastic process, and provide key mathematical results related to first passage times which make the characterization of entry and exit behavior quite simple and straightforward (especially for competitive industries). Besides providing an economic interpretation of a single firm's optimal entry and exit policies linked to the arrival of bad news, we compare our results with those obtained if demand follows a Geometric Brownian Motion, and we show that they are fundamentally different. Despite the stochastic process of the underlying variable has a continuous sample path in both cases, we demonstrate that in our setting the sample path of a firm's value experiences jumps whenever a growth/decline phase starts, irrespective of whether a firm is active or not. Further, in the case of competitive industries, the rates of entry and exit discontinuously fall to zero owing to the respective ends of a phase of growth and decline.

Keywords: Real Options, Irreversibility, Bad News Principle, Entry and Exit.

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1 Introduction

A large number of industries exhibit a cyclical evolution of demand. Typical microeconomics textbook examples include industries producing durable goods such as motor vehicles or electrical appliances.[1] Other examples of cyclical industries involve very diverse sectors such as construction, household furniture, carpets and rugs, wholesale trade, legal or child care services, hotels, railroad transportation, metalworking machinery, automobile repair, etc (Berman and Pfleeger 1997). The cyclical behavior of these industries typically reflects nothing but a high sensitiveness to the evolution of the state of the economy, which is well known to exhibit a cyclical but somewhat unpredictable pattern of evolution (Hamilton 1989).[2]

This paper studies irreversible (dis)investment decisions in industries whose demand follows random-length cycles such as the ones described above. In these industries, the dynamics of the underlying state variable (e.g., demand or profit) is not governed by the standard diffusion process traditionally employed by the real options literature, namely the Geometric Brownian Motion (GBM).[3] To address this shortcoming of the modern approach to investment decisions under uncertainty, we present a stochastic process that exhibits cyclical behavior in that the underlying state variable perpetually alternates between growth and decline phases at uncertain times. Our main objective is to characterize optimal/equilibrium entry and exit behavior in this setting, both for the case of a single firm and that of multiple atomistic firms. Because of the novelty of the stochastic process, a major contribution of the paper is to provide key mathematical results related to first passage times so as to make the analysis of entry and exit decisions quite simple and straightforward (including the cases where a firm is allowed to exit once it has entered the market). These results include expected discount factors to be used when discounting (one-shot) payoffs on the state space, or expected streams of discounted profits while transitioning from one demand/profit level to another one.

With these mathematical results at hand, we do not need to resort to the complex mathematical apparatus employed by the real options literature, namely dynamic

[1] For instance, see Pindyck and Rubinfeld (2005, pp. 40-41).

[2] Indeed, Hamilton's (1989) paper spawned an increasingly large literature devoted to the empirical estimation of time series that are assumed to experiment unobservable changes in its growth rate at some random dates.

[3] Two classic papers on real options theory are McDonald and Siegel (1986) and Dixit (1989). Pindyck (1991) or Dixit and Pindyck (1994) provide excellent surveys of the most relevant theoretical developments. See also the recent work by Riedel and Su (2006) for an elegant general approach to irreversible investment under uncertainty.

programming and stochastic calculus. More specifically, we are able to characterize optimal entry and exit policies, as well as the value of the options to invest and disinvest, by directly applying our theorems and using ordinary calculus. Our mathematical results are powerful enough to make further research on stochastically cyclical markets quite simple and accessible. To illustrate this point, we apply them to the analysis and characterization of entry and exit policies, both for a single firm and a competitive firm.

In a setting in which there is an alternation between growth and decline phases at uncertain times, there exists an option value of waiting to (dis)invest. Because the length of growth and decline phases is random, a firm has an incentive to wait and continuously update its information about the duration of the current phase without making any irreversible decision, and at the same time it can capitalize on favorable market evolutions. The existence of an option value of delaying (dis)investment has relevant conceptual implications for the theory of real options. In an influential paper, Abel and Eberly (1996) provided an economic interpretation of optimal (dis)investment policies in settings with fully ongoing uncertainty based on the Jorgensonian user cost of capital. We complement their approach by interpreting a single firm's optimal (dis)investment policies in the light of the "bad news principle of irreversible investment" spelled out by Bernanke (1983). According to this principle, a firm weighing whether to slightly delay (dis)investment when there is ongoing uncertainty should only care about the arrival of bad news and their adverse effect on payoffs.[4] In our view, Bernanke's (1983) insight is the conceptual pillar of real options theory, and we show why both for market entry and market exit, neither of which has been accomplished in GBM settings.

Another contribution of this paper is to draw implications for the valuation of stock prices. The stochastic process that we examine and the GBM have both a continuous sample path. However, unlike settings in which a firm's profit flow follows a single GBM, it can be proven in our setup that imperceptible changes in the profit flow collected by a firm may be accompanied by significant falls or rises in firm value. In particular, we show that firm value jumps upwards (downwards) whenever a growth (decline) phase starts. Although these two discontinuity results may be expected when a firm has an option to invest, it is worthwhile emphasizing that this holds *even if a firm has an option to disinvest*. This is important in that it gives a (robust) rationale

[4] The point is that an irreversible decision is costly in that it can be regretted *ex post*, which can happen only if bad news have arrived. As a direct consequence, a firm should consider only the effect of the arrival of bad news (and not good news) when contemplating a delay in the execution of an irreversible decision.

for using jump processes in the valuation of firms' stock prices regardless of whether a firm has an opportunity to invest or disinvest. More importantly, we show that this holds both for a single firm and for a (perfectly) competitive firm. An additional implication for competitive industries is that entry (exit) rates discontinuously fall from positive levels to zero owing to the end of a growth (decline) phase. Thus, entry and exit waves may be observed to suddenly vanish.

Our paper contributes to the recent and growing real options literature based on regime-switching models, which was started off by Hassett and Metcalf's (1999) pioneering work. The closest paper to ours is Drifill, Raybaudi and Sola (2003), which numerically analyzes how the value of a single firm's investment opportunity is affected by the existence of regime shifts. Our paper differs from their paper in several respects. From a technical standpoint, we characterize some general properties of a regime-switching stochastic process such as the (conditionally) expected stream of discounted profits harvested while the process transitions from one state to another. From a conceptual standpoint, we solve the model analytically, which allows us to provide an economic interpretation linked to Bernanke's (1983) "bad news principle" for the entry and exit policies of a single firm. In addition, we provide a rationale for the use of jump processes in the valuation of stock prices when the underlying state variable is continuous. Lastly, they do not deal with perfectly competitive industries, whose analysis turns out to be quite simple given our results for computing expected profit streams with an upper and a lower barrier. Indeed, we show that the results (and reasoning) in Leahy (1993) directly extend to stochastically cyclical markets without any loss of substance. In particular, a competitive firm that acted myopically by ignoring all future entry (exit) would behave correctly by entering (exiting) at the optimal entry (exit) threshold of a single firm.

The remainder of the paper is organized as follows. Section 2 describes the stochastic process that constitutes the starting point of our analysis. Given the novelty of the process, Section 3 provides several mathematical results such as expected discount factors on the state space and present value calculations, which makes some of the results in subsequent sections quite straightforward. Section 4 and 5 respectively characterize a single firm's optimal entry and exit policies, relate them to the "bad news principle," and analyze their implications for firm valuation. Section 6 briefly examines combined entry and exit decisions by a single firm. In turn, Section 7 examines the main properties of the entry and exit dynamics of a competitive industry in which demand follows random-length cycles, while Section 8 concludes. Proofs of the results not proven in the text can be found in two appendices.

2 The model

In this section we construct a stochastic process with continuous sample paths that represents the random evolution of a certain variable Π.[5] For the sake of concreteness, $\Pi(t)$ denotes instantaneous profit at time t, although it could certainly represent any other variable such as demand or price of a product. Let the dynamics of flow profits be such that $d\Pi = \alpha(t)\Pi dt$, where $\{\alpha(t), t \geq 0\}$ is a Markov chain with states $\{\alpha_1, \alpha_2\} \in \Re_{++} \times \Re_{--}$. It is assumed for convenience that the chain starts at state α_1 (i.e., $\alpha(0) = \alpha_1$), while the transition probabilities of this process are as follows. On the one hand, if the chain is in state α_1 at time $t \geq 0$, then the probability that it moves to state α_2 between times t and $t + dt$ is

$$\Pr(\alpha(t+dt) = \alpha_2|\alpha(t) = \alpha_1) = \lambda_1 dt + o(dt).$$

On the other hand, if the chain is in state α_2 at time $t > 0$, then the probability that it moves to state α_1 between times t and $t + dt$ is

$$\Pr(\alpha(t+dt) = \alpha_1\,|\alpha(t) = \alpha_2\,) = \lambda_2 dt + o(dt).$$

Letting $\widetilde{\tau}_i$ denote the inter-arrival time of event $i \in \{1, 2, ...\}$ (where an event is a change in the state of the chain), we have that $\{\widetilde{\tau}_i\}_{i=1}^{\infty}$ is a sequence of exponential random variables such that the inter-arrival times with odd (even) subscripts are exponentially distributed with rate $\lambda_1 > 0$ ($\lambda_2 > 0$). We define $T_i = T_{i-1} + \tau_i$ for all $i \in \{1, 2, ...\}$, where the initial date is $T_0 = 0$, and we refer to each T_i as a (realized) switching date.

Figure 1 shows two sample paths of the process we have described (for different parameter values). As seen in the figure, $\Pi(t)$ grows exponentially at rate $\alpha_1 > 0$ during the random length period (T_{i-1}, T_i) $(i = 1, 3, ...)$, and decreases exponentially at rate $\alpha_2 < 0$ during (T_{i-1}, T_i) $(i = 2, 4, ...)$. The sample path is continuous because $\lim_{t\uparrow T_i} \Pi(t) = \lim_{t\downarrow T_i} \Pi(t)$, although the path will exhibit a kink at any realized switching date T_i $(i = 1, 2, ...)$, and it will be almost everywhere differentiable. Lastly, it holds that $\Pi(t) > 0$ for any $t \geq 0$ and any set of realizations of the random variables involved because we assume that $\Pi(0) > 0$.

We say that the process (or, more concretely, market) is in a growth phase when it is characterized by a positive instantaneous growth rate; otherwise, we say that

[5] This process is the continuous-time limit of that put forward by Bagwell and Staiger (1997) in their analysis of collusive pricing over the business cycle.

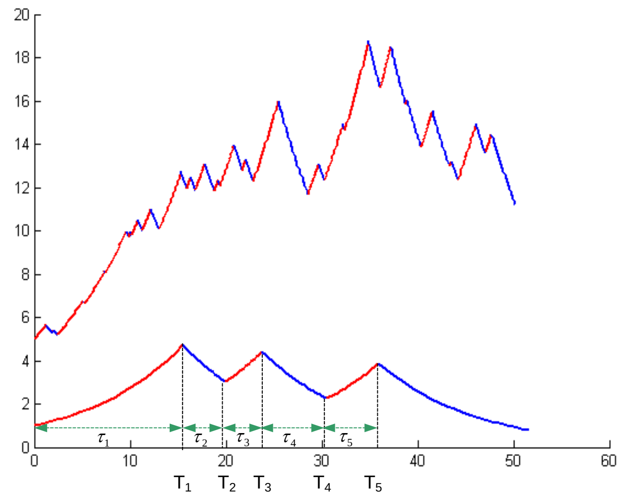

Figure 1: Two sample paths

the process is in a decline phase. Because of the i.i.d. exponential random variables, the current state of the market is clearly summarized by the current level of the flow profit and the type of phase through which the market is currently going.

This paper is concerned with (dis)investment decisions given the random temporal evolution of a variable that determines the profitability of a (dis)investment opportunity. We focus on the cases in which there exists a single decision-maker or many atomistic decision-makers.[6] Given these two settings, the paper studies a firm's decisions to enter and exit a market given that the flow of profits made by a firm if active in the market follows the stochastic process previously described. Specifically, at each date, a firm does not know when the next upturns or downturns will happen, although it knows the current level of the profit flow and whether the cycle is growing or declining. A firm also observes realized switching dates immediately after they arrive. Firms are assumed to be risk-neutral and use a constant discount rate $r > 0$. If a firm enters the market, then it is assumed to incur a positive sunk cost K and in return starts operating immediately (i.e., there is no time-to-build). Similarly, if a firm exits the market, then it is assumed to recover a non-negative value S, which can be the salvage or redeployment value of the asset. We first study market entry and exit decisions in isolation, and then combined entry and exit decisions (in the spirit of Dixit 1989) as an extension. We start with the more complicated case of a single firm (Sections 4-6) and conclude with multiple atomistic firms (Section 7), but

[6] It is not difficult to consider few decision-makers, so we omit oligopolistic applications for the sake of brevity.

before we derive several mathematical results that make these analyses quite simple.

3 Mathematical preliminaries

In this section we derive some relevant mathematical results regarding the stochastic process defined in Section 2 (proofs can be found in Appendix A).[7] They are useful properties for working on the state space of the process and they will be used in Sections 4-7.

Our first result deals with the expected values of discounted streams of flow profits, but before proceeding to computing them, it is useful to introduce some notation. In particular, let

$$\rho_1 \equiv \frac{\lambda_1 + \lambda_2 + r - \alpha_2}{(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) - \lambda_1 \lambda_2}$$

and

$$\rho_2 \equiv \frac{\lambda_1 + \lambda_2 + r - \alpha_1}{(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) - \lambda_1 \lambda_2},$$

and assume that

$$(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) > \lambda_1 \lambda_2 \tag{1}$$

so that we have an economically meaningful setting (otherwise, streams of discounted profits fail to be integrable). A necessary condition for (1) to hold is that $r + \lambda_1 > \alpha_1$.

We now deal with the expected stream of discounted profits if the firm is active in the market forever given the current state π_0.[8] We also allow the process to (temporarily) stay at level $\overline{\pi} \in (0, \infty]$ if it ever reaches such level in a growth phase, at least until the process starts declining. Further, we allow the process to (temporarily) stay at level $\underline{\pi} \in [0, \overline{\pi})$ if it ever reaches such level in a decline phase, at least until the process starts growing.[9] To this end, let $\overline{\mathcal{E}}(\pi_0 \,|\overline{\pi}, \underline{\pi})$ denote the expected stream

[7] Our working paper version (Ruiz-Aliseda and Wu 2007) extends all the results in this section to markets that follow a random number of random-length cycles under the assumption that the realized number of cycles can never be observed. In particular, this situation is captured by assuming that $\Pr(\alpha(t+dt) = \alpha_1 \,|\alpha(t) = \alpha_2) = \widetilde{\Lambda}\lambda_2 dt + o(dt)$, where $\widetilde{\Lambda}$ is a Bernoulli random variable that is independently drawn every time the chain leaves state α_1 (with probability $p_0 \in [0,1]$ that $\Lambda = 1$ and probability $1 - p_0 \in [0,1]$ that $\Lambda = 0$), and the additional assumption that every draw of this random variable is unobservable as long as the chain does not leave state α_2. (Equivalently, state α_1 would always be transient, whereas state α_2 could be either transient or absorbing in this more general setting. In particular, such a state would become absorbing once the chain had been $\widetilde{n}$ times in state α_1, where $\widetilde{n}$ is a geometrically distributed random variable with parameter $1 - p_0$ whose draw is unobservable. If state α_2 had not yet become absorbing and rather were transient, then the transition probability would be $\Pr(\alpha(t+dt) = \alpha_1 \,|\alpha(t) = \alpha_2) = \lambda_2 dt + o(dt)$.)

[8] Throughout, we denote the current level of the profit flow by π_0, whereas we denote growth and decline phases by an upper bar and a lower bar, respectively.

[9] Although not conventional, the notation $\overline{\pi} = \infty$ ($\underline{\pi} = 0$) is to be understood as the process not

of discounted profits if the process is in a growth phase at π_0, given the upper bound $\overline{\pi}$ and the lower bound $\underline{\pi}$.[10] Similarly, let $\underline{\mathcal{E}}(\pi_0 \,|\overline{\pi}, \underline{\pi})$ denote the expected stream of discounted profits if the process is in a decline phase at π_0, given the upper bound $\overline{\pi}$ and the lower bound $\underline{\pi}$. Then we have the following result regarding expected streams of discounted profits:

Theorem 1 *Suppose that the firm is operating at π_0 and is active forever. Suppose also that a process in a growth phase stays at level $\overline{\pi} \in (0, \infty]$ if it ever reaches such level, and that it leaves $\overline{\pi}$ only once the process starts declining. In addition, suppose that a process in a decline phase stays at level $\underline{\pi} \in [0, \overline{\pi})$ if it ever reaches such level, and that it leaves $\underline{\pi}$ only once the process starts growing. Then the expected stream of discounted profits if the process is in a growth phase is*

$$\overline{\mathcal{E}}(\pi_0 \,|\overline{\pi}, \underline{\pi}) = \xi(\overline{\pi}, \underline{\pi})\pi_0^{\beta_1} + \delta_1 \zeta(\overline{\pi}, \underline{\pi})\pi_0^{\beta_2} + \rho_1 \pi_0,$$

while the expected stream of discounted profits if the process is in a decline phase is

$$\underline{\mathcal{E}}(\pi_0 \,|\overline{\pi}, \underline{\pi}) = \delta_2 \xi(\overline{\pi}, \underline{\pi})\pi_0^{\beta_1} + \zeta(\overline{\pi}, \underline{\pi})\pi_0^{\beta_2} + \rho_2 \pi_0,$$

where $\beta_1 \equiv \dfrac{\alpha_1(r+\lambda_2) + \alpha_2(r+\lambda_1) - \sqrt{\Delta}}{2\alpha_1\alpha_2} > 1$, $\beta_2 \equiv \dfrac{\alpha_1(r+\lambda_2) + \alpha_2(r+\lambda_1) + \sqrt{\Delta}}{2\alpha_1\alpha_2} < 0$, $\Delta \equiv [\alpha_1(\lambda_2 + r) - \alpha_2(\lambda_1 + r)]^2 + 4\alpha_1\alpha_2\lambda_1\lambda_2 > 0$, $\delta_1 \equiv \dfrac{\lambda_2 + r - \alpha_2\beta_2}{\lambda_2} \in (0,1)$, $\delta_2 \equiv \dfrac{r + \lambda_1 - \alpha_1\beta_1}{\lambda_1} \in (0,1)$, $\xi(\overline{\pi}, \underline{\pi}) \equiv \dfrac{\delta_1 \overline{\pi}^{\beta_2} \rho_2 \underline{\pi} - \rho_1 \overline{\pi}\,\underline{\pi}^{\beta_2}}{\beta_1(\underline{\pi}^{\beta_2}\overline{\pi}^{\beta_1} - \delta_1\delta_2 \underline{\pi}^{\beta_1}\overline{\pi}^{\beta_2})}$ *and* $\zeta(\overline{\pi}, \underline{\pi}) \equiv \dfrac{\delta_2 \underline{\pi}^{\beta_1} \rho_1 \overline{\pi} - \rho_2 \underline{\pi}\,\overline{\pi}^{\beta_1}}{\beta_2(\underline{\pi}^{\beta_2}\overline{\pi}^{\beta_1} - \delta_1\delta_2 \underline{\pi}^{\beta_1}\overline{\pi}^{\beta_2})}$.

Theorem 1 is useful when analyzing perfectly competitive industries in which free entry and exit lead to profits being bounded above and below (the bounds being determined endogenously). In competitive industries in which entry or exit do not take place or in industries in which there exists a single firm with the opportunity to (dis)invest, it is necessary to analyze the stochastic process subject to just one or no barrier at all. Because $\beta_1 > 1$ and $\beta_2 < 0$, it holds that $\xi(\overline{\pi}, 0) = -\dfrac{\rho_1 \overline{\pi}^{1-\beta_1}}{\beta_1}$ and $\zeta(\infty, \underline{\pi}) = -\dfrac{\rho_2 \underline{\pi}^{1-\beta_2}}{\beta_2}$, so we have the following results based on Theorem 1:

having an upper (lower) bound, since the process can never hit level $\overline{\pi} = \infty$ ($\underline{\pi} = 0$).

[10] Expectations are conditional upon the current state being π_0 and upon whether the process is growing or not.

Corollary 1 *It holds that* $\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},0)=\rho_1\pi_0-\dfrac{\rho_1\overline{\pi}}{\beta_1}\left(\dfrac{\pi_0}{\overline{\pi}}\right)^{\beta_1}$ *and* $\underline{\mathcal{E}}(\pi_0\,|\overline{\pi},0)=\rho_2\pi_0-\dfrac{\delta_2\rho_1\overline{\pi}}{\beta_1}\left(\dfrac{\pi_0}{\overline{\pi}}\right)^{\beta_1}$*, while* $\overline{\mathcal{E}}(\pi_0\,|\infty,\underline{\pi})=\rho_1\pi_0-\dfrac{\delta_1\rho_2\underline{\pi}}{\beta_2}\left(\dfrac{\pi_0}{\underline{\pi}}\right)^{\beta_2}$ *and* $\underline{\mathcal{E}}(\pi_0\,|\infty,\underline{\pi})=\rho_2\pi_0-\dfrac{\rho_2\underline{\pi}}{\beta_2}\left(\dfrac{\pi_0}{\underline{\pi}}\right)^{\beta_2}$*. In addition,* $\overline{\mathcal{E}}(\pi_0\,|\infty,0)=\rho_1\pi_0$ *and* $\underline{\mathcal{E}}(\pi_0\,|\infty,0)=\rho_2\pi_0$*.*

Henceforth, we focus on results that are useful for industries that are not competitive, so we let $\underline{\pi}=0$ and $\overline{\pi}=\infty$ unless otherwise stated. In order to properly discount one-shot payoffs—such as investment costs or scrap values—attained when the process reaches a certain state, it is also necessary to derive the (conditionally) expected discounted value of a claim to a dollar at the first date at which the process hits a certain state π^* from above or below, starting from π_0. Such value is commonly referred to as "the expected discount factor," and we stick to this terminology in the remainder of the paper. The expected discount factor to be used when the dollar is achieved the first time the process reaches a certain state from below takes the following forms:

Theorem 2 (i) *Suppose that the process is in a growth phase and that the current state is* π_0*. Then the expected discounted value of a claim to a dollar when the process first hits* $\pi^*>\pi_0$ *from below is*

$$\overline{\varphi}_1(\pi_0,\pi^*)=\left(\frac{\pi_0}{\pi^*}\right)^{\beta_1}.$$

(ii) *Suppose that the process is in a decline phase and that the current state is* π_0*. Then the expected discounted value of a claim to a dollar when the process first hits* $\pi^*\geq\pi_0$ *from below is*

$$\underline{\varphi}_1(\pi_0,\pi^*)=\delta_2\left(\frac{\pi_0}{\pi^*}\right)^{\beta_1}.$$

Figure 2 provides a visual illustration of the problem. The process starts growing from π_0 and after several cycles first hits π^* from below at the (random) first passage time T^*. Theorem 2 shows that the expected discounted value of a claim to a dollar attained at the random time T^* is given by $\overline{\varphi}_1(\pi_0,\pi^*)$.

Theorem 2 deals with expected discount factors that are useful when discounting one-shot payoffs that are attained when the process first hits a certain state *from below*. In turn, Theorem 3 deals with the discounting of one-shot payoffs achieved when the process first hits a certain level *from above*:

Theorem 3 (i) *Suppose that the process is in a decline phase and that the current state is* π_0*. Then the expected discounted value of a claim to a dollar when the process*

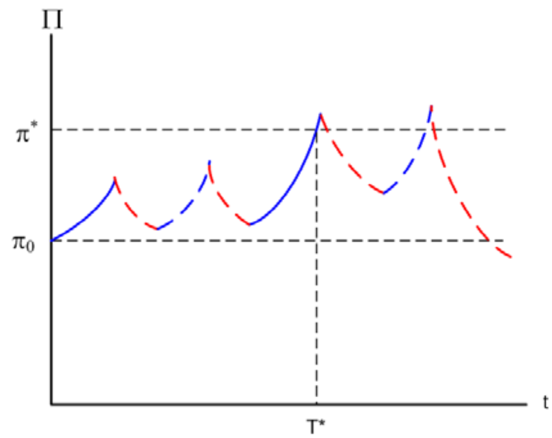

Figure 2: Expected discounted factor conditional upon the process growing at π_0

first hits $\pi^* < \pi_0$ *from above is*

$$\underline{\varphi}_2(\pi_0, \pi^*) = \left(\frac{\pi_0}{\pi^*}\right)^{\beta_2}.$$

(ii) *Suppose that the process is in a growth phase and that the current state is* π_0*. Then the expected discounted value of a claim to a dollar when the process first hits* $\pi^* \leq \pi_0$ *from above is*

$$\overline{\varphi}_2(\pi_0, \pi^*) = \delta_1 \left(\frac{\pi_0}{\pi^*}\right)^{\beta_2}.$$

This problem is represented in Figure 3, which depicts a situation in which the process is in a decline phase and the current state is π_0. It can be observed that, after several cycles, the process first hits π^{**} from above at the (random) first-passage time T^{**}. Theorem 3 shows that the expected discounted value of a claim to a dollar attained at time T^{**} is given by $\underline{\varphi}_2(\pi_0, \pi^{**})$.

We can draw a useful corollary from the previous two theorems:

Corollary 2 (i) *Suppose that the process is in a growth phase and that the current state is* π_0*. Then the expected discounted value of a claim to a dollar when the process first hits* π_0 *from above is* $\overline{\varphi}_2(\pi_0, \pi_0) = \delta_1 \in (0, 1)$*.*

(ii) *Suppose that the process is in a decline phase and that the current state is* π_0*. Then the expected discounted value of a claim to a dollar when the process first hits* π_0 *from below is* $\underline{\varphi}_1(\pi_0, \pi_0) = \delta_2 \in (0, 1)$*.*

To conclude with our results in this section, notice that the expected stream of discounted profits derived in Theorem 1 is based on the hypothesis that an active

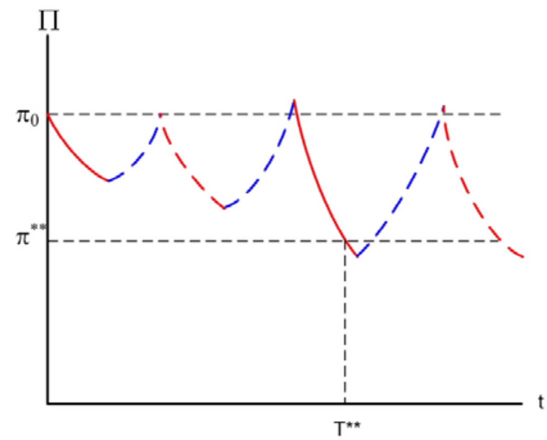

Figure 3: Expected discounted factor conditional upon the process declining at π_0

firm never becomes inactive, which is relevant for completely irreversible investment decisions. Sometimes, as when the firm foresees exiting at some random time in the future, it is also necessary to compute the (conditionally) expected stream of discounted profits harvested until a certain state is first hit by a process subject to no barriers. The following result deals with the expected discounted profit stream while transitioning from the current state π_0 to another one denoted by π^*.

Theorem 4 (*i*) *Suppose that the process is in a decline phase and that the current state is π_0. Then the expected stream of discounted profits while the process transitions from π_0 until it first hits $\pi^* < \pi_0$ from above is*

$$\underline{\mathcal{E}}(\pi_0, \pi^*) = \rho_2 \pi_0 \left[1 - \left(\frac{\pi_0}{\pi^*} \right)^{\beta_2 - 1} \right].$$

(*ii*) *Suppose that the process is in a growth phase and that the current state is π_0. Then the expected stream of discounted profits while the process transitions from π_0 until it first hits $\pi^* \leq \pi_0$ from above is*

$$\overline{\mathcal{E}}(\pi_0, \pi^*) = \rho_1 \pi_0 - \rho_2 \delta_1 \pi_0 \left(\frac{\pi_0}{\pi^*} \right)^{\beta_2 - 1}.$$

Figure 3 illustrates the situation faced by the firm when computing the expected stream of discounted profits while the process transitions from π_0 until it first hits $\pi^{**} \leq \pi_0$ from above, which happens at the first-passage time T^{**}. The formula for computing such expected payoff is given by $\underline{\mathcal{E}}(\pi_0, \pi^{**})$.

It is worth noting that the last statement in Corollary 1 also follows from Theorem 4, since we have that $\underline{\mathcal{E}}(\pi_0, \pi^*) = \underline{\mathcal{E}}(\pi_0 \,|\infty, 0)$ and $\overline{\mathcal{E}}(\pi_0, \pi^*) = \overline{\mathcal{E}}(\pi_0 \,|\infty, 0)$ for $\pi^* = 0$.

This theorem is also useful in drawing a simple but relevant result that will be used when dealing with disinvestment decisions. Letting $\gamma_1 \equiv \rho_1 - \rho_2\delta_1$, which (using the fact that $\rho_1 = \dfrac{\rho_2(r + \lambda_2 - \alpha_2) - 1}{\lambda_2}$) can be rewritten as

$$\gamma_1 = \frac{\rho_2[\lambda_2(1 - \delta_1) + r - \alpha_2] - 1}{\lambda_2}, \tag{2}$$

we have that Theorem 4 leads to the following result:

Corollary 3 *Suppose that the process is in a growth phase and that the current state is* π_0*. Then the expected stream of discounted profits while the process transitions from* π_0 *until it first hits* π_0 *from above is* $\overline{\mathcal{E}}(\pi_0, \pi_0) = \gamma_1\pi_0$*.*

4 Entry decision under uncertainty

The purpose of this section is to characterize the main properties of a single firm's optimal investment behavior and analyze its implications for firm valuation. For this reason, we assume that the firm is not initially active in a stochastically cyclical market such as the one described in Section 2. If the firm decides to undertake the investment and incur a sunk cost $K > 0$, then it is assumed to operate indefinitely, i.e., the value S of the outside option equals 0. (We study combined entry and exit in Section 6.)

The stochastic process of Section 2 is Markovian and homogeneous, so the firm's optimal investment rule for each phase of a cycle is stationary. The firm's entry problem is even simpler because investment does not take place while the market is declining, except for corner solutions which are (implicitly) ruled out to make the analysis nontrivial. This intuitive result is formally stated as follows:

Lemma 1 *The firm's optimal investment policy calls for no investment while the market is declining.*

Proof. See Appendix B. ■

To see why the lemma holds intuitively, suppose that the firm's optimal entry rule called for investment during a decline phase. Given that any profit level that is reached in a declining phase must have been reached in a growth phase, it is clear that the firm could have done better by investing at the same level in the growth phase. The reason is that, in the worst-case scenario, the market would suddenly stop growing and start declining at such level, so the firm should expect to gain more if the market were growing than if it invested immediately in the downturn. This would entail a contradiction.

Therefore, Lemma 1 implies that it suffices to pay attention to phases in which the market is in growth when solving for the firm's optimal investment threshold. Thus, suppose that the market is currently growing and denote the current state of the market by π_0. The firm simply chooses a threshold π_E such that it enters the market the first time the process hits such threshold from below. Hence, the firm solves the following optimization problem:

$$\begin{aligned}\max_{\pi_E} \overline{V}_E(\pi_E|\pi_0) &= [\overline{\mathcal{E}}(\pi_E, 0) - K]\overline{\varphi}_1(\pi_0, \pi_E) \\ &= (\rho_1 \pi_E - K)\left(\frac{\pi_0}{\pi_E}\right)^{\beta_1},\end{aligned}$$

where the last equality makes use of Theorems 2 and 4. Thus, the firm achieves an expected net payoff of $\overline{\mathcal{E}}(\pi_E, 0) - K$ the first time the market reaches level π_E starting from state π_0. For this reason, $\overline{\varphi}_1(\pi_0, \pi_E)$ is the expected discount factor that must be used when discounting this payoff on the state space.

Because $\overline{V}_E(\pi_E|\pi_0)$ is strictly quasi-concave and

$$\frac{\partial \overline{V}_E(\pi_E|\pi_0)}{\partial \pi_E} = [\frac{\rho_1(1-\beta_1)\pi_E + \beta_1 K}{\pi_E}]\left(\frac{\pi_0}{\pi_E}\right)^{\beta_1},$$

easy manipulations lead to the optimal investment threshold and the value of the investment opportunity:

Proposition 1 *Suppose that the market is growing and that the firm is currently inactive. Then the firm's optimal entry rule is to invest as soon as the market reaches state*

$$\pi_E^* = \frac{1}{\rho_1}\frac{\beta_1}{\beta_1 - 1}K, \tag{3}$$

where $\beta_1 = \dfrac{\alpha_1(r+\lambda_2) + \alpha_2(r+\lambda_1) - \sqrt{\Delta}}{2\alpha_1\alpha_2} > 1$. *The value of the firm is*

$$\overline{V}_E^*(\pi_0) \equiv \overline{V}_E(\pi_E^*|\pi_0) = \begin{cases} \left(\dfrac{K}{\beta_1 - 1}\right)^{1-\beta_1}\left(\dfrac{\rho_1\pi_0}{\beta_1}\right)^{\beta_1} & \text{if } \pi_0 \leq \pi_E^* \\ \rho_1\pi_0 - K & \text{if } \pi_0 > \pi_E^* \end{cases}.$$

The optimal investment threshold and the value of the option to invest *conditional upon the market being in a growth phase* are very similar to those derived in a GBM-based setup (c.f. Dixit and Pindyck, 1994, pp. 142-143). Similarly, Proposition 1 implies that the firm chooses to invest when the expected net present value of its investment exceeds $\frac{K}{\beta_1 - 1} > 0$, the opportunity cost of exercising the option to invest.

A major contribution of this paper is to improve our conceptual understanding regarding the optimality of a "wait-and-see" approach. In particular, unlike traditional real options models based on Ito processes, a single firm's problem can be given the following economic interpretation linked to Bernanke's (1983) "bad news principle of irreversible investment":

Proposition 2 *The optimal investment threshold* π_E^* *satisfies the following equation:*

$$\pi_E^* = rK + \lambda_1[\delta_2(\rho_1\pi_E^* - K) - (\rho_2\pi_E^* - K)]. \tag{4}$$

Proof. See Appendix B. ■

Proposition 2 states that the firm must equate the marginal cost and the marginal value of waiting to invest. The left hand side of equation (4) is the profit flow forgone by delaying entry a short period of time, $\pi_E^* dt$, while the right hand side of (4) quantifies the marginal value of waiting, which consists of two components. The first part, $rKdt$, is the part of the investment cost saved by waiting an infinitesimal unit of time. The second component, $\lambda_1 dt[\delta_2(\rho_1\pi_E^* - K) - (\rho_2\pi_E^* - K)]$, is the marginal option value of waiting and stems from Bernanke's "bad news principle of irreversible investment."[11] When the process reaches π_E^*, waiting to invest allows the firm to avoid making a poor investment in case the process switches to decline immediately. This happens with conditional probability $\lambda_1 dt$, and would only bring an expected stream of discounted profits of $\rho_2\pi_E^* - K = \underline{\mathcal{E}}(\pi_E^*, 0) - K$, at the expense of sacrificing the option to invest, which would yield an expected payoff of $\delta_2(\rho_1\pi_E^* - K) = \delta_2\left[\overline{\mathcal{E}}(\pi_E^*, 0) - K\right]$ (accounting for the random-length period of time elapsed until the profit flow grows back to the investment threshold, π_E^*).

To conclude with this section, it only remains to show that the value of an investment opportunity experiences an upward (downward) jump when a decline phase stops (starts). This is a relevant result that gives a rationale for using jump processes in the valuation of stock even if the underlying state variable follows a continuous stochastic process. Note that if the firm has not invested, the value of an inactive

[11] Such principle states that irreversibility yields no advantages but implies some costs because the firm cannot recoup its investment if market conditions turn out to be adverse, which creates the asymmetry that the firm cares only about adverse events—which would not be regrettable were investment reversible—but not favorable ones.

firm switches from $\overline{V}_E^*(\pi_0)$ to $\underline{V}_E^*(\pi_0)$ whenever the market switches from growth to decline (where $\underline{V}_E^*(\pi_0)$ denotes the firm's value when the market is declining at π_0). Because the firm never invests while the market is declining, it follows from Corollary 2 that $\underline{V}_E^*(\pi_0) = \delta_2 \overline{V}_E^*(\pi_0) < \overline{V}_E^*(\pi_0)$. Although the instantaneous profit stays (roughly) at the same level, the firm's value jumps down (up) significantly whenever the market suddenly stops growing (declining).[12] This result that contrasts with that obtained in a context in which the stochastic evolution of profit is governed by a GBM is formally stated as follows:

Proposition 3 *Let π_0 be the current profit flow level. If the firm is inactive, then $\overline{V}_E^*(\pi_0) > \underline{V}_E^*(\pi_0)$.*

5 Exit decision under uncertainty

In this section, a single firm is assumed to be already active in the market. Although in principle it can operate in the market forever, its (indivisible) asset has a one-time opportunity cost of $S > 0$. Moreover, the firm cannot reenter in the future if it exits, i.e., $K = \infty$.

Not surprisingly, and contrary to the entry problem, it can be shown that exit takes place only if the market is declining,[13] so the homogeneity of the (Markov) stochastic process implies that the firm simply chooses a threshold π_X such that the firm exits the market the first time the process hits such threshold from above. Formally, the firm solves the following problem:

$$\begin{aligned} \max_{\pi_X} \underline{V}_X(\pi_X|\pi_0) &= \underline{\mathcal{E}}(\pi_0, \pi_X) + S\underline{\varphi}_2(\pi_0, \pi_X) \\ &= \rho_2 \pi_0 \left[1 - \left(\frac{\pi_0}{\pi_X}\right)^{\beta_2 - 1}\right] + S\left(\frac{\pi_0}{\pi_X}\right)^{\beta_2}. \end{aligned}$$

Thus, starting from a level π_0 at which the market is declining, the firm collects a discounted stream of profits until state $\pi_X < \pi_0$ is first reached (from above). The value of such discounted profit stream while transitioning from π_0 to π_X is

$$\underline{\mathcal{E}}(\pi_0, \pi_X) = \rho_2 \pi_0 \left[1 - (\pi_0/\pi_X)^{\beta_2 - 1}\right],$$

[12] It is worth remarking that this result holds even if the firm becomes active, since $\rho_1 > \rho_2$.

[13] The argument is identical in spirit to the one used for entry, and it makes use of the fact that $r > \dfrac{\beta_2}{(\beta_2 - 1)\rho_2}$, which is shown in the proof of Proposition 6.

as shown by Theorem 4. However, there is an additional source of value, since the firm seizes the outside option, whose value is S, when state π_X is first reached. The proper expected discount factor, given that the market is in a decline phase, is

$$\underline{\varphi}_2(\pi_0, \pi_X) = (\pi_0/\pi_X)^{\beta_2} ,$$

as shown by Theorem 3.

Noticing that $\underline{V}_X(\pi_X|\pi_0)$ is strictly quasi-concave and

$$\frac{\partial \underline{V}_X(\pi_X|\pi_0)}{\partial \pi_X} = [\frac{\rho_2(\beta_2 - 1)\pi_X - \beta_2 S}{\pi_X}] \left(\frac{\pi_0}{\pi_X}\right)^{\beta_2} ,$$

it is easy to prove the following results:

Proposition 4 *Suppose that the market is declining and that the firm is currently active. Then the firm's optimal exit rule is to disinvest as soon as the market reaches state*

$$\pi_X^* = \frac{1}{\rho_2}\frac{\beta_2}{\beta_2 - 1}S, \tag{5}$$

where $\beta_2 = \dfrac{\alpha_1(r+\lambda_2) + \alpha_2(r+\lambda_1) + \sqrt{\Delta}}{2\alpha_1\alpha_2} < 0$. *The value of the firm is*

$$\underline{V}_X^*(\pi_0) \equiv \underline{V}_X(\pi_X^*|\pi_0) = \begin{cases} \rho_2\pi_0 + (S - \rho_2\pi_X^*)\left(\dfrac{\pi_0}{\pi_X^*}\right)^{\beta_2} & \text{if } \pi_0 \geq \pi_X^* \\ S & \text{if } \pi_0 < \pi_X^* \end{cases} .$$

Similarly to the canonical real options literature, Proposition 4 implies that the firm exits when the expected net present value of its disinvestment falls below $S/(\beta_2 - 1) < 0$, since there exists an opportunity cost for exercising the option to disinvest. As with the entry problem, the value of the option to disinvest *conditional upon the market being in a decline phase* is quite similar to that derived in a GBM setting. The same applies to the optimal disinvestment threshold. Paralleling the entry case, a single firm's optimal exit rule lends itself to an interpretation in the light of the "bad news principle:"

Proposition 5 *The optimal disinvestment threshold* π_X^* *satisfies the following equation:*

$$rS = \pi_X^* + \lambda_2\left(\gamma_1\pi_X^* + \delta_1 S - S\right). \tag{6}$$

Proof. See Appendix B. ■

When deciding whether or not to exit the market, the firm must compare the marginal value and the marginal cost of waiting. By delaying exit a short time period of length dt, the firm forgoes earning an interest of (approximately) $rSdt$. In turn, the marginal value of waiting to disinvest consists of two components. Delaying exit a little bit allows the firm to reap a flow of profits equal to $\pi_X^* dt$. There is also a marginal option value of waiting to disinvest, which stems from avoiding making a poor disinvestment decision. With conditional probability $\lambda_2 dt$, the market switches to growth, so waiting would allow the firm to remain in operation and keep the option to exit in the future alive, which is worth $\gamma_1 \pi_X^* + \delta_1 S$, at the expense of sacrificing S.[14]

To conclude with this section, we show that the value of a disinvestment opportunity experiences an upward (downward) jump if a declining phase ends (begins). Our next proposition establishes this result, based on a different argument to the one that gives rise to Proposition 3. Thus, while the jump in the firm's value for a single potential entrant is simply due to the discount factor related to the delay in entry caused by a change from growth to decline, the abrupt change of value for an active firm is the result of a trade-off between the expected stream of discounted profits reaped and the delayed recovery of the outside value. More specifically, if the market is declining at $\pi_0 > \pi_X^*$, the firm has a value equal to $\underline{V}_X^*(\pi_0)$. However, if the market switches to growth at π_0, then the firm's value function becomes $\overline{V}_X^*(\pi_0) = \gamma_1 \pi_0 + \delta_1 \underline{V}_X^*(\pi_0)$, because the firm never exits a growing market and hence must wait at least until the profit flow comes back to the same state. The first term, $\gamma_1 \pi_0$, denotes the expected stream of discounted profits collected while the market moves from state π_0 until it first comes back to π_0 from above (see Corollary 3). The second term, $\delta_1 \underline{V}_X^*(\pi_0)$, denotes the expected continuation value after the market declines back to state π_0 (see Corollary 2). By comparing $\underline{V}_X^*(\pi_0)$ and $\overline{V}_X^*(\pi_0)$, we have the following result that does not obtain in a GBM setting:

Proposition 6 *Let π_0 be the current state of the market. If the firm is active, then $\overline{V}_X^*(\pi_0) > \underline{V}_X^*(\pi_0)$.*

Proof. See Appendix B. ■

[14]Note from Corollary 3 that $\gamma_1 \pi_X^*$ quantifies the expected stream of discounted profits while the market transitions from a state π_X^* in which the market is growing until such state is first hit from above. In turn, note from Corollary 2 that $\delta_1 S$ quantifies the expected discounted value of getting an asset worth S when the market first reaches state π_X^* from above, given that the market is currently at a growing phase whose state is π_X^*.

6 Entry and exit decisions under uncertainty

This section extends the analysis of the previous two sections by allowing for combined entry and exit decisions made by a single firm. Reentry is not permitted, and we assume that $K > S$ to avoid "money pumps." The exit decision has just been studied (see Proposition 4), so it simply remains to examine how the entry decision is affected by the fact that an active firm exits the first time the process hits state $\pi_X^* = \dfrac{\beta_2 S}{(\beta_2 - 1)\rho_2}$ from above. It is standard to show that entry takes place only if the market is in growth, so taking into account that the value of an active firm when the state is π_0 equals

$$\underline{V}_X^*(\pi_0) = \rho_2\pi_0 + (S - \rho_2\pi_X^*)\,(\pi_0/\pi_X^*)^{\beta_2}\,,$$

and assuming that the entry threshold π_E chosen by the firm exceeds π_X^* (as can be shown to hold at the optimal solution), we have that the value of an inactive firm at state π_0 if the market is growing is:

$$\overline{V}_E(\pi_E|\pi_0) = [\overline{\mathcal{E}}(\pi_E, \pi_E) + \underline{V}_X^*(\pi_E)\overline{\varphi}_2\,(\pi_E, \pi_E) - K]\overline{\varphi}_1\,(\pi_0, \pi_E)\,.$$

Thus, the firm attains an expected payoff when the market first reaches level π_E starting from state π_0 (see bracketed terms), which must be discounted back to the current date using the expected discount factor $\overline{\varphi}_1\,(\pi_0, \pi_E)$. Note that the payoff that the firm expects to gain at the time of entry equals the sum of two terms minus the investment cost: $\overline{\mathcal{E}}(\pi_E, \pi_E)$ is the expected stream of discounted profits until the market first reaches level π_E from above, while $\underline{V}_X^*(\pi_E)\overline{\varphi}_2\,(\pi_E, \pi_E)$ is the properly discounted value of a firm active in a declining market whose state is π_E.

Using Corollary 3 (taking into account that $\gamma_1 = \rho_1 - \delta_1\rho_2$, since $\rho_2 = \dfrac{1 + \lambda_2\rho_1}{r + \lambda_2 - \alpha_2}$), Corollary 2 and Theorem 2 allows us to rewrite the above expression as follows:

$$\overline{V}_E(\pi_E|\pi_0) = \left[\rho_1\pi_E + \delta_1(S - \rho_2\pi_X^*)\left(\frac{\pi_E}{\pi_X^*}\right)^{\beta_2} - K\right]\left(\frac{\pi_0}{\pi_E}\right)^{\beta_1}. \tag{7}$$

Because $\overline{V}_E(\pi_E|\pi_0)$ is strictly quasi-concave for $\pi_E > \pi_X^*$ and

$$\frac{\partial \overline{V}_E(\pi_E|\pi_0)}{\partial \pi_E} = \left[\frac{(1-\beta_1)\rho_1\pi_E + (\beta_2 - \beta_1)\delta_1(S - \rho_2\pi_X^*)\,(\pi_E/\pi_X^*)^{\beta_2} + \beta_1 K}{\pi_E}\right]\left(\frac{\pi_0}{\pi_E}\right)^{\beta_1},$$

it follows that the optimal entry threshold is the level π_E^* that solves the following equation:

$$(1-\beta_1)\rho_1\pi_E^* + (\beta_2-\beta_1)\delta_1(S-\rho_2\pi_X^*)\left(\frac{\pi_E^*}{\pi_X^*}\right)^{\beta_2} + \beta_1 K = 0. \tag{8}$$

It can be shown that there exists a unique value $\pi_E^* > \pi_X^*$ that solves the previous expression,[15] so plugging expression (8) into (7) and using that $S - \rho_2\pi_X^* = \dfrac{S}{1-\beta_2}$ yields after some manipulations:

$$\begin{aligned}\overline{V}_E^*(\pi_0) &\equiv \overline{V}_E(\pi_E^*|\pi_0) = \left[\rho_1\pi_E^* + \delta_1(S-\rho_2\pi_X^*)\left(\frac{\pi_E^*}{\pi_X^*}\right)^{\beta_2} - K\right]\left(\frac{\pi_0}{\pi_E^*}\right)^{\beta_1} \\ &= \left[\frac{K}{\beta_1-1} - \frac{\delta_1 S}{\beta_1-1}\left(\frac{\pi_E^*}{\pi_X^*}\right)^{\beta_2}\right]\left(\frac{\pi_0}{\pi_E^*}\right)^{\beta_1}.\end{aligned}$$

Taking into account Propositions 4 and 5, we now state the previous results formally.

Proposition 7 *Suppose that the market is growing and that the firm is currently inactive. If the firm can exit but cannot reenter, then the firm's optimal entry rule is to invest as soon as the market reaches the state π_E^* such that*

$$(1-\beta_1)\rho_1\pi_E^* + (\beta_2-\beta_1)\delta_1(S-\rho_2\pi_X^*)\left(\frac{\pi_E^*}{\pi_X^*}\right)^{\beta_2} + \beta_1 K = 0$$

holds. The value of the firm is

$$\overline{V}_E^*(\pi_0) = \begin{cases} \left[\dfrac{K}{\beta_1-1} - \dfrac{\delta_1 S}{\beta_1-1}\left(\dfrac{\pi_E^*}{\pi_X^*}\right)^{\beta_2}\right]\left(\dfrac{\pi_0}{\pi_E^*}\right)^{\beta_1} & \text{if } \pi_0 < \pi_E^* \\ \rho_1\pi_0 + \delta_1\left(S-\rho_2\pi_X^*\right)\left(\dfrac{\pi_0}{\pi_X}\right)^{\beta_2} - K & \text{if } \pi_0 \geq \pi_E^* \end{cases}.$$

As usual, the optimal entry threshold can be interpreted based on the bad news principle of irreversible investment:

Proposition 8 *The optimal entry threshold π_E^* satisfies the following equation:*

$$\pi_E^* = rK + \lambda_1\left[\delta_2\overline{V}_E^*(\pi_E^*) - (\underline{V}_X^*(\pi_E^*) - K)\right].$$

Proof. See Appendix B. ■

[15] The proof is somewhat involved and is omitted for the sake of brevity, but it can be made available from the authors upon request.

7 Entry and exit in competitive industries

In this section, we apply the results developed in Section 3 to perfectly competitive industries in which there are many firms that sell a homogeneous good and act as price-takers. We assume that any firm can operate a single unit of capacity whose investment cost is $K > 0$ and whose redeployment value equals $S \in [0, K)$. Operating costs are set equal to zero, and the demand elasticity is large enough so that any active firm produces at full capacity. We suppose that the (inverse) market demand equals $\Pi = \pi D(Q)$, where Π and Q respectively denote the market price and the number of active firms. Furthermore, $D(\cdot)$ is a known decreasing function, whereas π is assumed to follow the stochastic process introduced in Section 2.

As explained in much more detail in Leahy (1993), the stationarity of the stochastic process that determines the (short-run) market price at any time for a given Q implies that there will be a threshold $\overline{\pi} \leq \infty$ that will trigger entry by inactive firms as long as price is high enough (i.e., $\pi D(Q) \geq \overline{\pi}$). Similarly, there will be a threshold $\underline{\pi} \geq 0$ that will trigger exit by active firms as long as price is low enough (i.e., $\pi D(Q) \leq \underline{\pi}$). (The extent of entry and exit for a given number of (in)active firms will be given by the specific properties of $D(\cdot)$ so as to keep the price at a constant level.) Noting that, given our assumptions, a firm's profit coincides with the short-run market price, we have that the profit variable Π inherits exactly the same stochastic behavior of variable π except for two aspects. On the one hand, if π is going through a growth phase, then entry by other firms implies that Π stays constant at level $\overline{\pi} \in (0, \infty]$ if it ever reaches such state, and it leaves $\overline{\pi}$ only once the process for π starts declining. On the other hand, if π is going through a decline phase, then exit by other firms implies that Π stays constant at level $\underline{\pi} \in [0, \infty)$ if it ever reaches such state, and it leaves $\underline{\pi}$ only once the process for π starts growing. As a result, the variable Π follows the same process described in Section 2 except that it can never go higher than $\overline{\pi}$ and lower than $\underline{\pi}$.[16] If all firms have rational expectations about π and other firms' entry/exit rules, then these properties of the stochastic variable Π are known, and in addition, all firms know that enough entry (exit) takes place so as to keep Π at the upper (lower) barrier while π is growing (declining).

Theorem 1 is particularly powerful in this case in which we have to find out the values of $\overline{\pi}$ and $\underline{\pi}$. Clearly, free entry and exit imply that the values of $\overline{\pi}$ and $\underline{\pi}$ in

[16]The preceding discussion relies on the implicit assumption that entry never takes place while π is in a decline phase, whereas exit never takes place while π is in a growth phase. This can be shown to hold in any dynamic equilibrium of a competitive industry.

equilibrium must be such that

$$\overline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi}\,) = K \tag{9}$$

and

$$\underline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi}\,) = S. \tag{10}$$

When $S = 0$ and $K < \infty$, it is clear that $\underline{\pi} = 0$, so Corollary 1 applied on (9) implies that

$$\rho_1\overline{\pi} - \frac{\rho_1\overline{\pi}}{\beta_1}\left(\frac{\overline{\pi}}{\overline{\overline{\pi}}}\right)^{\beta_1} = K,$$

that is,

$$\overline{\pi} = \pi_E^* = \frac{1}{\rho_1}\frac{\beta_1}{\beta_1 - 1}K.$$

This is a similar result to that obtained when price follows a Geometric Brownian Motion: the competitive entry threshold coincides with that of a single firm. This shows that the results and reasoning in Leahy (1993) for diffusion processes (such as the GBM) extend to stochastically cyclical markets, albeit in quite a straightforward manner (i.e., as a direct application of Corollary 1). As discussed by Leahy (1993), competition reduces the value of the option to build one unit of capacity by reducing the value of installed capacity. These two effects cancel out each other in equilibrium, which explains why the competitive entry threshold coincides with that of a single firm. One more result that can be derived when $S = 0$ and $K < \infty$ is that $\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},0\,) > \underline{\mathcal{E}}(\pi_0\,|\overline{\pi},0\,)$,[17] so the value of an active firm experiences jumps upwards (downwards) whenever a growth (decline) phase starts, and hence entry rates discretely drop to zero if $\Pi = \overline{\pi}$ and all of a sudden a decline phase starts. As in the single firm case, this result does not obtain for diffusion processes.

Similarly, when $S > 0$ and $K = \infty$, we have that $\overline{\pi} = \infty$, so Corollary 1 applied

[17] Using the facts that $\delta_2 = (r + \lambda_1 - \alpha_1\beta_1)/\lambda_1 \in (0,1)$ and $1 + \lambda_1\rho_2 = (\lambda_1 + r - \alpha_1)\,\rho_1$ (by (35)), we have that

$$\begin{aligned}\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},0\,) - \underline{\mathcal{E}}(\pi_0\,|\overline{\pi},0\,) &= (\rho_1 - \rho_2)\pi_0 - \frac{(1-\delta_2)\rho_1\pi_0}{\beta_1}\left(\frac{\pi_0}{\overline{\pi}}\right)^{\beta_1 - 1} \geq (\rho_1 - \rho_2 - \frac{(1-\delta_2)\rho_1}{\beta_1})\pi_0 \\ &= (\lambda_1\rho_1 - \lambda_1\rho_2 - \alpha_1\rho_1 + \frac{r\rho_1}{\beta_1})\frac{\pi_0}{\lambda_1} = \left(\frac{\beta_1}{(\beta_1 - 1)\rho_1} - r\right)\frac{(\beta_1 - 1)\rho_1\pi_0}{\lambda_1\beta_1},\end{aligned}$$

so the result follows because $\dfrac{\beta_1}{(\beta_1 - 1)\,\rho_1} > r$ (as shown in the proof of Lemma 1).

on (10) implies that

$$\rho_2\underline{\pi} - \frac{\rho_2\underline{\pi}}{\beta_2}\left(\frac{\underline{\pi}}{\underline{\pi}}\right)^{\beta_2} = S,$$

so

$$\underline{\pi} = \pi_X^* = \frac{1}{\rho_2}\frac{\beta_2}{\beta_2 - 1}S.$$

Again, this is an identical result to that obtained when price follows a Geometric Brownian Motion: the competitive exit threshold coincides with that of a single firm (as in Leahy 1993). Furthermore, $\overline{\mathcal{E}}(\pi_0\,|\infty, \underline{\pi}) > \underline{\mathcal{E}}(\pi_0\,|\infty, \underline{\pi})$,[18] so the value of an active firm experiences jumps upwards (downwards) whenever a growth (decline) phase starts, and hence exit rates discretely drop to zero if $\Pi = \underline{\pi}$ and all of a sudden a growth phase starts.

We summarize all these results as follows:

Proposition 9 *Suppose that $S = 0$ and $K < \infty$. Then a competitive firm enters whenever the market reaches state $\pi_E^* = \frac{1}{\rho_1}\frac{\beta_1}{\beta_1 - 1}K$ from below, and entry rates discontinuously fall from positive levels to zero owing to the start of a decline phase. Suppose that $S > 0$ and $K = \infty$. Then a competitive firm exits whenever the market reaches state $\pi_X^* = \frac{1}{\rho_2}\frac{\beta_2}{\beta_2 - 1}K$ from above, and exit rates discontinuously fall from positive levels to zero owing to the start of a growth phase. It holds both when $S = 0$ and $K < \infty$ and when $S > 0$ and $K = \infty$ that the value of an active firm experiences jumps upwards (downwards) whenever a growth (decline) phase starts.*

When it holds that $0 < S < K < \infty$, it is no longer possible to provide a closed-form solution to $\overline{\pi}$ and $\underline{\pi}$ by solving the system of equations (9) and (10), but it is easy to show that $\pi^* \equiv \frac{\overline{\pi}}{\underline{\pi}}$ solves the following equation:

$$\pi^*\left(\frac{\frac{\delta_1\rho_2(\pi^*)^{\beta_2-1} - \rho_1}{\beta_1\left(1 - \delta_1\delta_2(\pi^*)^{\beta_2-\beta_1}\right)} + \frac{\delta_1\delta_2\rho_1(\pi^*)^{\beta_2-\beta_1} - \delta_1\rho_2(\pi^*)^{\beta_2-1}}{\beta_2\left(1 - \delta_1\delta_2(\pi^*)^{\beta_2-\beta_1}\right)} + \rho_1}{\frac{\delta_1\delta_2\rho_2(\pi^*)^{\beta_2-\beta_1} - \delta_2\rho_1(\pi^*)^{1-\beta_1}}{\beta_1\left(1 - \delta_1\delta_2(\pi^*)^{\beta_2-\beta_1}\right)} + \frac{\delta_2\rho_1(\pi^*)^{1-\beta_1} - \rho_2}{\beta_2\left(1 - \delta_1\delta_2(\pi^*)^{\beta_2-\beta_1}\right)} + \rho_2}\right) = \frac{K}{S}.$$

Numerical solutions can be easily obtained, and jumps in firm value also happen whenever a growth/decline phase starts. Also, entry and exit rates exhibit sudden

[18] The proof of this result parallels that of $\overline{\mathcal{E}}(\pi_0\,|\overline{\pi}, 0) > \underline{\mathcal{E}}(\pi_0\,|\overline{\pi}, 0)$ except that it uses the facts that $\delta_1 = (r + \lambda_2 - \alpha_2\beta_2)/\lambda_2 \in (0,1)$, $1 + \lambda_2\rho_1 = (\lambda_2 + r - \alpha_2)\,\rho_2$ and $r > \frac{\beta_2}{(\beta_2 - 1)\,\rho_2}$ (which is shown in the proof of Proposition 6).

drops to zero whenever a growth and decline phase ends, so entry and exit waves may be observed to suddenly vanish.

8 Conclusion

This paper has modeled the stochastic evolution of markets that exhibit a somewhat unpredictable cyclical behavior. We have studied a firm's optimal (dis)investment behavior in this type of markets in which flow profit alternates between growth and decline phases at random times, and we have shown that, even though the sample path of profits is continuous, firm value experiences jumps upwards (downwards) whenever a growth (decline) phase starts. This holds regardless of whether the firm has an option to invest or an option to disinvest. We have also provided an economic interpretation of a single firm's optimal entry/exit rules in the light of the bad news principle of irreversible investment, thus strengthening the conceptual foundation of the theory of real options. In addition, we have shown that entry (exit) rates in a competitive industry discontinuously fall from positive levels to zero owing to the end of a growth (decline) phase.

There are at least a couple of aspects that are worthwhile emphasizing about our framework and that may prove to be useful for future work on the properties of irreversible (dis)investment in stochastically cyclical markets. On the one hand, we would like to remark that our setting does not require making use of the heavy mathematical apparatus traditionally employed by the real options literature. For instance, direct applications of our theorems allow us to avoid using stochastic calculus and even dynamic programming techniques when solving (dis)investment models that involve lumpiness. As we have shown, (dis)investment timing problems can be directly formulated and solved using ordinary calculus and are amenable to an economic interpretation. Indeed, if reentry is not allowed, we have shown that it is relatively simple to solve analytically the problem faced by a single firm that must choose when to enter and when to exit. Extensions to settings with multiple firms are straightforward, as we show for the case of a competitive industry.

Another aspect worthwhile pointing out is that the model can be enriched in order to improve its explanatory power, possibly at the expense of analytical tractability. Thus, when a switching date is realized, the sample path of the process that governs profit evolution is assumed to change to a different but known growth rate. Assuming that the growth rate is random may lead to possibly different (dis)investment dynamics given a market evolution that need not be stochastically cyclical despite the regime shifts. This may prove fruitful for the construction of random processes that better represent the stochastic dynamics of a variety of economic and financial variables. Both theoretical and empirical work may benefit from pursuing this promising research avenue.

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Appendix A

Proof of Theorem 1. Given our (memoryless) assumptions on the independent random variables involved, $\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi})$ and $\underline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi})$ can be related as follows:

$$\begin{aligned}\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi}) \;=\;& \int_{\frac{1}{\alpha_1}\ln(\overline{\pi}/\pi_0)}^{\infty} \lambda_1 e^{-\lambda_1\tau_1}\left(\int_0^{\frac{1}{\alpha_1}\ln(\overline{\pi}/\pi_0)} \pi_0 e^{\alpha_1 s}e^{-rs}ds + \overline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi})e^{-\frac{r}{\alpha_1}\ln(\overline{\pi}/\pi_0)}\right)d\tau_1 + \\ & \int_0^{\frac{1}{\alpha_1}\ln(\overline{\pi}/\pi_0)} \lambda_1 e^{-\lambda_1\tau_1}\left(\int_0^{\tau_1}\pi_0 e^{\alpha_1 s}e^{-rs}ds + \underline{\mathcal{E}}(\pi_0 e^{\alpha_1\tau_1}\,|\overline{\pi},\underline{\pi})e^{-r\tau_1}\right)d\tau_1 \qquad (11)\end{aligned}$$

and

$$\begin{aligned}\underline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi}) \;=\;& \int_{\frac{1}{\alpha_2}\ln(\underline{\pi}/\pi_0)}^{\infty} \lambda_2 e^{-\lambda_2\tau_2}\left(\int_0^{\frac{1}{\alpha_2}\ln(\underline{\pi}/\pi_0)} \pi_0 e^{\alpha_2 s}e^{-rs}ds + \underline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi})e^{-\frac{r}{\alpha_2}\ln(\underline{\pi}/\pi_0)}\right)d\tau_2 + \\ & \int_0^{\frac{1}{\alpha_2}\ln(\underline{\pi}/\pi_0)} \lambda_2 e^{-\lambda_2\tau_2}\left(\int_0^{\tau_2}\pi_0 e^{\alpha_2 s}e^{-rs}ds + \overline{\mathcal{E}}(\pi_0 e^{\alpha_2\tau_2}\,|\overline{\pi},\underline{\pi})e^{-r\tau_2}\right)d\tau_2. \qquad (12)\end{aligned}$$

To see how $\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi})$ arises, note that there is some chance that the process will hit the barrier $\overline{\pi}$ during the current growth phase, at $\tau_1 = \dfrac{\ln(\overline{\pi}/\pi_0)}{\alpha_1}$. In such a case, the firm harvests a stream of profits that must be discounted back to the current date, together with an asset worth $\overline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi})e^{-\frac{r}{\alpha_1}\ln(\overline{\pi}/\pi_0)}$. However, if the process starts declining at some level lower than $\overline{\pi}$, then the firm gets a stream of discounted profits until the switching date τ_1, together with an asset worth $\underline{\mathcal{E}}(\pi_0 e^{\alpha_1\tau_1}\,|\overline{\pi},\underline{\pi})e^{-r\tau_1}$. Regarding $\underline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi})$, there is some chance that the process will hit the barrier $\underline{\pi}$ during the current decline phase (at $\tau_2 = \dfrac{\ln(\underline{\pi}/\pi_0)}{\alpha_2}$), so in such a case the firm harvests a stream of profits that must be discounted back to the current date, together with an asset worth $\underline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi})e^{-\frac{r}{\alpha_2}\ln(\underline{\pi}/\pi_0)}$. However, if the process starts growing at some level higher than $\underline{\pi}$, then the firm gets a stream of discounted profits until the switching date τ_2, together with an asset worth $\overline{\mathcal{E}}(\pi_0 e^{\alpha_2\tau_2}\,|\overline{\pi},\underline{\pi})e^{-r\tau_2}$.

To solve the system of functional equations that consists of (11) and (12), guess that $\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi}) = C\pi_0^{\mu_1} + F\pi_0^{\mu_2} + M\pi_0$ and $\underline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi}) = H\pi_0^{\mu_1} + D\pi_0^{\mu_2} + N\pi_0$ (where μ_1 and μ_2 are constants to be found out, whereas C, F, M, H, D and N do not depend on π_0 although any may depend on $\overline{\pi}$ or $\underline{\pi}$).

Plugging the assumed functional forms of $\underline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi})$ and $\overline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi})$ into (11) and performing some manipulations yields:

$$
\begin{aligned}
\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi}) &= \int_{\frac{1}{\alpha_1}\ln(\overline{\pi}/\pi_0)}^{\infty} \lambda_1 e^{-\lambda_1\tau_1}\left(\int_0^{\frac{1}{\alpha_1}\ln(\overline{\pi}/\pi_0)} \pi_0 e^{\alpha_1 s}e^{-rs}ds + \overline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi})e^{-\frac{r}{\alpha_1}\ln(\overline{\pi}/\pi_0)}\right)d\tau_1 + \\
&\int_0^{\frac{1}{\alpha_1}\ln(\overline{\pi}/\pi_0)} \lambda_1 e^{-\lambda_1\tau_1}\left(\int_0^{\tau_1}\pi_0 e^{\alpha_1 s}e^{-rs}ds + \underline{\mathcal{E}}(\pi_0 e^{\alpha_1\tau_1}\,|\overline{\pi},\underline{\pi})e^{-r\tau_1}\right)d\tau_1 \\
&= \frac{\pi_0((\frac{\pi_0}{\overline{\pi}})^{\frac{\lambda_1}{\alpha_1}} - (\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}})}{r-\alpha_1} + \overline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi})\left(\frac{\pi_0}{\overline{\pi}}\right)^{\frac{r+\lambda_1}{\alpha_1}} + \\
&\frac{\pi_0(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{\lambda_1}{\alpha_1}})}{r-\alpha_1} - \frac{\lambda_1\pi_0(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}})}{(r-\alpha_1)(r+\lambda_1-\alpha_1)} + \frac{\lambda_1 H\pi_0^{\mu_1}(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1\mu_1}{\alpha_1}})}{r+\lambda_1-\alpha_1\mu_1} + \\
&\frac{\lambda_1 D\pi_0^{\mu_2}(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1\mu_2}{\alpha_1}})}{r+\lambda_1-\alpha_1\mu_2} + \frac{\lambda_1 N\pi_0(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}})}{r+\lambda_1-\alpha_1} \\
&= \frac{\pi_0(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}})}{r-\alpha_1} + (C\overline{\pi}^{\mu_1} + F\overline{\pi}^{\mu_2} + M\overline{\pi})\left(\frac{\pi_0}{\overline{\pi}}\right)^{\frac{r+\lambda_1}{\alpha_1}} - \\
&\frac{\lambda_1\pi_0(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}})}{(r-\alpha_1)(r+\lambda_1-\alpha_1)} + \frac{\lambda_1 H\pi_0^{\mu_1}(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1\mu_1}{\alpha_1}})}{r+\lambda_1-\alpha_1\mu_1} + \\
&\frac{\lambda_1 D\pi_0^{\mu_2}(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1\mu_2}{\alpha_1}})}{r+\lambda_1-\alpha_1\mu_2} + \frac{\lambda_1 N\pi_0(1-(\frac{\pi_0}{\overline{\pi}})^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}})}{r+\lambda_1-\alpha_1} \\
&= \frac{\lambda_1 H\pi_0^{\mu_1}}{r+\lambda_1-\alpha_1\mu_1} + (C - \frac{\lambda_1 H}{r+\lambda_1-\alpha_1\mu_1})\pi_0^{\mu_1}\left(\frac{\pi_0}{\overline{\pi}}\right)^{\frac{r+\lambda_1-\alpha_1\mu_1}{\alpha_1}} + \\
&\frac{\lambda_1 D\pi_0^{\mu_2}}{r+\lambda_1-\alpha_1\mu_2} + (F - \frac{\lambda_1 D}{r+\lambda_1-\alpha_1\mu_2})\pi_0^{\mu_2}\left(\frac{\pi_0}{\overline{\pi}}\right)^{\frac{r+\lambda_1-\alpha_1\mu_2}{\alpha_1}} + \\
&\frac{\pi_0(1+\lambda_1 N)}{r+\lambda_1-\alpha_1} + (M - \frac{1+\lambda_1 N}{r+\lambda_1-\alpha_1})\pi_0\left(\frac{\pi_0}{\overline{\pi}}\right)^{\frac{r+\lambda_1-\alpha_1}{\alpha_1}}.
\end{aligned}
$$

Because we supposed that $\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi}) = C\pi_0^{\mu_1} + F\pi_0^{\mu_2} + M\pi_0$, we must have that the following is satisfied:

$$C = \frac{\lambda_1 H}{r+\lambda_1-\alpha_1\mu_1}, \tag{13}$$

$$F = \frac{\lambda_1 D}{r+\lambda_1-\alpha_1\mu_2}, \tag{14}$$

and

$$M = \frac{1+\lambda_1 N}{r+\lambda_1-\alpha_1}. \tag{15}$$

On the other hand, plugging the assumed functional forms of $\overline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi})$ and

$\underline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi})$ into (12) and performing some manipulations yields:

$$\begin{aligned}
\underline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi}) &= \int_{\frac{1}{\alpha_2}\ln(\underline{\pi}/\pi_0)}^{\infty} \lambda_2 e^{-\lambda_2\tau_2}\left(\int_0^{\frac{1}{\alpha_2}\ln(\underline{\pi}/\pi_0)} \pi_0 e^{\alpha_2 s}e^{-rs}ds + \underline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi})e^{-\frac{r}{\alpha_2}\ln(\underline{\pi}/\pi_0)}\right)d\tau_2 + \\
&\int_0^{\frac{1}{\alpha_2}\ln(\underline{\pi}/\pi_0)} \lambda_2 e^{-\lambda_2\tau_2}\left(\int_0^{\tau_2} \pi_0 e^{\alpha_2 s}e^{-rs}ds + \overline{\mathcal{E}}(\pi_0 e^{\alpha_2\tau_2}\,|\overline{\pi},\underline{\pi})e^{-r\tau_2}\right)d\tau_2 \\
&= \frac{\pi_0(\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{\lambda_2}{\alpha_2}} - \left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}})}{r-\alpha_2} + \underline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi})\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2}{\alpha_2}} + \\
&\frac{\pi_0(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{\lambda_2}{\alpha_2}})}{r-\alpha_2} - \frac{\lambda_2\pi_0(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}})}{(r-\alpha_2)(r+\lambda_2-\alpha_2)} + \frac{\lambda_2 C\pi_0^{\mu_1}(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2\mu_1}{\alpha_2}})}{r+\lambda_2-\alpha_2\mu_1} + \\
&\frac{\lambda_2 F\pi_0^{\mu_2}(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2\mu_2}{\alpha_2}})}{r+\lambda_2-\alpha_2\mu_2} + \frac{\lambda_2 M\pi_0(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}})}{r+\lambda_2-\alpha_2} \\
&= \frac{\pi_0(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}})}{r-\alpha_2} + (H\underline{\pi}^{\mu_1} + D\underline{\pi}^{\mu_2} + N\underline{\pi})\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2}{\alpha_2}} - \\
&\frac{\lambda_2\pi_0(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}})}{(r-\alpha_2)(r+\lambda_2-\alpha_2)} + \frac{\lambda_2 C\pi_0^{\mu_1}(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2\mu_1}{\alpha_2}})}{r+\lambda_2-\alpha_2\mu_1} + \\
&\frac{\lambda_2 F\pi_0^{\mu_2}(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2\mu_2}{\alpha_2}})}{r+\lambda_2-\alpha_2\mu_2} + \frac{\lambda_2 M\pi_0(1-\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}})}{r+\lambda_2-\alpha_2} \\
&= \frac{\lambda_2 C\pi_0^{\mu_1}}{r+\lambda_2-\alpha_2\mu_1} + (H - \frac{\lambda_2 C}{r+\lambda_2-\alpha_2\mu_1})\pi_0^{\mu_1}\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2\mu_1}{\alpha_2}} + \\
&\frac{\lambda_2 F\pi_0^{\mu_2}}{r+\lambda_2-\alpha_2\mu_2} + (D - \frac{\lambda_2 F}{r+\lambda_2-\alpha_2\mu_2})\pi_0^{\mu_2}\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2\mu_2}{\alpha_2}} + \\
&\frac{\pi_0(1+\lambda_2 M)}{r+\lambda_2-\alpha_2} + (N - \frac{1+\lambda_2 M}{r+\lambda_2-\alpha_2})\pi_0\left(\frac{\pi_0}{\underline{\pi}}\right)^{\frac{r+\lambda_2-\alpha_2}{\alpha_2}}.
\end{aligned}$$

Because we supposed that $\underline{\mathcal{E}}(\pi_0\,|\overline{\pi},\underline{\pi}) = H\pi_0^{\mu_1} + D\pi_0^{\mu_2} + N\pi_0$, we must have that the following is satisfied:

$$H = \frac{\lambda_2 C}{r+\lambda_2-\alpha_2\mu_1}, \tag{16}$$

$$D = \frac{\lambda_2 F}{r+\lambda_2-\alpha_2\mu_2}, \tag{17}$$

and

$$N = \frac{1+\lambda_2 M}{r+\lambda_2-\alpha_2}. \tag{18}$$

Using (15) and (18) yields that $M = \rho_1$ and $N = \rho_2$. It is easy to show that we cannot have $C = 0$ (or $H = 0$), so (13) and (16) imply that μ_1 solves the following equation:

$$(r + \lambda_1 - \alpha_1\mu_1)(r + \lambda_2 - \alpha_2\mu_1) = \lambda_1\lambda_2.$$

There exist two values of μ_1 that solve this quadratic equation. Since $(r+\lambda_1-\alpha_1)(r+\lambda_2-\alpha_2) > \lambda_1\lambda_2$, one of the roots, β_1 say, can be easily shown to be greater than 1, whereas the other one, β_2 say, is negative. Note also that we cannot have that $F = 0$ (or $D = 0$), so (14) and (17) imply that μ_2 solves the following equation:

$$(r + \lambda_1 - \alpha_1\mu_2)(r + \lambda_2 - \alpha_2\mu_2) = \lambda_1\lambda_2.$$

In short, noting that $\Delta \equiv [\alpha_1 (\lambda_2 + r) - \alpha_2 (\lambda_1 + r)]^2 + 4\alpha_1\alpha_2\lambda_1\lambda_2 > 0$, we have that

$$\mu_1 = \beta_1 = \frac{\alpha_1(r + \lambda_2) + \alpha_2(r + \lambda_1) - \sqrt{\Delta}}{2\alpha_1\alpha_2} > 1$$

and

$$\mu_2 = \beta_2 = \frac{\alpha_1 (r + \lambda_1) + \alpha_2 (r + \lambda_1) + \sqrt{\Delta}}{2\alpha_1\alpha_2} < 0$$

(or equivalently, $\mu_1 = \beta_2$ and $\mu_2 = \beta_1$).[19]

It only remains to determine the values of C and D (and therefore those of H and F). To arrive at them, take into account that

$$\overline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi}) = \int_0^\infty \lambda_1 e^{-\lambda_1\tau_1}\left(\int_0^{\tau_1} \overline{\pi}e^{-rs}ds + \underline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi})e^{-r\tau_1}\right) d\tau_1,$$

so that using the hypotheses that $\overline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi}) = C\overline{\pi}^{\beta_1} + F\overline{\pi}^{\beta_2} + \rho_1\overline{\pi}$ and $\underline{\mathcal{E}}(\overline{\pi}\,|\overline{\pi},\underline{\pi}) = H\overline{\pi}^{\beta_1} + D\overline{\pi}^{\beta_2} + \rho_2\overline{\pi}$ yields after some algebra that

$$C\overline{\pi}^{\beta_1} + F\overline{\pi}^{\beta_2} + \rho_1\overline{\pi} = \frac{\lambda_1 H\overline{\pi}^{\beta_1}}{r + \lambda_1} + \frac{\lambda_1 D\overline{\pi}^{\beta_2}}{r + \lambda_1} + \frac{(1 + \lambda_1\rho_2)\overline{\pi}}{r + \lambda_1}.$$

Hence, using (14) (for $\mu_2 = \beta_2$) and (16) (for $\mu_1 = \beta_1$) yields that

$$\frac{[(r + \lambda_1)(r + \lambda_2 - \alpha_2\beta_1) - \lambda_1\lambda_2]C\overline{\pi}^{\beta_1}}{(r + \lambda_1)(r + \lambda_2 - \alpha_2\beta_1)} + \frac{\lambda_1\alpha_1\beta_2 D\overline{\pi}^{\beta_2}}{(r + \lambda_1 - \alpha_1\beta_2)(r + \lambda_1)} = \left(\frac{(1 + \lambda_1\rho_2)}{r + \lambda_1} - \rho_1\right)\overline{\pi}.$$

The facts that $(r + \lambda_1 - \alpha_1\beta_1)(r + \lambda_2 - \alpha_2\beta_1) = \lambda_1\lambda_2$ and $\rho_1 = \dfrac{1 + \lambda_1\rho_2}{\lambda_1 + r - \alpha_1}$ allow us

[19] It is straighforward to show that either $\mu_1 = \mu_2 = \beta_1$ or $\mu_1 = \mu_2 = \beta_2$ cannot be solutions.

to rewrite the above expression after some simple manipulations:

$$\beta_1 C\overline{\pi}^{\beta_1} + \frac{\lambda_1 \beta_2 D\overline{\pi}^{\beta_2}}{r+\lambda_1-\alpha_1\beta_2} = -\rho_1\overline{\pi}.$$

Note that the fact that $(r+\lambda_1-\alpha_1\beta_2)(r+\lambda_2-\alpha_2\beta_2)=\lambda_1\lambda_2$ leads to

$$\frac{\lambda_1}{r+\lambda_1-\alpha_1\beta_1} = \frac{r+\lambda_2-\alpha_2\beta_2}{\lambda_2} \equiv \delta_1,$$

so it holds that

$$\beta_1 C\overline{\pi}^{\beta_1} + \delta_1\beta_2 D\overline{\pi}^{\beta_2} = -\rho_1\overline{\pi}. \tag{19}$$

Equation (19) is part of a system of two equations whence we can get the values of C and D. To obtain the other equation, observe that it holds that

$$\underline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi}) = \int_0^\infty \lambda_2 e^{-\lambda_2\tau_2}\left(\int_0^{\tau_2}\underline{\pi}e^{-rs}ds + \overline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi})e^{-r\tau_2}\right)d\tau_2,$$

so using the hypotheses that $\underline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi}) = H\underline{\pi}^{\beta_1}+D\underline{\pi}^{\beta_2}+\rho_2\underline{\pi}$ and $\overline{\mathcal{E}}(\underline{\pi}\,|\overline{\pi},\underline{\pi}) = C\underline{\pi}^{\beta_1}+F\underline{\pi}^{\beta_2}+\rho_1\underline{\pi}$ yields after some algebra that

$$H\underline{\pi}^{\beta_1}+D\underline{\pi}^{\beta_2}+\rho_2\underline{\pi} = \frac{\lambda_2 C\underline{\pi}^{\beta_1}}{r+\lambda_2}+\frac{\lambda_2 F\underline{\pi}^{\beta_2}}{r+\lambda_2}+\frac{(1+\lambda_2\rho_1)\underline{\pi}}{r+\lambda_2}.$$

Hence, using (14) (for $\mu_2=\beta_2$) and (16) (for $\mu_1=\beta_1$) yields that

$$\frac{\lambda_2\alpha_2\beta_2 C\underline{\pi}^{\beta_1}}{(r+\lambda_2)(r+\lambda_2-\alpha_2\beta_2)}+\frac{[(r+\lambda_2)(r+\lambda_1-\alpha_1\beta_1)-\lambda_1\lambda_2]D\underline{\pi}^{\beta_2}}{(r+\lambda_2)(r+\lambda_1-\alpha_1\beta_1)} = \left(\frac{(1+\lambda_2\rho_1)}{r+\lambda_2}-\rho_2\right)\underline{\pi}.$$

Using the facts that $(r+\lambda_1-\alpha_1\beta_2)(r+\lambda_2-\alpha_2\beta_2)=\lambda_1\lambda_2$ and $\rho_2 = \dfrac{1+\lambda_2\rho_1}{\lambda_2+r-\alpha_2}$, we can rewrite the above expression after some simple manipulations as:

$$\frac{\lambda_2\beta_1 C\underline{\pi}^{\beta_1}}{r+\lambda_2-\alpha_2\beta_1}+\beta_2 D\underline{\pi}^{\beta_2} = -\rho_2\underline{\pi}.$$

Note that the fact that $(r+\lambda_1-\alpha_1\beta_1)(r+\lambda_2-\alpha_2\beta_1)=\lambda_1\lambda_2$ leads to

$$\frac{\lambda_2}{r+\lambda_2-\alpha_2\beta_2} = \frac{r+\lambda_1-\alpha_1\beta_1}{\lambda_1} \equiv \delta_2,$$

so

$$\delta_2\beta_2 C\underline{\pi}^{\beta_1}+\beta_2 D\underline{\pi}^{\beta_2} = -\rho_2\underline{\pi}. \tag{20}$$

Solving the system of equations that consists of (19) and (20) yields that

$$C=\frac{\delta_1\overline{\pi}^{\beta_2}\rho_2\underline{\pi}-\rho_1\overline{\pi}\underline{\pi}^{\beta_2}}{\beta_1(\underline{\pi}^{\beta_2}\overline{\pi}^{\beta_1}-\delta_1\delta_2\underline{\pi}^{\beta_1}\overline{\pi}^{\beta_2})}\equiv\xi(\overline{\pi},\underline{\pi}).$$

and

$$D=\frac{\delta_2\underline{\pi}^{\beta_1}\rho_1\overline{\pi}-\rho_2\underline{\pi}\overline{\pi}^{\beta_1}}{\beta_2(\underline{\pi}^{\beta_2}\overline{\pi}^{\beta_1}-\delta_1\delta_2\underline{\pi}^{\beta_1}\overline{\pi}^{\beta_2})}\equiv\zeta(\overline{\pi},\underline{\pi}).$$

Therefore, it follows that

$$H=\frac{\lambda_2 C}{r+\lambda_2-\alpha_2\beta_2}=\frac{(r+\lambda_1-\alpha_1\beta_1)C}{\lambda_1}=\delta_2\xi(\overline{\pi},\underline{\pi})$$

and

$$F=\frac{\lambda_1 D}{r+\lambda_1-\alpha_1\beta_1}=\delta_1\zeta(\overline{\pi},\underline{\pi}).$$

It only remains to show that $\delta_1\in(0,1)$ and $\delta_2\in(0,1)$. Note that

$$\delta_1=\frac{r+\lambda_2-\alpha_2\beta_2}{\lambda_2}=\frac{\alpha_1(r+\lambda_2)-\alpha_2(r+\lambda_1)-\sqrt{\Delta}}{2\alpha_1\lambda_2}\in(0,1),$$

whereas

$$\delta_2=\frac{r+\lambda_1-\alpha_1\beta_1}{\lambda_1}=-\frac{\alpha_1(r+\lambda_2)-\alpha_2(r+\lambda_1)-\sqrt{\Delta}}{2\alpha_2\lambda_1}\in(0,1).$$

■

Proof of Theorem 2. Let $\overline{\varphi}_1(\pi_0,\pi^*)$ denote the expected discounted value of a claim to a dollar when the process first hits π^* from below, conditional on the process being in a growth phase and on its state being $\pi_0<\pi^*$. Also, let $\underline{\varphi}_1(\pi_0,\pi^*)$ denote the expected discounted value of a claim to a dollar when the process first hits π^* from below, conditional on the process being in a decline phase and on the current state being π_0. Given our (memoryless) assumptions on the independent random variables involved, we can relate $\overline{\varphi}_1(\pi_0,\pi^*)$ and $\underline{\varphi}_1(\pi_0,\pi^*)$ as follows:

$$\begin{aligned}\overline{\varphi}_1(\pi_0,\pi^*) &= \int_0^{\frac{1}{\alpha_1}\ln(\pi^*/\pi_0)}\lambda_1 e^{-\lambda_1\tau_1}\underline{\varphi}_1(\pi_0 e^{\alpha_1\tau_1},\pi^*)e^{-r\tau_1}d\tau_1+\\ &\quad \int_{\frac{1}{\alpha_1}\ln(\pi^*/\pi_0)}^{\infty}\lambda_1 e^{-\lambda_1\tau_1}e^{-r\frac{1}{\alpha_1}\ln(\pi^*/\pi_0)}d\tau_1\end{aligned}\tag{21}$$

and

$$\underline{\varphi}_1(\pi_0,\pi^*) = \int_0^\infty \lambda_2 e^{-\lambda_2\tau_2}\overline{\varphi}_1(\pi_0 e^{\alpha_2\tau_2},\pi^*)e^{-r\tau_2}d\tau_2. \tag{22}$$

To understand the relation between $\overline{\varphi}_1(\pi_0,\pi^*)$ and $\underline{\varphi}_1(\pi_0,\pi^*)$, note that if the realized length of the growth phase τ_1 is smaller than $\frac{1}{\alpha_1}\ln(\frac{\pi^*}{\pi_0})$ (the time it would take to hit state π^* from below during the current growth phase), then the firm would acquire an asset whose discounted value would be $\underline{\varphi}_1(\pi_0 e^{\alpha_1\tau_1},\pi^*)e^{-r\tau_1}$; if τ_1 were greater than $\frac{1}{\alpha_1}\ln(\frac{\pi^*}{\pi_0})$, then the firm would acquire the discounted value of a dollar. As for $\underline{\varphi}_1(\pi_0,\pi^*)$, note that the process will start growing at some random future date τ_2, and hence the firm will acquire at such date an asset whose discounted value is $\overline{\varphi}_1(\pi_0 e^{\alpha_2\tau_2},\pi^*)e^{-r\tau_2}$.

To solve the system of functional equations that consists of (21) and (22), guess that $\overline{\varphi}_1(\pi_0,\pi^*) = Y\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1}$ (where Y and θ_1 are constants to be found out), and use such functional form for $\overline{\varphi}_1(\pi_0,\pi^*)$ in (22) so as to get:

$$\underline{\varphi}_1(\pi_0,\pi^*) = \frac{Y\lambda_2}{r+\lambda_2-\alpha_2\theta_1}\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1}. \tag{23}$$

Therefore, we have

$$\underline{\varphi}_1(\pi_0 e^{\alpha_1\tau_1},\pi^*) = \frac{Y\lambda_2 e^{\alpha_1\theta_1\tau_1}}{r+\lambda_2-\alpha_2\theta_1}\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1},$$

so (21) becomes:

$$\begin{aligned}\overline{\varphi}_1(\pi_0,\pi^*) &= \frac{Y\lambda_1\lambda_2}{r+\lambda_2-\alpha_2\theta_1}\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1}\int_0^{\frac{1}{\alpha_1}\ln(\pi^*/\pi_0)} e^{-(r+\lambda_1-\alpha_1\theta_1)\tau_1}d\tau_1 + \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_1}{\alpha_1}}\\ &= \frac{Y\lambda_1\lambda_2}{(r+\lambda_2-\alpha_2\theta_1)(r+\lambda_1-\alpha_1\theta_1)}\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1}[1-\left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_1-\alpha_1\theta_1}{\alpha_1}}] + \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_1}{\alpha_1}}\end{aligned}$$

We assumed that $\overline{\varphi}_1(\pi_0,\pi^*) = Y\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1}$, so the following must hold:

$$Y\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1} - \left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_1+r}{\alpha_1}} = \frac{Y\lambda_1\lambda_2}{(r+\lambda_2-\alpha_2\theta_1)(r+\lambda_1-\alpha_1\theta_1)}[\left(\frac{\pi_0}{\pi^*}\right)^{\theta_1} - \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_1}{\alpha_1}}].$$

Therefore, $Y = 1$, whereas θ_1 must satisfy the following equation:

$$(r+\lambda_2-\alpha_2\theta_1)(r+\lambda_1-\alpha_1\theta_1) = \lambda_1\lambda_2. \tag{24}$$

There exist two values of θ_1 that solve this quadratic equation, which we labeled before as β_1 and β_2. Clearly, only $\beta_1 > 1$ can be an admissible solution, so $\theta_1 = \beta_1$. Also, note that expression (23) with $Y = 1$ and $\theta_1 = \beta_1$ implies that

$$\underline{\varphi}_1(\pi_0, \pi^*) = \frac{\lambda_2}{r + \lambda_2 - \alpha_2\beta_1}\left(\frac{\pi_0}{\pi^*}\right)^{\beta_1} = \left(\frac{r + \lambda_1 - \alpha_1\beta_1}{\lambda_1}\right)\left(\frac{\pi_0}{\pi^*}\right)^{\beta_1},$$

since $(r + \lambda_2 - \alpha_2\beta_1)(r + \lambda_1 - \alpha_1\beta_1) = \lambda_1\lambda_2$. ■

Proof of Theorem 3. Let $\overline{\varphi}_2(\pi_0, \pi^*)$ denote the expected discounted value of a claim to a dollar when the process first hits π^* from above, conditional on the process being in a growth phase and on the current state being $\pi_0 \geq \pi^*$. In addition, let $\underline{\varphi}_2(\pi_0, \pi^*)$ denote the expected discounted value of a claim to a dollar when the process first hits π^* from above, conditional on the process being in a decline phase and on the current state being $\pi_0 > \pi^*$. Then we have the following:

$$\overline{\varphi}_2(\pi_0, \pi^*) = \int_0^\infty \lambda_1 e^{-\lambda_1\tau_1}\underline{\varphi}_2(\pi_0 e^{\alpha_1\tau_1}, \pi^*)e^{-r\tau_1}d\tau_1$$

and

$$\begin{aligned}\underline{\varphi}_2(\pi_0, \pi^*) \quad = \quad & \int_0^{\frac{1}{\alpha_2}\ln(\pi^*/\pi_0)} \lambda_2 e^{-\lambda_2\tau_2}\overline{\varphi}_2(\pi_0 e^{\alpha_2\tau_2}, \pi^*)e^{-r\tau_2}d\tau_2 + \\ & \int_{\frac{1}{\alpha_2}\ln(\pi^*/\pi_0)}^\infty \lambda_2 e^{-\lambda_2\tau_2}e^{-r\frac{1}{\alpha_2}\ln(\pi^*/\pi_0)}d\tau_2.\end{aligned}$$

To understand the relation between $\overline{\varphi}_2(\pi_0, \pi^*)$ and $\underline{\varphi}_2(\pi_0, \pi^*)$, note that if the process is going through a growth phase at π_0, then state $\pi^* \leq \pi_0$ will never be hit in such a phase. Hence, when a switching date is realized at τ_1, the firm will get $\underline{\varphi}_2(\pi_0 e^{\alpha_1\tau_1}, \pi^*)e^{-r\tau_1}$. However, if the process is going through a declining phase at π_0, the process will grow at some random future date τ_2, so two situations must be distinguished. On the one hand, if the process starts declining before reaching π^* at the random date τ_2, then the firm will attain $\overline{\varphi}_2(\pi_0 e^{\alpha_2\tau_2}, \pi^*)e^{-r\tau_2}$. On the other hand, if the declining process goes all the way down to π^*, then the firm will attain the discounted value of the dollar as soon as state π^* is reached.

Suppose that $\overline{\varphi}_2(\pi_0, \pi^*) = A\left(\frac{\pi_0}{\pi^*}\right)^{\theta_2}$ (where A, and θ_2 are constants to be found out). In this case, we have

$$\begin{aligned}
\underline{\varphi}_2(\pi_0,\pi^*) &= \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_2}{\alpha_2}} + \int_0^{\frac{1}{\alpha_2}\ln(\pi^*/\pi_0)} \lambda_2 e^{-\lambda_2\tau_2}\left[A\left(\frac{\pi_0 e^{\alpha_2\tau_2}}{\pi^*}\right)^{\theta_2}\right] e^{-r\tau_2}d\tau_2 \\
&= \frac{\lambda_2 A}{\lambda_2+r-\alpha_2\theta_2}\left(\frac{\pi_0}{\pi^*}\right)^{\theta_2} - \frac{\lambda_2 A}{\lambda_2+r-\alpha_2\theta_2}\left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_2}{\alpha_2}} + \left(\frac{\pi_0}{\pi^*}\right)^{\frac{r+\lambda_2}{\alpha_2}} \\
&= \frac{\lambda_2 A}{\lambda_2+r-\alpha_2\theta_2}\left(\frac{\pi_0}{\pi^*}\right)^{\theta_2} + \left(\frac{\lambda_2(1-A)+r-\alpha_2\theta_2}{\lambda_2+r-\alpha_2\theta_2}\right)\left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_2+r}{\alpha_2}}. \quad (25)
\end{aligned}$$

In addition,

$$\begin{aligned}
\overline{\varphi}_2(\pi_0,\pi^*) &= \int_0^\infty \lambda_1 e^{-(r+\lambda_1)\tau_1}\left[\begin{array}{c}\frac{\lambda_2 A}{\lambda_2+r-\alpha_2\theta_2}\left(\frac{\pi_0 e^{\alpha_1\tau_1}}{\pi^*}\right)^{\theta_2} + \\ \left(\frac{\lambda_2(1-A)+r-\alpha_2\theta_2}{\lambda_2+r-\alpha_2\theta_2}\right)\left(\frac{\pi_0 e^{\alpha_1\tau_1}}{\pi^*}\right)^{\frac{\lambda_2+r}{\alpha_2}}\end{array}\right] d\tau_1 \\
&= \frac{\lambda_1\lambda_2 A}{(r+\lambda_2-\alpha_2\theta_2)(r+\lambda_1-\alpha_1\theta_2)}\left(\frac{\pi_0}{\pi^*}\right)^{\theta_2} + \\
&\quad \left[\frac{\lambda_2(1-A)+r-\alpha_2\theta_2}{r+\lambda_2-\alpha_2\theta_2}\right]\left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_2+r}{\alpha_2}}\left(\frac{\lambda_1}{r+\lambda_1-\frac{\alpha_1}{\alpha_2}(\lambda_2+r)}\right)
\end{aligned}$$

Because we supposed that $\overline{\varphi}_2(\pi_0,\pi^*) = A\left(\frac{\pi_0}{\pi^*}\right)^{\theta_2}$, we must have that the following two equations hold:

$$\frac{\lambda_1\lambda_2 A}{(r+\lambda_2-\alpha_2\theta_2)(r+\lambda_1-\alpha_1\theta_2)} = A, \quad (26)$$

and

$$\frac{\lambda_2(1-A)+r-\alpha_2\theta_2}{\lambda_2+r-\alpha_2\theta_2} = 0. \quad (27)$$

Equation (26) implies that θ_2 solves

$$(\lambda_2+r-\alpha_2\theta_2)(\lambda_1+r-\alpha_1\theta_2) = \lambda_1\lambda_2.$$

We have shown before that the roots of this quadratic equation are $\beta_1 > 1$ and $\beta_2 < 0$. Only the negative root can be admissible now, so $\theta_2 = \beta_2$. In addition, equation (27) with $\theta_2 = \beta_2$ implies that

$$A = \frac{r+\lambda_2-\alpha_2\beta_2}{\lambda_2}. \quad (28)$$

Therefore, $\overline{\varphi}_2(\pi_0,\pi^*) = \delta_1\left(\frac{\pi_0}{\pi^*}\right)^{\beta_2}$. Finally, making use of (25) and (27) with $\theta_2 = \beta_2$, together with expression (28), yields that

$$\underline{\varphi}_2(\pi_0, \pi^*) = \left(\frac{\pi_0}{\pi^*}\right)^{\beta_2}.$$

■

Proof of Theorem 4. Let $\overline{\mathcal{E}}(\pi_0, \pi^*)$ denote the expected value of the stream of discounted profits collected while the process transitions from π_0 until it first hits π^* from above, conditional on the process being in a growth phase and on the current state being $\pi_0 \geq \pi^*$. Also, let $\underline{\mathcal{E}}(\pi_0, \pi^*)$ be the expected stream of discounted profits collected while the process transitions from π_0 until it first hits π^* from above, conditional on the process being in a decline phase and on the current state being $\pi_0 > \pi^*$. We can relate $\overline{\mathcal{E}}(\pi_0, \pi^*)$ and $\underline{\mathcal{E}}(\pi_0, \pi^*)$ as follows:

$$\overline{\mathcal{E}}(\pi_0, \pi^*) = \int_0^{\infty} \lambda_1 e^{-\lambda_1 \tau_1} [\int_0^{\tau_1} \pi_0 e^{\alpha_1 s} e^{-rs} ds + \underline{\mathcal{E}}(\pi_0 e^{\alpha_1 \tau_1}, \pi^*) e^{-r\tau_1}] d\tau_1 \tag{29}$$

and

$$\underline{\mathcal{E}}(\pi_0, \pi^*) = \int_0^{\frac{1}{\alpha_2}\ln(\pi^*/\pi_0)} \lambda_2 e^{-\lambda_2 \tau_2} [\int_0^{\tau_2} \pi_0 e^{\alpha_2 s} e^{-rs} ds + \overline{\mathcal{E}}(\pi_0 e^{\alpha_2 \tau_2}, \pi^*) e^{-r\tau_2}] d\tau_2 + \int_{\frac{1}{\alpha_2}\ln(\pi^*/\pi_0)}^{\infty} \lambda_2 e^{-\lambda_2 \tau_2} (\int_0^{\frac{1}{\alpha_2}\ln(\pi^*/\pi_0)} \pi_0 e^{\alpha_2 s} e^{-rs} ds) d\tau_2. \tag{30}$$

To see how $\overline{\mathcal{E}}(\pi_0, \pi^*)$ arises, note that the process will start declining at some random future date τ_1, so the firm gets a stream of discounted profits since the current date until then, and also the discounted value of an asset worth $\underline{\mathcal{E}}(\pi_0 e^{\alpha_1 \tau_1}, \pi^*)$. As for $\underline{\mathcal{E}}(\pi_0, \pi^*)$, we have that the process grows at some random future date τ_2, although it may start growing before hitting π^* (from above). If it does not, then the firm reaps a stream of discounted profits until π^* is hit at time $\dfrac{\ln(\pi^*/\pi_0)}{\alpha_2}$. If the process starts growing before hitting π^*, then the firm reaps a discounted profit stream until the process stops declining, and the discounted value of an asset worth $\overline{\mathcal{E}}(\pi_0 e^{\alpha_2 \tau_2}, \pi^*)$ at such switching date.

Let us suppose that $\overline{\mathcal{E}}(\pi_0, \pi^*) = E(\pi_0)^{\theta_6} + G(\pi_0)^{\theta_7}$ (where θ_6 and θ_7 are constants, whereas E and G do not depend on π_0 although they may depend on π^*), so plugging the assumed functional form of $\overline{\mathcal{E}}(\pi_0, \pi^*)$ into (30) and performing several manipulations yields:

$$
\begin{aligned}
\underline{\mathcal{E}}(\pi_0,\pi^*) &= \frac{\pi_0(1-\left(\frac{\pi_0}{\pi^*}\right)^{\frac{(r-\alpha_2)}{\alpha_2}})\left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_2}{\alpha_2}}}{r-\alpha_2}+\frac{\pi_0}{r+\lambda_2-\alpha_2}-\frac{\pi_0\left(\frac{\pi_0}{\pi^*}\right)^{\frac{\lambda_2}{\alpha_2}}}{r-\alpha_2}+ \\
&\quad \frac{\lambda_2\pi_0\left(\frac{\pi_0}{\pi^*}\right)^{\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{(r-\alpha_2)(r+\lambda_2-\alpha_2)}+\frac{\lambda_2 E\pi_0^{\theta_6}(1-\left(\frac{\pi_0}{\pi^*}\right)^{\frac{(r+\lambda_2-\alpha_2\theta_6)}{\alpha_2}})}{(r+\lambda_2-\alpha_2\theta_6)}+ \\
&\quad \frac{\lambda_2 G\pi_0^{\theta_7}(1-\left(\frac{\pi_0}{\pi^*}\right)^{\frac{(r+\lambda_2-\alpha_2\theta_7)}{\alpha_2}})}{(r+\lambda_2-\alpha_2\theta_7)} \\
&= \left(\frac{1}{r+\lambda_2-\alpha_2}\right)\pi_0+\frac{\lambda_2 E\pi_0^{\theta_6}}{r+\lambda_2-\alpha_2\theta_6}+\frac{\lambda_2 G\pi_0^{\theta_7}}{r+\lambda_2-\alpha_2\theta_7}- \\
&\quad \left(\frac{(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2}+\frac{\lambda_2 E(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2\theta_6)}{\alpha_2}}}{r+\lambda_2-\alpha_2\theta_6}+\frac{\lambda_2 G(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2\theta_7)}{\alpha_2}}}{r+\lambda_2-\alpha_2\theta_7}\right)(\pi_0)^{\frac{(r+\lambda_2)}{\alpha_2}}.
\end{aligned}
$$

Substitute this into (29) so as to get the following after some manipulations:

$$
\begin{aligned}
\overline{\mathcal{E}}(\pi_0,\pi^*) &= \left(1+\frac{\lambda_1}{r+\lambda_2-\alpha_2}\right)\frac{\pi_0}{(r+\lambda_1-\alpha_1)}- \\
&\quad \frac{\lambda_1\alpha_2(\pi_0)^{\frac{(r+\lambda_2)}{\alpha_2}}}{(\alpha_2(r+\lambda_1)-\alpha_1(r+\lambda_2))}\left(\frac{(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{(r+\lambda_2-\alpha_2)}+\frac{\lambda_2 E(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2\theta_6)}{\alpha_2}}}{(r+\lambda_2-\alpha_2\theta_6)}+\frac{\lambda_2 G(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2\theta_7)}{\alpha_2}}}{(r+\lambda_2-\alpha_2\theta_7)}\right)+ \\
&\quad \frac{\lambda_2 E(\pi_0)^{\theta_6}\lambda_1}{(r+\lambda_2-\alpha_2\theta_6)(r+\lambda_1-\alpha_1\theta_6)}+\frac{\lambda_2 G(\pi_0)^{\theta_7}\lambda_1}{(r+\lambda_2-\alpha_2\theta_7)(r+\lambda_1-\alpha_1\theta_7)}.
\end{aligned}
$$

Assume that $\theta_6 = 1$ so that the assumption that $\overline{\mathcal{E}}(\pi_0,\pi^*) = E\,(\pi_0)^{\theta_6} + G\,(\pi_0)^{\theta_7}$ implies that the following must hold:

$$E = \frac{(r+\lambda_2-\alpha_2)(r+\lambda_1-\alpha_2)-\lambda_1\lambda_2}{(r-\alpha_2)((r+\lambda_2-\alpha_2)(r+\lambda_1-\alpha_1)-\lambda_1\lambda_2)} = \rho_1, \tag{31}$$

$$G = -\frac{(r+\lambda_2-\alpha_2\theta_7)}{\lambda_2\,(\pi^*)^{(\theta_7-1)}}\left(\frac{1+\lambda_2\rho_1}{r+\lambda_2-\alpha_2}\right), \tag{32}$$

and

$$(r+\lambda_2-\alpha_2\theta_7)(r+\lambda_1-\alpha_1\theta_7) = \lambda_1\lambda_2.$$

We must clearly have that $\theta_7 = \beta_2 < 0$, so using this result as well as expressions

(31) and (32) yields after very laborious manipulations:

$$\begin{aligned}
\underline{\mathcal{E}}(\pi_0,\pi^*) &= \left(\frac{1+\lambda_2\rho_1}{r+\lambda_2-\alpha_2}\right)\pi_0 - \\
&\left(\begin{array}{c}\frac{(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2}+\frac{\lambda_2 E(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2}+ \\ \frac{\lambda_2 G(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2\beta_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2\beta_2}\end{array}\right)(\pi_0)^{\frac{(r+\lambda_2)}{\alpha_2}}+\frac{\lambda_2 G\pi_0^{\beta_2}}{r+\lambda_2-\alpha_2\beta_2} \\
&= \left(\frac{1+\lambda_2\rho_1}{r+\lambda_2-\alpha_2}\right)\pi_0 - \pi_0\left(\frac{\pi_0}{\pi^*}\right)^{\beta_2-1}\left(\frac{1+\lambda_2\rho_1}{r+\lambda_2-\alpha_2}\right) \\
&= \rho_2\pi_0\left[1-\left(\frac{\pi_0}{\pi^*}\right)^{\beta_2-1}\right],
\end{aligned}$$

where the second equality makes use of the fact that

$$\frac{(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2}+\frac{\lambda_2 E\,(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2}+\frac{\lambda_2 G\,(\pi^*)^{-\frac{(r+\lambda_2-\alpha_2\beta_2)}{\alpha_2}}}{r+\lambda_2-\alpha_2\beta_2}=0,$$

and the third equality uses

$$\frac{1+\lambda_2\rho_1}{r+\lambda_2-\alpha_2}=\rho_2. \tag{33}$$

In turn, note that

$$\begin{aligned}
\overline{\mathcal{E}}(\pi_0,\pi^*) &= E\pi_0+G\,(\pi_0)^{\beta_2} \\
&= \rho_1\pi_0-\frac{(r+\lambda_2-\alpha_2\beta_2)\pi_0}{\lambda_2}\left(\frac{1+\lambda_2\rho_1}{r+\lambda_2-\alpha_2}\right)\left(\frac{\pi_0}{\pi^*}\right)^{\beta_2-1} \\
&= \rho_1\pi_0-\rho_2\delta_1\pi_0\left(\frac{\pi_0}{\pi^*}\right)^{\beta_2-1},
\end{aligned}$$

where the last equality makes use of (33) and the definition of δ_1. ■

Appendix B

Proof of Lemma 1.

Let $\overline{\pi}_E^*$ denote the threshold that triggers investment when the process is above or at such state for the first time given that the market is growing. In addition, let $\underline{\pi}_E^*$ denote the threshold that triggers investment when the process is above or at such state for the first time given that the market is declining. We claim that $\underline{\pi}_E^* \geq \overline{\pi}_E^*$,[20] so suppose to the contrary that $\underline{\pi}_E^* < \overline{\pi}_E^*$, and consider states such that the firm does

[20] Note that this implies that the firm invests only if the market is growing owing to the continuous sample path of the stochastic process.

not invest immediately if the market is in growth, but such that the arrival of the next switching date would trigger immediate investment: formally, $\pi_0 \in [\underline{\pi}_E^*, \overline{\pi}_E^*)$. In this case, the dynamics of the value of the investment opportunity while the market is growing, denoted by $\overline{V}_E^*(\pi_0)$, are given by the following Bellman equation:

$$\overline{V}_E^*(\pi_0) = \max\{\rho_1\pi_0 - K, (1-rdt)[\lambda_1 dt(\rho_2(\pi_0+\alpha_1\pi_0 dt)-K)+(1-\lambda_1 dt)\overline{V}_E^*(\pi_0+\alpha_1\pi_0 dt)]\}.$$

On the waiting region, a Taylor expansion and straightforward manipulations ignoring terms of order higher than dt yield

$$(r+\lambda_1)\overline{V}_E^*(\pi_0) = \alpha_1\pi_0\frac{d\overline{V}_E^*(\pi_0)}{d\pi_0} + \lambda_1(\rho_2\pi_0 - K).$$

The solution to this differential equation is

$$\overline{V}_E^*(\pi_0) = \frac{\lambda_1\rho_2\pi_0}{r+\lambda_1-\alpha_1} - \frac{\lambda_1 K}{r+\lambda_1} + B(\pi_0)^{\frac{r+\lambda_1}{\alpha_1}}, \tag{34}$$

where B is a constant. In particular, we know that $\overline{V}_E^*(\overline{\pi}_E^*) = \rho_1\overline{\pi}_E^* - K$, whence one can get the value of B and plug it into expression (34) so as to get

$$\overline{V}_E^*(\pi_0\,|\overline{\pi}_E^*) = \frac{\lambda_1\rho_2\pi_0}{r+\lambda_1-\alpha_1} - \frac{\lambda_1 K}{r+\lambda_1} + [\rho_1\overline{\pi}_E^* - K + \frac{\lambda_1 K}{r+\lambda_1} - \frac{\lambda_1\rho_2\overline{\pi}_E^*}{r+\lambda_1-\alpha_1}]\left(\frac{\pi_0}{\overline{\pi}_E^*}\right)^{\frac{r+\lambda_1}{\alpha_1}}.$$

Using the fact that

$$(r+\lambda_1-\alpha_1)\rho_1 = 1+\lambda_1\rho_2 \tag{35}$$

and maximizing $\overline{V}_E^*(\pi_0\,|\overline{\pi}_E^*)$ with respect to $\overline{\pi}_E^*$ yields that $\overline{\pi}_E^* = rK$ (since $\overline{V}_E^*(\pi_0\,|\overline{\pi}_E^*)$ is strictly quasi-concave). As a result, we have that the value of an optimally managed investment opportunity if the market is growing at $\pi_0 \in [\underline{\pi}_E^*, \overline{\pi}_E^*)$ equals

$$\overline{V}_E^*(\pi_0\,|rK) = \frac{\lambda_1\rho_2\pi_0}{r+\lambda_1-\alpha_1} - \frac{\lambda_1 K}{r+\lambda_1} + \frac{\alpha_1 rK}{(r+\lambda_1-\alpha_1)(r+\lambda_1)}\left(\frac{\pi_0}{rK}\right)^{\frac{r+\lambda_1}{\alpha_1}}. \tag{36}$$

Once the value of $\overline{\pi}_E^*$ has been found out, it simply remains to find out that of $\underline{\pi}_E^*$. To this end, let us examine the value of the investment opportunity in a growing market for $\pi_0 < \underline{\pi}_E^*$. Let $\underline{V}_E^*(\pi_0)$ denote the value of the investment opportunity if the market is declining, and note that the dynamics of $\underline{V}_E^*(\pi_0)$ are given by

$$(r+\lambda_2)\underline{V}_E^*(\pi_0) = \alpha_2\pi_0\frac{d\underline{V}_E^*(\pi_0)}{d\pi_0} + \lambda_2\overline{V}_E^*(\pi_0), \tag{37}$$

since the firm is in the waiting region during the decline phase. Similarly, $\overline{V}_E^*(\pi_0)$ satisfies the following differential equation for $\pi_0 < \underline{\pi}_E^*$:

$$(r + \lambda_1)\overline{V}_E^*(\pi_0) = \alpha_1 \pi_0 \frac{d\overline{V}_E^*(\pi_0)}{d\pi_0} + \lambda_1 \underline{V}_E^*(\pi_0), \tag{38}$$

since the firm does not invest right away if the market switches from growth to decline. Solving the system of differential equations comprised by (37) and (38), and using the boundary condition that $\overline{V}_E^*(0) = 0$ leads to the following solution:

$$\overline{V}_E^*(\pi_0) = R(\pi_0)^{\beta_1}, \tag{39}$$

where R is a constant to be found out, and $\beta_1 = \dfrac{\alpha_1(r+\lambda_2) + \alpha_2(r+\lambda_1) - \sqrt{\Delta}}{2\alpha_1\alpha_2} > 1$. Evaluating (36) at $\pi_0 = \underline{\pi}_E^*$, we have that the boundary condition from which R can be derived is

$$R(\underline{\pi}_E^*)^{\beta_1} = \frac{\lambda_1 \rho_2 \underline{\pi}_E^*}{r + \lambda_1 - \alpha_1} - \frac{\lambda_1 K}{r + \lambda_1} + \frac{\alpha_1 r K}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} \left(\frac{\underline{\pi}_E^*}{rK}\right)^{\frac{r+\lambda_1}{\alpha_1}},$$

so expression (39) becomes

$$\overline{V}_E^*(\pi_0 \,|\underline{\pi}_E^*) = \left[\frac{\lambda_1 \rho_2 \underline{\pi}_E^*}{r + \lambda_1 - \alpha_1} - \frac{\lambda_1 K}{r + \lambda_1} + \frac{\alpha_1 r K}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} \left(\frac{\underline{\pi}_E^*}{rK}\right)^{\frac{r+\lambda_1}{\alpha_1}}\right] \left(\frac{\pi_0}{\underline{\pi}_E^*}\right)^{\beta_1}. \tag{40}$$

Performing some manipulations, we have that the derivative of $\overline{V}_E^*(\pi_0 \,|\underline{\pi}_E^*)$ with respect to $\underline{\pi}_E^*$ is

$$\frac{\partial \overline{V}_E^*(\pi_0 \,|\underline{\pi}_E^*)}{\partial \underline{\pi}_E^*} = \left[\frac{\lambda_1 \rho_2 (1 - \beta_1) \underline{\pi}_E^*}{r + \lambda_1 - \alpha_1} + \frac{rK(r + \lambda_1 - \alpha_1 \beta_1)}{(r + \lambda_1 - \alpha_1)(r + \lambda_1)} \left(\frac{\underline{\pi}_E^*}{rK}\right)^{\frac{r+\lambda_1}{\alpha_1}} + \frac{\beta_1 \lambda_1 K}{r + \lambda_1}\right] \frac{(\pi_0)^{\beta_1}}{(\underline{\pi}_E^*)^{\beta_1 + 1}}.$$

We claim that $\dfrac{\partial^2 \overline{V}_E^*(\pi_0 \,|\underline{\pi}_E^*)}{\partial (\underline{\pi}_E^*)^2} < 0$ whenever $\dfrac{\partial \overline{V}_E^*(\pi_0 \,|\underline{\pi}_E^*)}{\partial \underline{\pi}_E^*} = 0$ holds (i.e., we claim that $\overline{V}_E^*(\pi_0 \,|\underline{\pi}_E^*)$ is strictly quasi-concave). For the value of $\underline{\pi}_E^*$ such that

$\dfrac{\partial \overline{V}^*_E(\pi_0\,|\underline{\pi}^*_E)}{\partial \underline{\pi}^*_E} = 0$ holds, we have that

$$\begin{aligned}\frac{\partial^2 \overline{V}^*_E(\pi_0\,|\underline{\pi}^*_E)}{\partial (\underline{\pi}^*_E)^2} &= \left[\frac{\lambda_1\rho_2(1-\beta_1)}{r+\lambda_1-\alpha_1} + \frac{rK(r+\lambda_1-\alpha_1\beta_1)}{\alpha_1(r+\lambda_1-\alpha_1)\underline{\pi}^*_E}\left(\frac{\underline{\pi}^*_E}{rK}\right)^{\frac{r+\lambda_1}{\alpha_1}}\right]\frac{(\pi_0)^{\beta_1}}{(\underline{\pi}^*_E)^{\beta_1+1}} \\ &= \frac{\lambda_1[(\beta_1-1)\rho_2\underline{\pi}^*_E - \beta_1 K](\pi_0)^{\beta_1}}{\alpha_1(\underline{\pi}^*_E)^{\beta_1+2}} \\ &< \frac{\lambda_1\beta_1(\pi_0)^{\beta_1}}{\alpha_1(\underline{\pi}^*_E)^{\beta_1+2}}\left(\frac{\underline{\pi}^*_E - rK}{r}\right) < 0,\end{aligned}$$

where the last equality follows because $\dfrac{\partial \overline{V}^*_E(\pi_0\,|\underline{\pi}^*_E)}{\partial \underline{\pi}^*_E} = 0$, the first inequality holds because $\rho_2 < \rho_1 < \dfrac{\beta_1}{r(\beta_1-1)}$,[21] and the last inequality follows since $\underline{\pi}^*_E < rK$. Evaluate $\dfrac{\partial \overline{V}^*_E(\pi_0\,|\underline{\pi}^*_E)}{\partial \underline{\pi}^*_E}$ at $\underline{\pi}^*_E = rK$ so as to get

$$\begin{aligned}\frac{\partial \overline{V}^*_E(\pi_0\,|rK)}{\partial \underline{\pi}^*_E} &= \left[\frac{\lambda_1\rho_2(1-\beta_1)(r+\lambda_1)rK}{(r+\lambda_1)(r+\lambda_1-\alpha_1)} + \frac{rK(r+\lambda_1-\alpha_1\beta_1)}{(r+\lambda_1-\alpha_1)(r+\lambda_1)} + \frac{\beta_1\lambda_1(r+\lambda_1-\alpha_1)K}{(r+\lambda_1)(r+\lambda_1-\alpha_1)}\right]\times \\ &\quad \frac{(\pi_0)^{\beta_1}}{(\underline{\pi}^*_E)^{\beta_1+1}} \\ &= \frac{[(1-\beta_1)r\rho_1+\beta_1]\,K(\pi_0)^{\beta_1}}{(\underline{\pi}^*_E)^{\beta_1+1}} > 0,\end{aligned}$$

since $\lambda_1\rho_2 = (\lambda_1 + r - \alpha_1)\,\rho_1 - 1$ by expression (35) and $\rho_1 < \dfrac{\beta_1}{r(\beta_1-1)}$ was just shown to always hold. This proves that $\overline{V}^*_E(\pi_0\,|\underline{\pi}^*_E)$ is an increasing function for $\underline{\pi}^*_E < rK$, and (40) implies that the payoff expected by the firm would be bounded

[21] To show that $\rho_1 < \dfrac{\beta_1}{r\,(\beta_1-1)}$, note that

$$(r+\lambda_1-\alpha_1)\,(r+\lambda_2-\alpha_2) - \lambda_1\lambda_2 = \alpha_1\alpha_2\,(1-\beta_1)\,(1-\beta_2)\,,$$

so some algebraic manipulations yield that ρ_1 can be rewritten as follows:

$$\rho_1 = \frac{(r-\alpha_2)\,(r+\lambda_1+\lambda_2-\alpha_2)}{\alpha_1\alpha_2\,(r-\alpha_2)\,(1-\beta_1)\,(1-\beta_2)} = \frac{1}{r}\left[\frac{r(r+\lambda_1+\lambda_2)-r\alpha_2}{\alpha_1\alpha_2\,(1-\beta_1)\,(1-\beta_2)}\right] = \frac{1}{r}\left[\frac{\beta_1\beta_2-\frac{r}{\alpha_1}}{(1-\beta_1)\,(1-\beta_2)}\right],$$

where the last equality makes use of the fact that $\alpha_1\alpha_2\beta_1\beta_2 = r\,(r+\lambda_1+\lambda_2)$. As a result, it follows that

$$\frac{(\beta_1-1)\,\rho_1 r - \beta_1}{\rho_1\,(\beta_1-1)} = \frac{\beta_1\beta_2 - \frac{r}{\alpha_1} - \beta_1\,(\beta_2-1)}{\rho_1\,(\beta_1-1)\,(\beta_2-1)} < \frac{\alpha_1\beta_1 - r}{\alpha_1\rho_1\,(\beta_1-1)\,(\beta_2-1)} < 0,$$

since $\beta_1 > 1$, $\beta_2 < 0$ and $\alpha_1\beta_1 - r = \lambda_1\,(1-\delta_2) > 0$. Hence, we must have that $\rho_1 < \dfrac{\beta_1}{r\,(\beta_1-1)}$.

above by

$$\begin{aligned}\overline{V}_E^*(\pi_0\,|rK\,) &= \left[\frac{\lambda_1\rho_2 rK}{r+\lambda_1-\alpha_1} - \frac{\lambda_1 K}{r+\lambda_1} + \frac{\alpha_1 rK}{(r+\lambda_1-\alpha_1)(r+\lambda_1)}\right]\left(\frac{\pi_0}{rK}\right)^{\beta_1} \\ &= (\rho_1 rK - K)\left(\frac{\pi_0}{rK}\right)^{\beta_1},\end{aligned}$$

which is the (expected) payoff if the firm does not invest during the market decline and it invests the first time the market reaches state rK, conditional upon the current state being $\pi_0 < rK$. As we show below (see Proposition 1), the firm's maximal payoff conditional upon investing only if the market is growing is $\max_{\pi_E}(\rho_1\pi_E - K)\,(\pi_0/\pi_E)^{\beta_1}$. The fact that

$$\overline{V}_E^*(\pi_0\,|rK\,) = (\rho_1 rK - K)\left(\frac{\pi_0}{rK}\right)^{\beta_1} < \max_{\pi_E}(\rho_1\pi_E - K)\left(\frac{\pi_0}{\pi_E}\right)^{\beta_1}$$

contradicts the optimality of investing during the market decline, which concludes the proof. ■

Proof of Proposition 2. Note that expression (3) holds if and only if

$$\rho_1(\alpha_1\beta_1 - \alpha_1)\pi_E^* = \alpha_1\beta_1 K \tag{41}$$

is satisfied. Because we have that $\alpha_1\beta_1 = \lambda_1\,(1-\delta_2) + r$, condition (41) is equivalent to

$$(r-\alpha_1)\rho_1\pi_E^* = rK + \lambda_1\,(1-\delta_2)\,(K - \rho_1\pi_E^*).$$

It is simple to show that it holds that $(r-\alpha_1)\,\rho_1 = 1 + \lambda_1\,(\rho_2 - \rho_1)$, so plugging this equality into the previous expression yields the desired result after canceling some terms and rearranging:

$$\pi_E^* = rK + \lambda_1[\delta_2(\rho_1\pi_E^* - K) - (\rho_2\pi_E^* - K)].$$

■

Proof of Proposition 5. Note that equation (5) holds if and only if

$$\pi_X^*\alpha_2\rho_2(\beta_2 - 1) - \alpha_2\beta_2 S = 0$$

is satisfied. Adding up rS on both sides of this expression, and using the facts that $r - \alpha_2\beta_2 = \lambda_2\,(\delta_1 - 1)$ (see definition of δ_1 in Theorem 3) and $1 + \lambda_2\gamma_1 = \rho_2[\lambda_2\,(1-\delta_1) + r - \alpha_2]$ (so $1 + \lambda_2\gamma_1 = \alpha_2\rho_2(\beta_2 - 1)$) leads to the desired result. ■

Proof of Proposition 6. First, we prove that $\overline{V}_X^* (\pi_X^*) - \underline{V}_X^* (\pi_X^*) = \gamma_1 \pi_X^* - (1-\delta_1) S > 0$. By Proposition 5,

$$\gamma_1 \pi_X^* + \delta_1 S - S = \frac{rS - \pi_X^*}{\lambda_2},$$

so we simply have to show that $r > \dfrac{\beta_2}{\rho_2(\beta_2 - 1)}$ holds. To prove this, notice that

$$(r + \lambda_1 - \alpha_1)(r + \lambda_2 - \alpha_2) - \lambda_1 \lambda_2 = \alpha_1 \alpha_2 (1-\beta_1)(1-\beta_2),$$

so we have that

$$\rho_2 = \frac{1}{r}[\frac{r(r + \lambda_1 + \lambda_2) - r\alpha_1}{\alpha_1 \alpha_2 (1-\beta_1)(1-\beta_2)}] = \frac{1}{r}[\frac{\beta_1 \beta_2 - \frac{r}{\alpha_2}}{(1-\beta_1)(1-\beta_2)}],$$

where the last equality follows because it holds that $\alpha_1 \alpha_2 \beta_1 \beta_2 = r(r + \lambda_1 + \lambda_2)$. As a result, using the fact that $\alpha_2 \beta_2 - r = \lambda_2 (1-\delta_1)$ (see definition of δ_1 in Theorem 3) yields that

$$\frac{(\beta_2 - 1)\rho_2 r - \beta_2}{(\beta_2 - 1)\rho_2} = \frac{\beta_1 \beta_2 - \frac{r}{\alpha_2} - \beta_2(\beta_1 - 1)}{\rho_2(\beta_1 - 1)(\beta_2 - 1)} = \frac{\lambda_2(1-\delta_1)}{\alpha_2 \rho_2 (\beta_1 - 1)(\beta_2 - 1)},$$

whence it is clear that $\dfrac{\rho_2 r(\beta_2 - 1) - \beta_2}{\rho_2(\beta_2 - 1)} = r - \dfrac{\beta_2}{(\beta_2 - 1)\rho_2} > 0$, since $\delta_1 \in (0,1)$, $\beta_1 > 1$ and $\beta_2 < 0$.

Therefore, we must have that $\overline{V}_X^* (\pi_X^*) - \underline{V}_X^* (\pi_X^*) > 0$, and to complete the proof it suffices to show that $\overline{V}_X^* (\pi_0) - \underline{V}_X^* (\pi_0) > \overline{V}_X^* (\pi_X^*) - \underline{V}_X^* (\pi_X^*)$ for $\pi_0 > \pi_X^*$. To prove that $\overline{V}_X^* (\pi_0) - \underline{V}_X^* (\pi_0)$ is increasing for $\pi_0 > \pi_X^*$, let $\pi_0 > \pi_X^*$ and use the definition of γ_1 in (2) so that

$$\begin{aligned}
\overline{V}_X^* (\pi_0) - \underline{V}_X^* (\pi_0) &= \gamma_1 \pi_0 - (1-\delta_1) \underline{V}_X^* (\pi_0) \\
&= \frac{\{\rho_2[\lambda_2(1-\delta_1) + r - \alpha_2] - 1\}\pi_0}{\lambda_2} - (1-\delta_1) \underline{V}_X^* (\pi_0) \\
&= (1-\delta_1)(\rho_2 \pi_0 - \underline{V}_X^* (\pi_0)) + \frac{[\rho_2 (r - \alpha_2) - 1]\pi_0}{\lambda_2} \\
&= (1-\delta_1)(\rho_2 \pi_X^* - S)\left(\frac{\pi_0}{\pi_X^*}\right)^{\beta_2} + \frac{[\rho_2 (r - \alpha_2) - 1]\pi_0}{\lambda_2} \\
&= \frac{(1-\delta_1) S}{\beta_2 - 1}\left(\frac{\pi_0}{\pi_X^*}\right)^{\beta_2} + \frac{[\rho_2 (r - \alpha_2) - 1]\pi_0}{\lambda_2},
\end{aligned}$$

where the last two equalities follow from Proposition 4. Note that

$$\rho_2(r-\alpha_2)-1=\frac{\lambda_2(\alpha_1-\alpha_2)\rho_2}{(\lambda_1+\lambda_2+r-\alpha_1)},$$

so

$$\overline{V}_X^*(\pi_0)-\underline{V}_X^*(\pi_0)=\frac{(1-\delta_1)S}{\beta_2-1}\left(\frac{\pi_X^*}{\pi_0}\right)^{-\beta_2}+\frac{(\alpha_1-\alpha_2)\rho_2\pi_0}{(\lambda_1+\lambda_2+r-\alpha_1)}.$$

Hence, $\overline{V}_X^*(\pi_0)-\underline{V}_X^*(\pi_0)$ is clearly increasing in π_0 (since $\delta_1\in(0,1)$ and $\beta_2<0$), which shows that $\overline{V}_X^*(\pi_0)-\underline{V}_X^*(\pi_0)>\overline{V}_X^*(\pi_X^*)-\underline{V}_X^*(\pi_X^*)$ for $\pi_0>\pi_X^*$. ■

Proof of Proposition 8. To show that

$$\pi_E^*=rK+\lambda_1\left[\delta_2\overline{V}_E^*(\pi_E^*)-(\underline{V}_X^*(\pi_E^*)-K)\right],$$

use

$$\overline{V}_E^*(\pi_E^*)=\frac{K}{\beta_1-1}-\frac{\delta_1 S}{\beta_1-1}\left(\frac{\pi_E^*}{\pi_X^*}\right)^{\beta_2}$$

and

$$\underline{V}_X^*(\pi_E^*)=\rho_2\pi_E^*-\frac{S}{\beta_2-1}\left(\frac{\pi_E^*}{\pi_X^*}\right)^{\beta_2}$$

so that

$$\begin{aligned}
& rK+\lambda_1\left[\delta_2\overline{V}_E^*(\pi_E^*)-(\underline{V}_X^*(\pi_E^*)-K)\right]\\
=\;& (r+\lambda_1)K-\lambda_1\rho_2\pi_E^*+\lambda_1\left[\delta_2\left(\frac{K}{\beta_1-1}-\frac{\delta_1 S(\pi_E^*/\pi_X^*)^{\beta_2}}{\beta_1-1}\right)+\frac{S(\pi_E^*/\pi_X^*)^{\beta_2}}{\beta_2-1}\right]\\
=\;& \pi_E^*+(r+\lambda_1)K-(\lambda_1+r-\alpha_1)\rho_1\left[\frac{\beta_1 K+\delta_1\frac{S(\beta_2-\beta_1)\left(\pi_E^*/\pi_X^*\right)^{\beta_2}}{1-\beta_2}}{(\beta_1-1)\rho_1}\right]+\\
& \lambda_1\delta_2\left(\frac{K}{\beta_1-1}-\frac{\delta_1 S(\pi_E^*/\pi_X^*)^{\beta_2}}{\beta_1-1}\right)+\frac{\lambda_1 S(\pi_E^*/\pi_X^*)^{\beta_2}}{\beta_2-1}\\
=\;& \pi_E^*+K\left[(\lambda_1+r)-\frac{(\lambda_1+r-\alpha_1)\beta_1}{\beta_1-1}+\frac{\lambda_1+r-\alpha_1\beta_1}{\beta_1-1}\right]+\\
& \frac{\delta_1 S(\pi_E^*/\pi_X^*)^{\beta_2}}{\beta_2-1}\left[\frac{(\lambda_1+r-\alpha_1)(\beta_2-\beta_1)}{\beta_1-1}-\frac{(\lambda_1+r-\alpha_1\beta_1)(\beta_2-1)}{\beta_1-1}+\frac{\lambda_1}{\delta_1}\right]\\
=\;& \pi_E^*+\frac{\delta_1 S(\pi_E^*/\pi_X^*)^{\beta_2}}{\beta_2-1}\left[\frac{\lambda_1}{\delta_1}+\alpha_1\beta_2-(\lambda_1+r)\right]\\
=\;& \pi_E^*,
\end{aligned}$$

where we have made use of the facts that $\lambda_1 \rho_2 = \rho_1(\lambda_1 + r - \alpha_1) - 1$ (by (35)),

$$\pi_E^* = \frac{\beta_1 K + \delta_1 \frac{S(\beta_2 - \beta_1)\left(\pi_E^*/\pi_X^*\right)^{\beta_2}}{1-\beta_2}}{(\beta_1 - 1)\,\rho_1}$$

(by (8)), $\lambda_1 \delta_2 = \lambda_1 + r - \alpha_1 \beta_1$, $\delta_1 = \dfrac{\lambda_2 + r - \alpha_2 \beta_2}{\lambda_2}$ and (24). ■