

# Identification of Time and Risk Preferences in Buy Price Auctions

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## Abstract

Buy price auctions merge a posted price option with a standard bidding mechanisms, and have been used by various online auction sites including eBay and GMAC. A buyer in a buy price auction can accept the buy price to win with certainty and end the auction early. Intuitively, the buy price option may be appealing to bidders who are risk averse or impatient to obtain the good, and a number of authors have examined how such mechanisms can increase the seller's expected revenue over standard auctions. We show that data from buy price auctions can be used to identify bidders' risk aversion and time preferences. We develop a private value model of bidder behavior in a buy price auction with a temporary buy price. In our setup, bidders arrive stochastically over time, and the auction proceeds as a second-price sealed bid auction after the buy price disappears. Upon arrival, a bidder in our model is allowed to act immediately (i.e. accept the buy price if it is still available, or place a bid) or wait and act later. Allowing for general forms of risk aversion and impatience, we first characterize equilibria in cutoff strategies and describe the condition under which all symmetric pure-strategy subgame-perfect Bayesian Nash equilibria are in cutoff strategies. Given sufficient exogenous variation in auction characteristics such as reserve and buy prices and in auction lengths, we then show that the arrival rate, valuation distribution, utility function, and time-discounting function in our model are all nonparametrically identified. We also develop extensions of the identification results for settings in which the variation in auction characteristics is more limited.

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# 1 Introduction

This paper studies identification of bidder preferences in single unit buy price (BP) auctions. BP auctions merge a posted price selling environment with an auction environment, and have been used by eBay (in their “Buy-it-Now” auctions), GMAC, and other organizations as an alternative to standard first or second price auctions. We show that data from BP auctions can be particularly informative about risk aversion and time preferences among potential bidders, in a way that standard auctions are not. As a result, it is possible to recover bidder preferences from widely available observational data, or carry out experiments to obtain appropriate data.

There is a large theoretical literature that shows how BP auctions can increase expected revenue over standard auctions; see Budish and Takeyama (2001), Mathews (2004), Mathews and Katzman (2006), Hidvégi, Wang and Whinston (2006), Gallien and Gupta (2007), Wang, Montgomery, and Srinivasan (2008), and Reynolds and Wooders (2009). BP auctions allow some or all potential bidders to purchase the item immediately at a posted buy price. If this does not happen, a standard auction is held. Intuitively, the buy price option may be appealing to bidders who are risk averse or impatient to obtain the good. The existing theoretical models typically assume risk aversion, impatience, or both, on the part of bidders, and show that BP mechanisms can increase expected revenue to the seller. Since bidders’ decisions in BP auctions depend on their risk aversion and their impatience, one might conjecture that observed data from BP auctions could be informative about risk aversion and impatience.

Many of the existing theoretical models of BP auctions abstract significantly from the specific mechanisms used in practice. For example, some of the models are purely static, whereas in practice many BP auctions have two phases, a buy price phase and a bidding phase, with particular rules about when the bidding phase starts and how long it lasts. The models of Mathews (2004) and Gallien and Gupta (2007) do feature sequential arrival of bidders and an auction format closely modeled on eBay auctions, but impose specific parametric forms on time discounting or risk-aversion. We first develop a theoretical model for a BP auction which captures some key dynamic features of real-world BP auctions and allows for general forms of both risk aversion and impatience, but leads to a tractable equilibrium and relatively straightforward identification results. In our model, bidders have independent private values and arrive according to a time-varying Poisson process. Any potential bidder who arrives in the buy price phase can purchase the good at the buy price (thereby winning the item and ending the auction), can bid (thereby initiating the bidding phase), or can wait. Potential bidders who arrive during the bidding phase (or bidders who arrived during the buy price phase and have waited) can place bids. The bidding phase lasts for a fixed amount of time and is modeled as a second-price sealed bid auction.<sup>1</sup> We describe conditions under

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<sup>1</sup>Our fixed length bidding phase differs from the setup of Mathews (2004) and Gallien and Gupta (2007), who assume a fixed overall length of the auction (similar to eBay). The reason we consider this alternative is because it makes our basic identification results most straightforward. In Section 5, we extend our identification results to eBay-style models.

which all symmetric, pure-strategy, subgame-perfect Bayesian Nash equilibria (BNE) of this game are in cutoff strategies, where a potential bidder arriving in the buy price phase accepts the posted price if her valuation is sufficiently high.

Having characterized equilibrium strategies for potential bidders in the auction, we consider identification. In our model, bidders are heterogeneous in their valuations, but have common utility and time-discounting functions. This allows bidders to be risk averse, impatient, or both. Our setup imposes some restrictions on the nature of bidder preferences, but under these assumptions, we show that the arrival rate function, the distribution of valuations, the utility function, and the time-discounting function are nonparametrically identified under an assumption of exogenous variation in the auction setup (e.g. reserve and buy prices), and some support conditions. The assumption that reserve and buy prices vary exogeneously is somewhat strong, but provides a natural starting point for identification analysis and could be relaxed in various ways. Our results could also be used by sellers (or economists) who wish to experiment with reserve and buy prices in order to learn about the preferences of buyers. We also show that the model is overidentified, in the sense that it imposes testable restrictions on the distribution of observed data. In addition, if the support conditions are not fully met, then certain local versions of the structural objects are identified.

Although our model captures many features of real-world BP auctions, it does differ in some details from the BP auctions used by both eBay and those used by GMAC. We show that under some additional assumptions, primarily used to guarantee that bidders use cutoff strategies immediately upon arrival, our identification arguments can be extended to these two cases. The extended identification results for eBay-style Buy-it-Now auctions are being used in ongoing empirical work (Ackerberg, Hirano, and Shahriar, 2006).

Our findings contribute to the literature on identification of auction models, and more generally to the literature on recovering risk aversion and other features of preferences from revealed behavior. Beginning with Guerre, Perrigne, and Vuong (2000), Li, Perrigne, and Vuong (2002), and Athey and Haile (2002), a large literature has emerged exploring identification in various auction formats.<sup>2</sup> If bidders are risk averse, identification becomes much more challenging in these formats; see, e.g. Campo, Guerre, Perrigne, and Vuong (2010), Bajari and Hortacsu (2005), Campo (2006), Perrigne and Vuong (2007), Lu and Perrigne (2008), Athey and Haile (2007), and Guerre, Perrigne, and Vuong (2009). To our knowledge, our paper is the first to examine identification of bidder preferences in BP auctions, and our identification results indicate that these auctions can provide considerable information on bidder risk preferences. Moreover, our results show that bidder propensities to accept buy prices can be used to infer *both* their risk aversion and time preferences. Some of our identification arguments may also be useful more generally in the context

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<sup>2</sup>See Haile and Tamer (2003), Bajari and Hortacsu (2003), Song (2004), Adams (2004, 2007, 2009), Gonzales, Hasker, and Sickles (2004), Canals-Cerda and Percy (2008), Nekipelov (2008), Zeithammer and Adams (2009), and Backus and Lewis (2009), among others. Athey and Haile (2007) provides a survey of identification results for auction models.

of the literature on identification of risk preferences when there is unobserved heterogeneity (see Chiappori, Gandhi, Salanie, and Salanie (2009) and the references there).

## 2 Model and Equilibrium

We start with a simple continuous time, independent private value (IPV) BP auction.<sup>3</sup> The auction starts at time 0, has a reserve price (minimum bid)  $r \in [0, \infty)$ , and a buy price  $p \in [r, \infty)$ . At times  $t > 0$ , potential bidders arrive at the auction according to a Poisson process with rate  $\lambda(t)$ . These potential bidders have private valuations  $v$  drawn independently from the distribution  $F_V(v)$ .

There are two phases in the auction. At any time  $t$  in the the first phase (the buy price phase), any potential bidder who has previously arrived at the auction can take one of the following actions:

1. Immediately purchase the object at  $p$  (Accept the BP). In this case, the auction ends.
2. Submit a sealed bid  $b > r$  for the object (Reject the BP). In this case, the buy price phase ends, and the auction immediately enters the second phase (the bidding phase).

The bidding phase lasts for fixed length  $\tau > 0$ . During the bidding phase, potential bidders no longer have the option to purchase the object immediately at  $p$ . Other potential bidders who either have already arrived, or who arrive during the bidding phase, can also submit a sealed bid  $b > r$  for the object. These sealed bids can be placed at any time during the bidding phase.

At the end of the bidding phase, the auction ends and the object is awarded to the bidder who has placed the highest sealed bid. The winning price is the maximum of either the reserve price  $r$  or the highest sealed bid of the other bidders. We assume that bidders do not directly observe the actions or arrivals of other bidders. However, we assume that any bidder who is present at the auction at  $t$  knows whether the auction is currently in the buy price phase or the bidding phase, and if the latter, that the bidder knows at what point in time the auction entered the bidding phase.

In the terminology of Gallien and Gupta (2007), our auction features a “temporary buyout option,” which disappears once the buy price is rejected and the bidding phase begins.<sup>4</sup> However, as long as no potential bidder accepts or rejects the BP, the auction continues indefinitely. There are a number of possible variations on this mechanism. For example, we could consider a design where if by time  $T - \tau$ , no bidder has accepted or rejected the BP, the auction automatically enters the bidding phase. Alternatively, we could fix the overall length of the auction at  $T$  (unless the BP is accepted). In this case, the length of the bidding phase is  $T - t$ , where  $t$  is the point in the auction at which the BP is rejected. This corresponds to the setup of eBay’s “Buy-it-Now auctions” and

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<sup>3</sup>See Shahriar (2008) for a model of BP auctions with common values.

<sup>4</sup>In contrast, under Gallien and Gupta’s “permanent buyout option” scheme, the option to purchase the good at price  $p$  remains for the entire duration of the auction. This was used by Yahoo! on their now defunct auction site.

the models of Mathews (2004) and Gallien and Gupta (2007). We could also consider alternative forms for the bidding phase, for example by explicitly modeling eBay’s proxy bidding system. We begin with our stylized setup because it simplifies the equilibrium analysis and leads more directly to identification results. In Section 5, we consider identification under some of these alternative BP auction designs and are able to extend our results under additional assumptions.

Consider a bidder who arrives at time  $t$  with a valuation  $v$  for the object. We assume that if this bidder wins the object at time  $t^*$  and pays price  $p^*$ , she obtains payoff

$$\delta(t^* - t)U(v - p^*),$$

where  $U(\cdot)$  is a utility function and  $\delta(\cdot)$  is a function capturing impatience, i.e. the idea that a bidder would prefer to win the object earlier. If the bidder does not obtain the object, she obtains utility 0.  $\lambda(\cdot)$ ,  $F_V(\cdot)$ ,  $U(\cdot)$  and  $\delta(\cdot)$  are primitives of our model; the variables  $(p, r, \tau)$  characterize the auction setup. We make the following assumptions on these primitives.

**Assumption 1** *The model primitives  $\{\lambda(\cdot), F_V(\cdot), U(\cdot), \delta(\cdot)\}$  satisfy:*

1.  $\{\lambda(\cdot), F_V(\cdot), U(\cdot), \delta(\cdot)\}$  are common knowledge to all potential bidders;
2.  $\lambda(\cdot)$  is twice continuously differentiable and satisfies  $0 < \lambda(t) < \infty$  for all  $t \geq 0$ ;
3.  $F_V(\cdot)$  is twice continuously differentiable on  $[0, \infty)$ .  $F_V(0) = 0$ ,  $F_V(v) > 0$  for all  $v > 0$ ,  $\int_0^\infty F_V(v)dv = 1$ ;
4.  $U(\cdot)$  is twice continuously differentiable;
5.  $U'(\cdot) > \epsilon$  for some  $\epsilon > 0$ ;
6.  $-B < U''(\cdot) \leq 0$  for some  $0 < B < \infty$  (weak risk-aversion);
7.  $\delta'(\cdot) < 0$  (strict impatience),  $\delta(\cdot) > 0$ ;
8.  $U(0) = 0$ ,  $U'(0) = 1$ ,  $\delta(0) = 1$  (normalizations).

We additionally make the following assumption about bidder behavior.

**Assumption 2** *Bidders do not play weakly dominated strategies.*

The bidding phase is essentially a second-price sealed bid auction, which generally have unusual equilibria in weakly dominated strategies (Milgrom (1981), Plum (1992), Blume and Heidhues (2004)). Assumption 2 is a simple way to rule these unusual equilibria out, and it ensures that we

get unique equilibrium play in the bidding phase where bidders follow the weakly dominant strategy of submitting bids equal to their valuations (or not submitting a bid if  $v < r$ ).

With these assumptions, we can state the following

**Proposition 1** *Under Assumptions 1 and 2, any symmetric, pure strategy, perfect Bayesian-Nash equilibrium (BNE) of this auction game has the following properties:*

1. *Potential bidders with  $v < r$  never take any action;*
2. *Potential bidders with  $v > r$  who arrive during the buy price phase **immediately** either a) “Accept the BP,” i.e. purchase the good at  $p$ ; or b) “Reject the BP” by placing a sealed bid equal to  $v$ ;*
3. *Potential bidders with  $v > r$  who arrive during the bidding phase place a sealed bid equal to  $v$  at some point before the end of the auction.*

**Proof:** *Appendix A.*  $\square$

Part 2 of the Proposition states that in equilibrium, potential bidders with  $v > r$  arriving during the buy price phase *do not wait*—they either accept or reject the BP immediately. The incentives to act immediately in equilibrium arise from two sources. First, waiting delays the time at which the bidder may potentially win the item, generating lower utility due to impatience. Second, waiting engenders more competition from other potential bidders for the object. For example, delaying accepting the BP incurs the risk that another potential bidder will enter and accept the BP first. Delaying rejecting the BP lengthens the time until the end of the auction (since the bidding phase has fixed length  $\tau$ ), increasing the expected number of competitors that the bidder will face in the sealed-bid auction.

Part 3 (combined with Part 2) of the Proposition implies that we get the well known 2nd-price sealed bid auction outcome for any auction that enters the bidding phase. Specifically, the bidder with the highest valuation wins the object at the valuation of the second highest bidder, or  $r$  when there are no other bids placed.

## 2.1 Optimal BP Decision and Conditions for a Cutoff Equilibrium

Proposition 1 does not fully characterize the BNE, as it does not specify whether a potential bidder arriving during the buy price phase (with  $v > r$ ) will accept or reject the BP. We now characterize this decision. Consider such a bidder who arrives at  $t$ . If the bidder accepts the BP option immediately, she will obtain payoff

$$U^A(v, p) := U(v - p).$$

If the bidder rejects the BP and places a sealed bid at time  $t$ , then she will win the object if she has placed the highest sealed bid by time  $t + \tau$ , and pay a price equal to the valuation of the next highest bidder (if there is another bid), or equal to  $r$  if there are no other bids placed. Let

$$\gamma = \int_t^{t+\tau} \lambda(s) ds,$$

so the number of other bidders who arrive after  $t$  is  $\text{Poisson}(\gamma)$ . (Note that  $\gamma$  is a function of  $t$  and  $\tau$ , but we suppress this in the notation.) Then the bidder's expected utility from rejecting the BP is

$$U^R(v, r, \tau, t) := \delta(\tau) \cdot \left\{ e^{-\gamma} U(v - r) + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} F_V^n(v) E_n [U(v - \max\{r, Y\} | Y \leq v)] \right\},$$

where  $F_V^n(v) = [F_V(v)]^n$  and  $E_n$  is the expectation when  $Y$  has CDF  $F^n$ . In this formulation,  $n$  represents the number of other bidders that arrive after the BP is rejected, and  $Y$  represents the maximum of these other bidders' valuations. Note that  $U^R(v, r, \tau, t)$  does not depend on  $p$ , because Proposition 1 implies that any bidder arriving prior to  $t$  with  $v > r$  would have already either accepted or rejected the BP. Hence, a bidder who arrives while the BP is still available knows, in equilibrium, that no prior arriving bidder has  $v > r$ .

The following proposition provides a simpler expression for  $U^R(v, r, \tau, t)$  which we will use extensively in the sequel.

**Proposition 2**

$$U^R(v, r, \tau, t) = \delta(\tau) \left( \alpha(r, \tau, t) U(v - r) + \int_r^v U(v - y) h(y, \tau, t) dy \right),$$

where

$$\begin{aligned} \alpha(r, \tau, t) &= \exp(\gamma F_V(r) - \gamma), \\ h(y, \tau, t) &= \exp(\gamma F_V(y) - \gamma) \gamma f_V(y), \end{aligned}$$

and  $\alpha(r, \tau, t)$ ,  $h(y, \tau, t)$  satisfy:

$$\alpha(r, \tau, t) + \int_r^{\infty} h(y, \tau, t) dy = 1, \quad \text{and} \quad \frac{\partial \alpha(r, \tau, t)}{\partial r} = h(r, \tau, t).$$

**Proof:** Appendix A.  $\square$

From the perspective of a bidder rejecting the BP at  $t$ ,  $\alpha(r, \tau, t)$  is the probability that no other bidder will arrive during the bidding phase with a valuation greater than  $r$ , and  $h(y, \tau, t)$  is the density of the maximum of the valuations of bidders who arrive during the bidding phase.

With expressions for  $U^A(v, p)$  and  $U^R(v, r, \tau, t)$  in hand, we can now characterize the choice of whether to accept or reject the BP. In particular, we examine conditions under which this decision depends on a bidder's valuation  $v$  in a particularly simple way: the bidder accepts the BP option if her valuation is above a cutoff value and rejects the BP (i.e. initiates bidding) otherwise. We call this a "cutoff strategy."

Given Proposition 1, bidders with  $v > r$  who arrive during the buy price phase immediately either accept or reject the BP. Clearly, in any BNE, the bidder must accept the BP if and only if

$$U^A(v, p) \geq U^R(v, r, \tau, t)$$

or

$$U(v - p) \geq \delta(\tau) \left( \alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)dy \right).$$

Define

$$M(v, r, \tau, t) = U^{-1} \left( \delta(\tau) \left( \alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)dy \right) \right), \quad \forall r, v \geq r, t, \tau.$$

$M(v, r, \tau, t)$  is the certainty equivalent of the random outcome obtained by rejecting the BP option. Whether or not the BP decision follows a cutoff strategy depends on how the certainty equivalent varies with  $v$ . Consider the following assumption:

**Assumption 3** For some  $\epsilon > 0$ ,  $M_v(v, r, \tau, t) < 1 - \epsilon$ ,  $\forall r, v \geq r, t, \tau$ .

This is a sufficient condition for equilibrium BP decisions to follow cutoff strategies.<sup>5</sup>

**Proposition 3** Under Assumptions 1, 2, and 3, in any symmetric, pure strategy, perfect BNE, there exists a finite valued cutoff function  $c(p, r, \tau, t)$ , implicitly defined by the equation

$$U(c(p, r, \tau, t) - p) = \delta(\tau) \left( \alpha(r, \tau, t)U(c(p, r, \tau, t) - r) + \int_r^{c(p, r, \tau, t)} U(c(p, r, \tau, t) - y)h(y, \tau, t)dy \right), \quad (1)$$

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<sup>5</sup>A necessary condition for the accept/reject decision to follow a cutoff strategy for any  $p, r, t$ , and  $\tau$  is that  $M_v(v, r, \tau, t) \leq 1$ .

such that a potential bidder who arrives at  $t$  during the buy price phase with  $v > r$  immediately accepts the BP if  $v > c(p, r, \tau, t)$ , immediately rejects the BP if  $v < c(p, r, \tau, t)$ , and is indifferent between immediately accepting and immediately rejecting the BP if  $v = c(p, r, \tau, t)$ . The cutoff function  $c(p, r, \tau, t)$  satisfies:

1.  $c_p(p, r, \tau, t) > 0$ ,  $c_r(p, r, \tau, t) < 0$ ,  $c_\tau(p, r, \tau, t) < 0$ ;
2.  $c(r, r, \tau, t) = r$ ,  $c(p, r, \tau, t) > p$  when  $p > r$ .

**Proof:** Appendix A.  $\square$

Assumption 3 is a high level assumption. In general, whether or not it holds will depend on the forms of  $U(\cdot)$ ,  $\alpha(r, \tau, t)$ ,  $h(y, \tau, t)$ , and  $\delta(\tau)$ . Some intuition can be obtained in the case where the bidder is risk neutral, i.e.  $U(x) = x$ . Then we have

$$M(v, r, \tau, t) = \delta(\tau) \left( \alpha(r, \tau, t)(v - r) + \int_r^v (v - y)h(y, \tau, t)dy \right),$$

$$M_v(v, r, \tau, t) = \delta(\tau) \left( \alpha(r, \tau, t) + \int_r^v h(y, \tau, t)dy \right) < \delta(\tau) < 1, \quad \forall t, \tau, v \geq r,$$

since  $\alpha(r, \tau, t) + \int_r^v h(y, \tau, t)dy < 1$ ,  $\delta(0) = 1$ , and  $\delta'(\tau) < 0$ . Hence under risk neutrality, equilibria always involve cutoff strategies, regardless of the forms of  $\alpha(r, \tau, t)$ ,  $h(y, \tau, t)$ , and  $\delta(\tau)$ . The intuition is fairly clear in the risk neutral case. When a bidder's valuation  $v$  increases from  $v^*$  to  $v^* + 1$ ,  $U^A(v, p)$  increases by 1. On the other hand,  $U^R(v, r, \tau, t)$  increases by less than 1 because of discounting, and because some of the utility gains from the valuation increase are lost to competing bidders with valuations between  $v^*$  and  $v^* + 1$ . Since the utility from accepting the BP option increases in  $v$  faster than the utility from rejecting the BP option, optimization implies a cutoff rule where bidders with high valuations accept the BP, and bidders with low valuations reject the BP.

It is possible to obtain more primitive conditions ensuring an equilibrium in cutoff strategies. For example, Appendix B shows that  $U'''(x) \leq 0$  is a sufficient condition for Assumption 3 to hold for any primitives  $\{\lambda(\cdot), F_V(\cdot), \delta(\cdot)\}$  satisfying our conditions. However, given a particular  $\{\lambda(\cdot), F_V(\cdot), \delta(\cdot)\}$ , there will generally be utility functions that do not satisfy  $U'''(x) \leq 0$  but do satisfy Assumption 3.

Regarding the properties of  $c(p, r, \tau, t)$ , it is intuitive that when the BP  $p$  increases, the cutoff increases, making a bidder less likely to accept the BP. When the reserve price  $r$  increases, the cutoff decreases because the expected utility from rejecting the BP decreases. When  $\tau$  increases, the expected utility from rejecting the BP decreases, and the cutoff decreases. There are two

reasons for this. First, increasing  $\tau$  increases the expected number of competitors entering in the bidding phase, lowering the expected utility from rejecting the buy price. Second, the expected utility from rejecting the BP decreases as  $\tau$  increases due to impatience. Property 2 simply states that when the BP exactly equals the reserve price, all entering bidders with  $v > r$  will accept the BP. As  $p$  increases above  $r$ ,  $c(p, r, \tau, t)$  also increases, and is strictly above  $p$ . Lastly, note that  $c_t(p, r, \tau, t)$  may be positive or negative (or 0), depending on how the Poisson rate  $\lambda(t)$  varies across  $t$ .

## 2.2 Inverse cutoff function $p(c, r, \tau, t)$

Given that the cutoff function  $c(p, r, \tau, t)$  is strictly increasing in  $p$ , we can invert it to obtain an inverse cutoff function  $p(c, r, \tau, t)$ . The inverse cutoff function tells us, for a given  $r, \tau$ , and  $t$ , what the BP would have to be for a bidder with valuation  $c$  to be indifferent between accepting and rejecting the BP. Following equation (1), the inverse cutoff function solves

$$U(c - p(c, r, \tau, t)) = \delta(\tau) \left( \alpha(r, \tau, t)U(c - r) + \int_r^c U(c - y)h(y, \tau, t)dy \right). \quad (2)$$

Unlike the cutoff function, we can explicitly solve out for the inverse cutoff function as a function of model primitives, i.e.

$$p(c, r, \tau, t) = c - U^{-1} \left( \delta(\tau) \left( \alpha(r, \tau, t)U(c - r) + \int_r^c U(c - y)h(y, \tau, t)dy \right) \right). \quad (3)$$

The inverse cutoff function and this alternative representation of the indifference condition will be useful in the identification arguments below. The following properties will also be helpful:

**Proposition 4** *The inverse cutoff function  $p(c, r, \tau, t)$  satisfies the following properties:*

1.  $0 < p_c(c, r, \tau, t) < 1$ ,  $p_r(c, r, \tau, t) > 0$ ,  $p_\tau(c, r, \tau, t) \geq 0$ ;
2.  $r \leq p(c, r, \tau, t) \leq c$ ,  $p(c, r, \tau, t) = c$  iff  $c = r$ ;
3.  $p_{cc}(c, r, \tau, t)$ ,  $p_{rr}(c, r, \tau, t)$ , and  $p_{cr}(c, r, \tau, t)$  exist and are bounded away from  $\infty$  and  $-\infty$ ;
4.  $p_c(z, z, \tau, t) = 1 - \delta(\tau)\alpha(z, \tau, t)$ ,  
 $p_r(z, z, \tau, t) = \delta(\tau)\alpha(z, \tau, t)$ ,  
 $p_{cc}(z, z, \tau, t) = -U''(0)\delta(\tau)\alpha(z, \tau, t)(1 - \delta(\tau)\alpha(z, \tau, t)) - \delta(\tau)h(z, \tau, t)$ ,  
 $p_{rr}(z, z, \tau, t) = -U''(0)\delta(\tau)\alpha(z, \tau, t)(1 - \delta(\tau)\alpha(z, \tau, t)) + \delta(\tau)\alpha'(z, \tau, t)$ ,  
 $p_{cr}(z, z, \tau, t) = U''(0)\delta(\tau)\alpha(z, \tau, t)(1 - \delta(\tau)\alpha(z, \tau, t))$ .

**Proof:** *Appendix A.*  $\square$

Property 4 in this proposition concerns the behavior of the inverse cutoff function when the buy price (and thus the cutoff) equals the reserve price (in which case all arriving bidders with  $v > r$  accept the BP). Since we assume that  $p \geq r$  (and thus  $c \geq r$ ), the derivatives when  $c = r$  should be interpreted as one-sided derivatives.

### 3 Identification

We now consider identification of the structural demand parameters  $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$  of this model. Heuristically, we suppose we have many independent observations of auctions with the same  $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$ , and exogenous variation in the reserve price ( $r$ ), buy price ( $p$ ), and bidding phase length ( $\tau$ ). Formally, we define random variables  $R, P, \Upsilon$ , whose realizations are  $r, p, \tau$ . Let  $F_{r,p,\tau}$  denote their joint distribution. Conditional on  $R = r, P = p, \Upsilon = \tau$ , we have a distribution for the auction outcomes determined by the (fixed) structural demand parameters, and the auction mechanism and equilibrium solution described in Section 2. Given knowledge of the joint distribution of  $R, P, \Upsilon$  and the auction outcomes, we want to recover the structural demand parameters.

The assumption that variation in the reserve price, buy price, and length is exogenous may be strong in some situations. Even if we view the identification analysis as conditional on auction-level covariates, the variation in  $r, p$ , and  $\tau$  could arise from unobserved (to the econometrician) differences across auctions that sellers take into account when choosing the auction features (Krasnokutskaya (2004), Asker (2008), Roberts (2009), Descarolis (2009)). It may be possible to relax this exogeneity assumption using instrumental variables techniques, but we do not pursue this in the current paper. However, the assumption may be credible in some markets where one believes the majority of variation in auction setup is due to seller characteristics rather than unobserved demand factors, or in experiments with true randomized variation (in the lab, field, or elsewhere).

Our single-unit theoretical model in Section 2 also embodies an assumption that there are no other auctions, either simultaneously, or in the future. This is often assumed in the auction literature (with some notable exceptions, e.g. Pesendorfer and Jofre-Benet (2003), Zeithammer (2006, 2007, 2009), Nekipelov (2008), Zeithammer and Adams (2009), and Backus and Lewis (2009)), but it may be questionable for some product categories on eBay and similar markets.

#### 3.1 Observational Model

Now we specify the auction outcomes that are observed. Let  $T_1$  be the time of the first action (either accepting or rejecting the BP) taken by any bidder in the auction. Let  $B = 0$  indicate that the first acting bidder rejected the BP, and let  $B = 1$  indicate that the first acting bidder

accepted the BP. The parameters of the model  $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$  determine a joint distribution for  $(T_1, B)$  given  $(P = p, R = r, \Upsilon = \tau)$ . Let  $F_1(\cdot|p, r, \tau)$  denote the conditional distribution of  $T_1$  given  $(P = p, R = r, \Upsilon = \tau)$ , and let  $\Pr(B = 1|p, r, \tau, t_1)$  denote the conditional probability of the BP option being accepted given  $(P = p, R = r, \Upsilon = \tau, T_1 = t_1, \cdot)$ .

Our basic identification results will only require that the outcomes  $T_1$  and  $B$  are observed (along with the “exogenous” variables  $(P = p, R = r, \Upsilon = \tau)$ ). This will be enough to identify  $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$ , using the implications of Propositions 1 and 3. In principle, we might observe other outcome variables, for example the final price in the auction, or the sealed bids placed by participants, or the proxy bids in eBay auctions.<sup>6</sup> In Sections 4 and 5 we examine the identifying power of some of these other outcome variables.

To derive the simplest version of our identification results, we make the following assumption on the support of  $(R, P, \Upsilon)$ :

**Assumption 4** *The marginal distribution of  $R$  has support  $[0, \infty)$  and the conditional distribution of  $P$  given  $R = r$  has support  $[r, \infty)$ . The conditional distribution of  $\Upsilon$  given  $(R = r, P = p)$  has support  $[0, \infty)$ .*

This support condition is relaxed in Section 4.

### 3.2 Identification of $\lambda(\cdot)$ and $F_V(\cdot)$

We begin by examining identification of the arrival rate and valuation distribution. Our arguments for identification of these two objects are similar to Canals-Cerda and Percy (2008), who consider identification in eBay auctions without buy prices (and without impatience or risk aversion) Recall that potential bidders arrive according to a Poisson process with arrival rate  $\lambda(t)$ , and by Proposition 1, if no other action has yet been taken, the arriving bidder takes an action if her valuation  $V \geq r$ . Hence the time of the first observed action  $T_1$  in an auction, given  $(P = p, R = r, \Upsilon = \tau)$ , has conditional hazard rate

$$\theta(t_1|p, r, \tau) = \lambda(t_1)(1 - F_V(r)), \quad (4)$$

To separately identify  $\lambda(\cdot)$  and  $F_V(\cdot)$ , note that when  $r = 0$ ,  $F_V(r) = 0$ , so

$$\lambda(t_1) = \theta(t_1|p, 0, \tau).$$

Since the Poisson intensity is bounded,  $T_1|P, R = 0$  has full support  $[0, \infty)$ , so conditioning on  $R = 0$  identifies  $\lambda(\cdot)$  on  $[0, \infty)$ .

Given that  $\lambda(\cdot)$  has been identified, we can identify  $F_V(\cdot)$  by using equation (4) for values of  $r$  other than 0. This identifies  $F_V(\cdot)$  on  $[0, \infty)$ .

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<sup>6</sup>In some cases, these other variables may be hard to interpret, e.g. eBay’s proxy bids.

Since  $\lambda(\cdot)$  and  $F_V(\cdot)$  are identified, it follows that  $\alpha(r, \tau, t_1)$  and  $h(y, \tau, t_1)$  are identified over their full supports, since we can form  $\gamma = \int_{t_1}^{t_1+\tau} \lambda(s)ds$  for any  $t_1$  and then directly construct

$$\begin{aligned}\alpha(r, \tau, t_1) &= \exp(\gamma F_V(r) - \gamma), \\ h(y, \tau, t_1) &= \exp(\gamma F_V(y) - \gamma) \gamma f_V(y).\end{aligned}$$

### 3.3 Identification of $c(p, r, \tau, t_1)$ and $p(c, r, \tau, t_1)$

To identify the cutoff function and its inverse, we use the observed distribution of  $B$  given  $(P = p, R = r, \Upsilon = \tau, T_1 = t_1)$ . For a bidder who takes an action at time  $t_1$ , the distribution of her valuation  $V$  is  $F_V$  truncated below at  $r$ . By Proposition 3, this bidder will accept the buy price if  $V \geq c(p, r, \tau, t_1)$ . Therefore,

$$\Pr(B = 1|p, r, \tau, t_1) = \frac{1 - F_V(c(p, r, \tau, t_1))}{1 - F_V(r)}.$$

Given knowledge of  $F_V$  and the conditional probability of  $B = 1$ , we can invert to obtain

$$c(p, r, \tau, t_1) = F_V^{-1}(1 - (1 - F_V(r)) \Pr(B = 1|p, r, \tau, t_1)).$$

This identifies the cutoff function on the joint support of  $(P, R, \Upsilon, T_1)$ . Having identified  $c(p, r, \tau, t_1)$ , we can then invert it to obtain the inverse cutoff function  $p(c, r, \tau, t_1)$  and the first and second derivatives of  $p(c, r, \tau, t_1)$  with respect to  $c$  and  $r$ .

### 3.4 Identification of $U(\cdot)$

We now consider identification of the utility function. By definition, the cutoff function  $c(p, r, \tau, t_1)$  gives the valuation of the bidder who is indifferent between accepting or rejecting the BP given  $(p, r, \tau, t_1)$ . Similarly, the inverse cutoff function  $p(c, r, \tau, t)$  gives the BP  $p$  that would make a bidder with valuation  $c$  indifferent between accepting or rejecting the BP. As noted in Section 2, this indifference condition can be written as

$$U(c - p(c, r, \tau, t_1)) = \delta(\tau) \left( \alpha(r, \tau, t_1) U(c - r) + \int_r^c U(c - y) h(y, \tau, t_1) dy \right). \quad (5)$$

This integral equation will hold for all  $t_1 \in [0, \infty)$ ,  $r \in [p, \infty)$ ,  $c \in [r, \infty)$ , and  $\tau \in (0, \infty)$ . To identify the utility function  $U(\cdot)$ , we need to show that given knowledge of  $\alpha(r, \tau, t_1)$ ,  $h(y, \tau, t_1)$  and  $p(c, r, \tau, t_1)$ , there is a unique utility function  $U(\cdot)$  that satisfies this integral equation.

Equation (5) is an integral equation with a vanishing delay term.<sup>7</sup> In general, such equations do

<sup>7</sup>The delay term is  $p(c, r, \tau, t)$  in  $U(c - p(c, r, \tau, t))$ . The delay is called ‘‘vanishing’’ because  $(c - p(c, r, \tau, t)) \rightarrow 0$  as  $c \rightarrow 0$ .

not have simple solutions. However, the delay term can be eliminated by differentiating the integral equation with respect to  $c$ :

$$U'(c - p(c, r, \tau, t_1))(1 - p_c(c, r, \tau, t_1)) = \delta(\tau) \left( \alpha(r, \tau, t_1)U'(c - r) + \int_r^c U'(c - y)h(y, \tau, t_1)dy \right); \quad (6)$$

and with respect to  $r$ :

$$U'(c - p(c, r, \tau, t_1))(-p_r(c, r, \tau, t_1)) = \delta(\tau) \left( -\alpha(r, \tau, t_1)U'(c - r) + U(c - r) \left( \frac{\partial \alpha(r, \tau, t_1)}{\partial r} - h(r, \tau, t_1) \right) \right); \quad (7)$$

$$U'(c - p(c, r, \tau, t_1))(-p_r(c, r, \tau, t_1)) = -\delta(\tau)\alpha(r, \tau, t_1)U'(c - r); \quad (8)$$

where the second line follows because  $\frac{\partial \alpha(r, \tau, t_1)}{\partial r} = h(r, \tau, t_1)$ .

Under our assumptions, each side of (8) is bounded away from 0. Dividing (6) by (8) cancels the delay term (and eliminates the impatience term), and differentiating the resulting equation with respect to  $r$  results in an ordinary first order linear differential equation in  $U'(\cdot)$ , i.e.

$$U''(c - r) = \frac{(\Phi_r(c, r, \tau, t_1) + h(r, \tau, t_1))}{\Phi(c, r, \tau, t_1)}U'(c - r), \quad (9)$$

where

$$\Phi(c, r, \tau, t_1) = \alpha(r, \tau, t_1) \left[ \frac{(1 - p_c(c, r, \tau, t_1))}{p_r(c, r, \tau, t_1)} - 1 \right].$$

Since  $h(r, \tau, t_1)$  and all the components of  $\Phi(c, r, \tau, t_1)$  have already been shown to be identified, we only need to consider whether there is a unique solution  $U(\cdot)$  to this differential equation (with  $U(0) = 0$  and  $U'(0) = 1$ ).

**Proposition 5** *Under Assumptions 1, 3, and 4, there is a unique  $U(\cdot)$  on support  $[0, \infty)$  satisfying (9). Hence,  $U(\cdot)$  is identified on support  $[0, \infty)$ .*

**Proof:** *Appendix A.*  $\square$

The intuition here is that first order linear differential equations like (9) typically have a unique solution given an initial condition (which in our case is  $U'(0) = 1$ ). Note that (9) holds for any values of  $(c(p, r, \tau, t_1), r, \tau, t_1)$ . So, for example, one can fix  $(r, \tau, t_1)$  and use variation in  $p$  across its support (i.e. in  $c(p, r, \tau, t_1)$ ) to trace out  $U(\cdot)$ . Since this can be done at any  $(r, \tau, t_1)$ , this will generate overidentifying restrictions, which will be discussed in the next section.

Equation (9) can be rewritten as

$$\frac{U''(c - r)}{U'(c - r)} = \frac{(\Phi_r(c, r, \tau, t_1) + h(r, \tau, t_1))}{\Phi(c, r, \tau, t_1)}. \quad (10)$$

This implies that to identify the Arrow-Pratt measure of risk aversion at a certain point, one only needs to compute the values of  $\Phi(c, r, \tau, t_1)$ ,  $\Phi_r(c, r, \tau, t_1)$ , and  $h(r, \tau, t_1)$  at that point. On a related note, we investigate how sensitive our identification results are to the support condition (Assumption 4) in Section 4.1.

### 3.5 Identification of $\delta(\cdot)$

Lastly, consider identification of the impatience function. Manipulating the indifference condition (5) gives us

$$\delta(\tau) = \frac{U(c - p(c, r, \tau, t_1))}{\left( \alpha(r, \tau, t_1)U(c - r) + \int_r^c U(c - y)h(y, \tau, t_1)dy \right)}. \quad (11)$$

Since all of the terms on the right-hand-side have already been shown to be identified, it is clear that  $\delta(\cdot)$  is identified. Since  $\tau$  varies over support  $(0, \infty)$  and we have normalized  $\delta(0) = 1$ ,  $\delta(\cdot)$  is identified on  $[0, \infty)$ .

## 4 Extensions

### 4.1 Relaxing Support Conditions

Assumption 4 requires the exogenous random variables  $(R, P, \Upsilon)$  to have large supports, which may be unrealistic in many applications. We now examine how restrictions on the supports affect our identification results. First consider a situation where all the auctions in the data set have a fixed bidding phase length  $\tau_0$ .

**Assumption 5** *The marginal distribution of  $R$  has support  $[0, \infty)$  and the conditional distribution of  $P$  given  $R = r$  has support  $[r, \infty)$ . The conditional distribution of  $\Upsilon$  given  $(R, P)$  has  $Pr(\Upsilon = \tau_0 | R, P) = 1$  almost surely.*

**Proposition 6** *Under Assumptions 1, 3, and 5,  $\{F_V(\cdot), \lambda(\cdot), U(\cdot)\}$  are identified over their full supports, and  $\delta(\tau)$  is identified at  $\tau_0$ .*

Intuitively, if the bidding phase has fixed length  $\tau_0$ , we can only identify  $\delta(\cdot)$  at that point. However, the other structural functions are identified over their entire supports by our previous arguments.

In practice, it may also be difficult to estimate  $\lambda(t)$  for large  $t$ , because this would require observing many auctions that last until time  $t$  without an action being taken. To capture this situation, we suppose that observations on auctions are truncated at some time  $\bar{T}$ :

**Assumption 6** *There is some  $\bar{T} > \tau_0$  such that we only observe  $(T_1, B)$  when  $T_1 < \bar{T}$ .*

In this case, even though we can only identify the Poisson process prior to time  $\bar{T}$ , we can still identify  $F_V(\cdot)$  and  $U(\cdot)$ . Specifically, the following result follows from the arguments in Section 3.

**Proposition 7** *Under Assumptions 1, 3, 5, and 6,  $\{F_V(\cdot), U(\cdot)\}$  are identified over their full supports,  $\lambda(\cdot)$  is identified on support  $[0, \bar{T})$ , and  $\delta(\tau)$  is identified at  $\tau_0$ .*

Next, we consider restricting the support of  $R$  to the bounded set  $[\underline{r}, \bar{r}]$ .

**Assumption 7** *The marginal distribution of  $R$  has support  $[\underline{r}, \bar{r}]$  and the conditional distribution of  $P$  given  $R = r$  has support  $[r, \infty)$ . The conditional distribution of  $\Upsilon$  given  $(R, P)$  has  $Pr(\Upsilon = \tau_0 | R, P) = 1$  almost surely. There is some  $\bar{T} > \tau_0$  such that we only observe  $(T_1, B)$  when  $T_1 < \bar{T}$ .*

**Proposition 8** *Under Assumptions 1, 3, and 7,  $\lambda(t)(1 - F_V(r))$  is identified on  $r \in [\underline{r}, \bar{r}]$  and  $t \in [0, \bar{T})$ ,  $\delta(\tau)$  is identified at  $\tau_0$ , and  $U(\cdot)$  is identified on  $[0, \bar{r} - \underline{r}]$ .*

**Proof:** Appendix A.  $\square$

In this case, since the reserve price never goes below  $\underline{r}$ , we cannot distinguish between a non-arrival and an arrival of a bidder with valuation below  $\underline{r}$ . As a consequence,  $\lambda(t)$  and  $F_V(r)$  are not separately identified. However, over limited supports, we can identify the composite function  $\lambda(t)(1 - F_V(r))$ , and through this  $\alpha(r, \tau, t_1)$ ,  $h(y, \tau, t_1)$  and  $c(p, r, \tau, t_1)$ . A slight modification of our original identification argument leads to the final result. The support on which  $U(\cdot)$  is identified depends on the range of the reserve price in the support of the data.

Lastly, we further restrict the support of the buy price  $P$ .

**Assumption 8** *The marginal distribution of  $R$  has support  $[\underline{r}, \bar{r}]$  and the conditional distribution of  $P$  given  $R = r$  has support containing  $[p_0 - \epsilon, p_0 + \epsilon]$  for some  $\epsilon > 0$ . The conditional distribution of  $\Upsilon$  given  $(R, P)$  has  $Pr(\Upsilon = \tau_0 | R, P) = 1$  almost surely. There is some  $\bar{T} > \tau_0$  such that we only observe  $(T_1, B)$  when  $T_1 < \bar{T}$ . There exists  $r^* \in (\underline{r}, \bar{r})$  and  $t^* < \bar{T} - \tau_0$  such that  $c(p_0, r^*, \tau_0, t^*) \in (\underline{r}, \bar{r})$ .*

**Proposition 9** *Under Assumptions 1, 3, and 8,  $\lambda(t)(1 - F_V(r))$  is identified on  $r \in [\underline{r}, \bar{r}]$ ,  $\delta(\tau)$  is identified at  $\tau_0$ , and  $\frac{U''(\cdot)}{U'(\cdot)}$  is identified at the point  $c(p_0, r^*, \tau_0, t^*) - r^*$ .*

**Proof:** Appendix A.  $\square$

The last condition of Assumption 8 requires the observed buy price  $p_0$  being low enough such that the cutoff at this buy price is within the range  $[\underline{r}, \bar{r}]$  (for some  $r^*$  and  $t^*$ ). This is needed to identify

the cutoff function. However, provided this holds, we only need a small amount of variation in the buy price to identify the Arrow-Pratt measure of risk aversion at a particular  $c(p_0, r^*, \tau_0, t^*) - r^*$ . If there is a set of points  $(r^*, t^*)$  such that  $r^* \in (\underline{r}, \bar{r})$ ,  $t^* < \bar{T} - \tau_0$ , and  $c(p_0, r^*, \tau_0, t^*) \in (\underline{r}, \bar{r})$ , then the Arrow-Pratt measure of risk aversion will be identified at all the corresponding values of  $c(p_0, r^*, \tau_0, t^*) - r^*$ .

In summary, we can relax our original support conditions in various ways and still obtain “local” identification of the structural objects. However, even Assumption 8 makes a significant joint support condition on  $r$  and  $p$ . Intuitively, to identify  $U(\cdot)$  and  $\delta(\cdot)$  locally, we need variation in the reserve price and we need that there are buy prices low enough that the equilibrium cutoff is sometimes in this range.

## 4.2 Testable Restrictions

Our model has a number of testable restrictions on the observed data (or functions of the observed data). These follow from the discussion in Sections 2 and 3 and include:

1. The conditional hazard  $\theta(t_1|p, r, \tau)$  does not depend on  $p$  or  $\tau$ ;
2. The ratio  $\frac{\theta(t_1|p, r, \tau)}{\theta(t_1|p, 0, \tau)}$  does not depend on  $p, \tau$ , or  $t_1$ ;
3.  $r \leq p(c, r, \tau, t_1) \leq c$ ,  $0 < p_c(c, r, \tau, t_1) < 1$ ,  $p_r(c, r, \tau, t_1) > 0$ ,  $p_\tau(c, r, \tau, t_1) > 0$ ,  $p_c(z, z, \tau, t_1) = 1 - \alpha(z, \tau, t_1)$ ,  $p_r(z, z, \tau, t_1) = \alpha(z, \tau, t_1)$ ;
4. The coefficient in the differential equation (9), i.e.

$$\frac{(\Phi_r(c, r, \tau, t_1) + h(r, \tau, t_1))}{\Phi(c, r, \tau, t_1)},$$

does not depend on  $\tau$  and  $t_1$ . The coefficient only depends on  $c$  and  $r$  through the difference  $c - r$ .<sup>8</sup>

These restrictions could be tested, and also imply that more flexible versions of the model can be identified. For example, restrictions 1 and 2 imply that one can identify a model where the distribution of a bidder’s valuation depends on their time of arrival, i.e.  $F_V(v, t)$ . Restriction 4 implies we could identify an extended model where a bidder’s utility function depends on their time of arrival, i.e.  $\delta(\tau)U(v - p, t)$ .<sup>9</sup> Ideally, we would like to find necessary and sufficient conditions

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<sup>8</sup>The fact that (9) implies

$$\frac{U''(c - r)}{U'(c - r)} = \frac{(\Phi_r(c, r, \tau, t_1) + h(r, \tau, t_1))}{\Phi(c, r, \tau, t_1)}$$

generates these restrictions.

<sup>9</sup>One could potentially investigate whether even more general models are identified or partially identified, e.g. the utility function  $U(\tau, v - p, t)$  or  $U(\tau, v, p, t)$ , or models where bidders are heterogeneous in their risk attitudes or their impatience rather than in their valuations. Even more challenging would be models where bidder heterogeneity cannot be summarized by a scalar.

on  $\theta(t_1|p, r, \tau)$  and  $\Pr(B = 1|p, r, \tau, t_1)$  (which can be estimated directly from data) for these to be rationalized by our model  $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$  (see, e.g. Aryal, Perrigne, and Vuong (2009)). However, this appears to be quite complicated in our context. For example, the inverse cutoff function (3) implies that the relationship between  $p(c, r, \tau, t_1)$  (and thus  $\Pr(B = 1|p, r, \tau, t_1)$ ) at two values of  $t_1$  depends in a complicated way on the arrival process between those two points in time (as an integrand through  $h(r, \tau, t_1)$  and then through the inverse utility function).

### 4.3 Additional Data on Final Prices

Our basic identification argument only uses data on  $T_1$ , the time of the first observed arrival, and  $B$ , the indicator for whether the BP option was accepted or rejected. This data identifies the arrival rate  $\lambda(t)$ , the valuation distribution  $F_V(v)$ , and the inverse cutoff equation  $p(c, r, \tau, t_1)$ . Using  $\lambda(t)$  and  $F_V(v)$ , we can identify the functions  $\alpha(r, \tau, t_1)$  and  $h(y, \tau, t_1)$  in our integral equation (5). Given knowledge of  $\alpha(r, \tau, t_1)$ ,  $h(y, \tau, t_1)$ , and  $p(c, r, \tau, t_1)$ , we then showed identification of the utility components  $U(\cdot)$  and  $\delta(\cdot)$ .

This subsection considers an alternative approach to identifying  $\alpha(r, \tau, t_1)$  and  $h(y, \tau, t_1)$ . Recall that  $\alpha(r, \tau, t_1)$  is the probability that, given rejection of the BP at  $t_1$ , no other bidder with valuation greater than  $r$  arrives during the bidding phase.  $h(y, \tau, t_1)$  is the density of the maximum valuation of all future bidders arriving during the bidding phase (or the reserve price).<sup>10</sup>

Instead of first recovering  $\lambda(t)$  and  $F_V(v)$ , we can use the final outcomes for auctions that enter the bidding phase to learn  $\alpha(r, \tau, t_1)$  and  $h(y, \tau, t_1)$ . This approach will be particularly useful for our extension to eBay “Buy-It-Now” auctions considered below.<sup>11</sup> We make the following assumption about winning prices.

**Assumption 9** *If an auction enters the bidding phase, the final price in the auction is the 2nd highest valuation of all bidders who arrived at the auction.*

Note that Assumption 9 will hold in equilibrium if the bidding phase consists of either a 2nd price-sealed bid auction, a button auction, or an eBay style proxy-bidding auction (up to bid increments).

Consider an auction with setup  $(p, r, \tau)$ . Suppose that at  $T_1 = t_1$ , an arriving bidder rejects the BP option ( $B = 0$ ). This bidder has value  $V$  distributed according to  $F_V$  truncated between  $r$  and  $c(p, r, \tau, t_1)$ . Let  $\tilde{Y}$  equal the highest valuation among bidders arriving *after* time  $t_1$ , or equal  $R$  if no further bidders arrive to the auction with valuations greater than  $r$ . Under our assumptions,  $\tilde{Y}$  is conditionally independent of  $V$ , i.e.

$$\tilde{Y} \perp V | P = p, R = r, \Upsilon = \tau, T_1 = t_1, B = 0.$$

<sup>10</sup>In equilibrium, no bidder who arrived prior to  $t_1$  has valuation  $> r$ .

<sup>11</sup>The approach detailed in this section is related to a large literature on estimation methods for dynamic models initiated by Hotz and Miller (1993).

Suppose we observe the random variable  $W$ , an indicator that the bidder who rejected the BP option ended up winning the auction. Note that  $W = 1(\tilde{Y} < V)$ , since the bidder who rejected the BP only wins if her valuation is higher than the bidders entering during the bidding phase.

Suppose that conditional on  $W = 1$ , we observe the final price in the auction,  $Z$ . Under Assumption 9,  $Z = \tilde{Y}$  if  $W = 1$ . In other words, if the bidder who rejected the BP option wins the auction, then the final price  $Z$  is equal to the highest valuation of bidders arriving after time  $t_1$  (or  $r$  if there are no such bidders with valuations greater than  $r$ ).

Since  $W$  is observed and  $Z$  is observed (given  $W = 1$ ), we can identify

$$Pr(W = 1|p, r, \tau, t_1, B = 0),$$

and

$$p_Z(z|W = 1, p, r, \tau, t_1, B = 0).$$

The first term is the probability that the BP rejector wins the auction. The second term is the distribution of the final price given that the BP rejector wins the auction. (In general the conditional distribution of  $Z$  will have point mass at  $r$ , so we interpret conditional densities as being with respect to the sum of Lebesgue measure and counting measure at  $r$ .)

By Bayes' Theorem, we can write

$$\begin{aligned} p_Z(z|W = 1, r, p, \tau, t_1, B = 0) &= \frac{p_y(z|r, p, \tau, t_1, B = 0)Pr(W = 1|z, r, p, \tau, t_1, B = 0)}{Pr(W = 1|r, p, \tau, t_1, B = 0)} \\ &= \frac{p_y(z|r, p, \tau, t_1, B = 0) (F_V(c(p, r, \tau, t_1)) - F_V(z))}{Pr(W = 1|r, p, \tau, t_1, B = 0)}, \end{aligned}$$

where  $p_y$  indicates the conditional density of  $\tilde{Y}$ . Assuming we have already identified  $F_V(v)$  as in Section 3, we can use this equation to recover  $p_y(z|r, p, \tau, t_1, B = 0)$ . This identifies  $\alpha(r, \tau, t_1)$  and  $h(y, \tau, t_1)$ , since

$$\alpha(r, \tau, t_1) = P(\tilde{Y} = r|r, p, \tau, t_1, B = 0),$$

and

$$h(y, \tau, t_1) = p_y(y|r, p, \tau, t_1, B = 0) \text{ for } r < y < c(p, r, \tau, t_1).$$

Assuming we have also identified  $c(p, r, \tau, t_1)$ <sup>12</sup>, we can now use the integral equation (5) to identify  $U(\cdot)$  and  $\delta(\cdot)$  (note that Equation (5) only depends on  $h(y, \tau, t_1)$  on the support  $c(p, r, \tau, t_1) > y > r$ ). This approach allows us to identify  $U(\cdot)$  and  $\delta(\cdot)$  without requiring that  $\lambda(\cdot)$  be separately identified at later points in the auction. Note that this additional data provides additional testable restrictions of the model. For example,  $p_y(y|r, p, \tau, t_1, B = 0)$  should not depend on  $p$ .

<sup>12</sup>Note that one does not need to know  $\lambda(\cdot)$  at later points in the auction to identify  $F_V(\cdot)$  and  $c(p, r, \tau, t_1)$  (for low  $t_1$ ), using the arguments of Section 3.

This argument only uses the final price conditional on the BP rejector winning the auction. In principle, there is more information that could be used for identifying structural objects and generating testable restrictions of our model. For example, we could use the final price regardless of who wins the auction. If we also observe all the bids in the bidding phase, and these bids represent bidders' valuations, then we could use this information as well.<sup>13</sup> If we also observe clickstream data measuring when a user first visited a particular auction, this could provide an alternative source of identification for arrival rates.

## 5 Applications

### 5.1 eBay's Buy-it-Now Auctions

eBay's popular Buy-it-Now auctions feature a buy price option that disappears as soon as any bidder places a bid, which our original model captures. However, in eBay's BP auction format, there is a fixed length for the overall auction. Since eBay's auctions end at some fixed time  $T$ , the bidding phase has length  $T - t_1$ , not a fixed length  $\tau$  as we assumed in our model. We call our BP auction model a "Fixed  $\tau$ " BP auction, whereas eBay's Buy-it-Now auction is a "Fixed  $T$ " BP auction. In a Fixed  $T$  auction, the environment is defined by  $(p, r, T)$ . The cutoff function for a bidder arriving at  $t_1$  is then  $c(p, r, T, t_1)$ , depending on  $p$ ,  $r$ , and the overall length of the auction  $T$ . There are a number of papers that have focused on empirically studying eBay (or related) Buy-it-Now auctions, including Wan, Teo, and Zhu (2003), Chan, Kadiyali, and Park (2007), and Anderson, Friedman, Milam, and Singh (2008). Akerberg, Hirano, and Shahriar (2006) estimate a parametric structural model related to the non-parametric model we describe here.

Generally speaking, identification of parameters in a Fixed  $T$  auction is similar to that in a Fixed  $\tau$  auction. Variation in reserve prices across auctions can trace out arrival rates and valuations, variation in  $p$  identifies the cutoff function  $c(p, r, T, t_1)$ , and an integral equation similar to (5), i.e.

$$U(c - p(c, r, T, t_1)) = \delta(T - t_1) \left( \alpha(r, T, t_1)U(c - r) + \int_r^c U(c - y)h(y, T, t_1)dy \right), \quad (12)$$

identifies  $U(\cdot)$  and  $\delta(\cdot)$ .

However, the Fixed  $T$  BP auction has a complication that does not arise in our original model. Consider a potential bidder arriving at some  $t$ , with a valuation greater than  $r$ , but less than  $c(p, r, T, t)$ . In our original model, such a bidder has an incentive to immediately reject the BP. This is because immediately rejecting ends the auction sooner, and minimizes the expected amount of competition that bidder faces in the bidding phase. In contrast, in a Fixed  $T$  model, this bidder

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<sup>13</sup>Depending on the context, bids may only provide bounds on valuations (Haile and Tamer (2003), Zeithammer and Adams (2009)).

*does not* have a strict incentive to immediately reject the BP, since immediately rejecting does not end the auction sooner or limit competition.<sup>14</sup> This can lead to multiple equilibria, because bidders are indifferent between rejecting the BP immediately, or waiting.<sup>15</sup> Then we may not be able to identify  $F_V(r)$ , since we are not certain that observed actions are being taken by bidders who have just arrived.<sup>16</sup>

We can resolve this problem by either modifying the model, or using a solution concept that ensures that bidders who reject the BP act immediately. Gallien and Gupta (2007) discuss restricting attention to trembling hand perfect BNE, where the trembles involve a bidder accidentally accepting the BP. This results in an equilibrium where bidders who want to reject the BP do so immediately. The same is true if one adds a small monitoring cost to the model that is incurred when one waits to reject the BP.<sup>17</sup> In both cases, our identification arguments in Sections 2 and 3 can be applied to Fixed  $T$  auctions.

Because we need to ensure that certain subsets of the bidders act immediately upon entering the auction,<sup>18</sup> we are also relying heavily on the assumption that auctions are isolated. If the same (or a similar) good were potentially available in other eBay auctions, then bidders might have an incentive to wait before taking an action. This suggests that it might be valuable to investigate how our identification arguments extend to more dynamic settings (e.g. Jofre-Bonet and Pesendorfer (2003), Zeithammer (2006), Backus and Lewis (2009)).<sup>19</sup>

While our extension to eBay’s Buy-it-Now auctions require some further strong assumptions about behavior of bidders, our approach to identification has a number of attractive features. First, recall that in Fixed  $\tau$  auctions, if  $\tau$  is fixed in the data at  $\tau_0$ , one can identify  $\delta(\cdot)$  only at  $\tau_0$ . In a Fixed  $T$  auction, even if  $T$  is fixed in the data at  $T_0$ , one can identify the impatience function  $\delta(\cdot)$  at more than one point (due to variation in  $t_1$  - see Equation (12)). Second, note that our identification arguments do not require data on eBay proxy bids. As noted by Zeithammer and Adams (2009) among others, these bids may be hard to interpret on eBay. We investigate many of these and

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<sup>14</sup>Moreover, there is no reason to act immediately to prevent another bidder from entering and accepting the BP. This is because in a model where bidders are only heterogeneous in their valuations, this other bidder would always win the auction phase.

<sup>15</sup>In the Fixed  $T$  environment, bidders with valuations *above* the cutoff  $c(p, r, T, t_1)$  do have an incentive to accept the BP immediately, as they do not want to lose the item to another arriving bidder.

<sup>16</sup>This point illustrates an interesting theoretical advantage of Fixed  $\tau$  BP auctions vs Fixed  $T$  BP auctions. Specifically, identification is easier in Fixed  $\tau$  auctions because of the stronger incentives for bidders to act immediately. It would be interesting to compare expected revenue across the two types of BP auctions, though that seems beyond the scope of the current paper.

<sup>17</sup>Another perturbation is another model in Gallien and Gupta (2007), where there is assumed to be a point mass of “desperate” bidders who are very impatient and accept the BP immediately if they arrive. This also creates incentives for normal bidder to reject the BP immediately. However, with this model, one would want to explicitly consider identification of the point mass of desperate bidders, and check whether this affects identification of the other model components. The same issue arises in a perturbation where one adds a small utility benefit of participating in the bidding phase.

<sup>18</sup>We could extend the argument to allow for an exogenous random delay before acting.

<sup>19</sup>In the eBay context it may also be hard to identify the arrival process late in the auction, because it requires data on auctions in which no action is taken until close to the end of the auction. Hence the alternative approach described in Section 4.3 to identify  $\alpha(r, T, t_1)$  and  $h(y, T, t_1)$  using final prices might be particularly useful here.

other issues in our ongoing empirical work on eBay’s Buy-it-Now auctions (Akerberg, Hirano, and Shahriar (2006)).

## 5.2 GMAC Buy Price Auctions

GMAC uses a type of BP auction to sell fleet cars (cars coming off lease) to auto dealers around the US. In these auctions, the seller (GMAC) sets a BP  $p$  and a reserve price  $r$ . There are three distinct phases of the auction. In the first phase, only the option to buy the car at  $p$  is available to bidders—they cannot place bids. After a fixed length of time  $\tilde{T}$ , the auction enters the second phase, in which bidders can either accept the BP or reject the BP by placing a regular bid. If any bidder rejects the BP, the BP disappears for all bidders, and the auction enters the third phase, where bidders can only place bids (using a proxy bidding system similar to eBay’s). The auction ends at a fixed point in time  $T$ .

Thus, GMAC fleet auctions are similar to eBay auctions, except they have an introductory phase of fixed length, in which bidders cannot place regular bids, and in which the BP continues to be available unless it is accepted. The setup of a GMAC auction is described by  $(p, r, T, \tilde{T})$ .

As with eBay auctions, bidders who plan to reject the BP do not have strict incentives to do so immediately upon arrival. (Bidders who arrive at  $t_1 < \tilde{T}$  cannot immediately reject the BP even if they want to.) We could use similar arguments to the eBay case to ensure that bidders act immediately (when they can do so). However, we can avoid these arguments by making use of the initial phase of the GMAC auction.

Consider a GMAC auction where  $p = r$ . For such auctions, *all* potential bidders have strict incentives to act immediately. In any pure strategy, symmetric, BNE, all arriving bidders with valuations greater than  $p = r$  immediately accept the BP, because waiting risks the possibility that another bidder will accept the BP instead.

Thus, if the distribution of  $P|R = r$  has positive probability of  $P = r$  for  $r \in [r, \bar{r}]$ , we can identify  $\lambda(t)(1 - F_V(r))$  on  $[r, \bar{r}]$  for all  $t$ . This is because the hazard rate of the first observed action, i.e.  $\theta(t_1|p, r, T, \tilde{T})$ , satisfies

$$\theta(t_1|p, r, T, \tilde{T}) = \lambda(t_1)(1 - F_V(r)) \text{ when } p = r. \tag{13}$$

To identify the cutoff function  $c(p, r, T, \tilde{T}, t_1)$ , we need to also consider auctions where  $p > r$ . When  $p \geq r$ , the hazard rate of the first observed action satisfies

$$\theta(t_1|p, r, T, \tilde{T}) = \lambda(t_1)(1 - F_V(c(p, r, T, \tilde{T}, t_1))) \text{ for } t_1 < \tilde{T}. \tag{14}$$

Note that (14) *only* holds when  $t_1 < \tilde{T}$ . At  $\tilde{T}$  and after, first actions can also be taken by bidders who arrived earlier and have been “waiting.” This complicates matters for  $t_1 \geq \tilde{T}$ , because the

hazard rate of first observed action then depends on equilibrium selection issues (i.e. how long people wait). Fortunately, we will be able to identify parts of the functions  $U(\cdot)$  and  $\delta(\cdot)$  using only data when  $t_1 < \tilde{T}$ .

We now show that  $c(p, r, T, \tilde{T}, t_1)$  is identified over limited domains. When  $t_1 < \tilde{T}$ , the cutoff function  $c(p, r, T, \tilde{T}, t_1)$  can be shown to satisfy the implicit equation

$$\theta(t_1|c(p, r, T, \tilde{T}, t_1), c(p, r, T, \tilde{T}, t_1), T, \tilde{T}) = \theta(t_1|p, r, T, \tilde{T}).$$

Intuitively, this says that the hazard rate in an auction at  $(t_1, p, r, T, \tilde{T})$  is equivalent to the hazard rate in a hypothetical auction where the reserve price and buy price are both set at  $c(p, r, T, \tilde{T}, t_1)$ . The hazard on the left hand side of the equation, i.e.  $\theta(t_1|k, k, T, \tilde{T})$  is strictly decreasing in  $k$ , and identified in the data between  $\theta(t_1|\bar{r}, \bar{r}, T, \tilde{T})$  and  $\theta(t_1|\underline{r}, \underline{r}, T, \tilde{T})$ . The hazard on the right-hand-side is identified over its entire support in the data. Hence,  $c(p, r, T, \tilde{T}, t_1)$  is identified as long as  $\theta(t_1|\bar{r}, \bar{r}, T, \tilde{T}) < \theta(t_1|p, r, T, \tilde{T}) < \theta(t_1|\underline{r}, \underline{r}, T, \tilde{T})$ . More intuitively, we can identify the cutoff function at all points where the cutoff is between  $\underline{r}$  and  $\bar{r}$ . Note that this implies that the inverse cutoff function  $p(c, r, T, \tilde{T}, t_1)$  is identified on  $c \in [\underline{r}, \bar{r}]$ . Recall that these results only hold when  $t_1 < \tilde{T}$ .

Next, we need to identify  $\alpha(r, T, \tilde{T}, t_1)$  and  $h(y, T, \tilde{T}, t_1)$  in the integral equation

$$U(c - p(c, r, T, \tilde{T}, t_1)) = \delta(T - t_1) \left( \alpha(r, T, \tilde{T}, t_1)U(c - r) + \int_r^c U(c - y)h(y, T, \tilde{T}, t_1)dy \right). \quad (15)$$

The right-hand-side measures the expected utility if the bidder does not accept the BP, but the  $\alpha(r, T, \tilde{T}, t_1)$  and  $h(y, T, \tilde{T}, t_1)$  functions are now a bit more complicated. Bidders arriving at  $t_1 < \tilde{T}$  know that there may be a set of bidders who arrived beforehand that may participate in the bidding phase. Moreover, this set of prior arriving bidders have valuation distributions truncated between  $r$  and the cutoff function (which depends on the time<sup>20</sup>). However, given 1)  $\lambda(t)(1 - F_V(r))$  is already identified over all  $t$  and  $r \in [\underline{r}, \bar{r}]$ , and 2)  $c(p, r, T, \tilde{T}, t_1)$  is already identified for all  $t < \tilde{T}$  and on domain  $[\underline{r}, \bar{r}]$ , one can show that  $\alpha(r, T, \tilde{T}, t_1)$  is identified on  $r \in [\underline{r}, \bar{r}]$  and that  $h(y, T, \tilde{T}, t_1)$  is identified on  $y \in [\underline{r}, \bar{r}]$ . This allows one to identify  $\delta(\cdot)$  on support  $[T, T - \tilde{T})$ , and  $U(\cdot)$  on support  $[0, \bar{r}]$ .

To summarize, we have shown that in the GMAC fleet auction mechanism, one can identify the structural parameters, at least over certain domains. In contrast to the eBay example, we did not have to perturb the model to incentivize BP rejectors to act immediately. The introductory period in GMAC fleet auctions is useful because in this period, we know that all BP rejectors must wait to place a bid. The problem is more severe on eBay, because we cannot observe how many BP

<sup>20</sup>Note that in this model, the equilibrium cutoff function should increase in  $t_1$  because of 1) strict discounting, and 2) since in equilibrium, the later the BP is still available, the less competition one can anticipate in the bidding phase.

rejectors wait, and how many act immediately.

## 6 Conclusion

A BP auction allows a bidder to avoid uncertainties regarding winning and winning price, and/or to obtain the item sooner. As a result, bidder behavior in a BP auction is affected by her risk and time preferences. The existing theoretical literature on BP auctions has shown that when bidders are known to be either impatient or risk averse, sellers can increase expected revenue by using BP auctions. Our paper takes a different perspective, focusing on the extent to which data from BP auctions allows a researcher (or seller) to identify bidders' risk aversion and time preferences.

Using general forms of bidders' valuation distributions, risk aversion and impatience, the paper develops an IPV model for an auction with a temporary BP. Bidders arrive at the auction stochastically according to a Poisson process. Upon arrival, a bidder can either accept the BP, reject it (by placing a sealed bid), or wait and act later. The BP is available as long as no bidder has placed a bid. As soon as a bid is placed the BP option disappears, and the auction proceeds as a second-price sealed bid auction. We first characterize equilibria in the model, and derive conditions under which any symmetric pure-strategy subgame-perfect Bayesian-Nash equilibrium involves bidders acting immediately upon arrival and using cutoff strategies. Then we study identification. Given sufficient variation in auction characteristics such as the reserve price, the buy price, and the length of the bidding phase, we show that the four unknown structural objects—the arrival rate, valuation distribution, utility function, and the time-discounting function—are all nonparametrically identified.

The paper also provides extensions of the results to cases where the variation in auction characteristics is limited. We show how local identification of the structural objects can still be obtained in these cases. Under some additional assumptions, we extend our identification results to the specific setup of eBay's Buy-it-Now auctions and GMAC's BP auctions for fleet cars.

Our paper contributes to the literature on identification in auctions, on BP auctions, and more generally to the literature on recovering risk and time preferences from observed behavior when there is unobserved heterogeneity. Possible future extensions could include endogenizing auction characteristics, allowing higher dimensional bidder heterogeneity, or considering a dynamic environment with sequential or simultaneous auctions.

# A Appendix: Proofs of Propositions

## A.1 Proof of Proposition 1

We first set up our model as a random-player game (Milchtaich 2004), with the random set of players described by a point process. For background on point processes, see Kallenberg (1983), Fristedt and Gray (1996) Ch. 29, and Kallenberg (2010) Ch. 12.

Let  $\Gamma = \mathbb{R}_+ \times \mathbb{R}_+$  be the set of possible bidder types (with the usual Borel sigma-algebra). We interpret  $\gamma = (a, v) \in \Gamma$  to mean that a player arrives at time  $a$  and has valuation  $v$ . In our model, valuations are drawn independently from a probability distribution with CDF  $F_v$  and potential bidders arrive according to arrival rate  $\lambda(t)$ . In a slight abuse of notation, let  $F_v$  also stand for the probability measure associated with  $F_v$  and let  $\lambda$  be the measure defined by  $\lambda(A) = \int_A \lambda(t) dt$  for measurable subsets  $A \subset \mathbb{R}_+$ . Let  $\Pi$  be the product measure  $\lambda \times F_v$ . By our assumptions, this defines a Poisson point process with sigma-finite intensity  $\Pi$ . This process selects a random integer-valued (point) measure on  $\Gamma$ , which can be simply represented by a set of player-types  $\{(a_1, v_1), \dots, (a_k, v_k)\}$ . We use  $\mathcal{N}(\Gamma)$  to denote the set of possible realizations of point measures on  $\Gamma$ .

By Kallenberg (1983), Theorem 11.1, a bidder who knows her own type has a posterior for her competitors that is also a Poisson point process with intensity  $\pi$ . (See also Miltaich (2004).) Thus, from the point of view of any individual bidder who has arrived, her competitors are generated by the same Poisson point process.

Next, we define bidder actions and strategies. At each time  $t$  and given the history of the auction up to time  $t$ , a bidder can accept the buy price, reject the buy price and bid, or wait, and can choose the amount of the bid. Let  $\mathcal{A}$  be the space of functions mapping the current state of the auction to the bidder decisions (with a suitable sigma-algebra). Let  $\mathcal{A}^T$  be the set of functions from time  $t \in [0, \infty) = \mathcal{T}$  into  $\mathcal{A}$ . We endow this space with the sigma-algebra generated by the evaluation maps  $e_t : \mathcal{A}^T \rightarrow \mathcal{A}$  which return the decision function at time  $t$ . Let  $S \subset \mathcal{A}^T$  be the subset of action functions that are feasible according to the rules of the auction, By Kallenberg (2010), Lemma 3.1, elements of  $S$  that are pointwise (in time) measurable with respect to the sigma-algebra on  $\mathcal{A}$  are measurable with respect to the sigma-algebra on  $\mathcal{A}^T$  restricted to  $S$ .

A strategy  $\sigma : \Gamma \rightarrow S$  maps type into state- and time-contingent actions. We assume it is measurable with respect to the usual sigma-algebra on  $\Gamma$  and the sigma-algebra described above for  $S$ . We restrict attention to symmetric, pure strategies, so (in equilibrium) a single  $\sigma$  will represent the strategy profile of players, and will generate a random element  $\Sigma \in \mathcal{N}(S)$  which is the collection of action profiles of all the bidders. Let  $P_{\Pi\sigma}$  denote the distribution of  $\Sigma$  induced by the distribution  $\Pi$  over player times and the symmetric strategy  $\sigma$ .

From bidder  $i$ 's perspective, we can also think of the random point measure  $\Sigma^{-i}$  describing the

state-contingent action functions of players other than  $i$ . In the arguments to follow, it will be useful to regard  $\Sigma^{-i}$  both as a list of other players' strategies, and as a point measure, so that  $\Sigma^{-i}(R)$  gives the number of other players who play strategies belonging to the set  $R \subset S$ . By the conditioning properties of the Poisson process mentioned above, and the assumption that bidders' valuations are independent, the distribution of  $\Sigma^{-i}$  is also given by  $P_{\Pi\sigma}$ . Bidder  $i$ 's realized utility in the auction game is given by

$$U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}),$$

where the specific form of the utility function could be derived from the rules of the game as described in the main text.

Consider a candidate equilibrium of this game  $\sigma$ . We want to show by contradiction that under our assumptions,  $\sigma$  must satisfy the 3 statements of the Proposition. We first show Statement 1, then Statement 3, and finally Statement 2.

*Statement 1: Potential bidders with  $v_i < r$  never take any action.*

This is straightforward to show given that Assumption 2 rules out weakly dominated strategies. Suppose  $\sigma$  prescribes that a bidder with some  $v_i < r$ , arriving at  $a_i$ , either accepts the BP or places a bid at any point in time  $t \geq a_i$ . (Recall that bids must be  $\geq r$ .) This strategy is weakly dominated by the strategy  $\sigma^*$  that is identical to  $\sigma$  except that it prescribes that a bidder with  $(a_i, v_i)$  never takes any action:  $\sigma^*$  ensures that this bidder obtains a 0 utility realization,  $\sigma$  never gives such a bidder a strictly positive utility realization, and there are feasible opponent strategy profiles under which  $\sigma$  leads to a strictly negative utility outcome. (For example, suppose all other bidders follow a strategy of never taking any action). Hence,  $\sigma$  cannot be an equilibrium strategy in our game.

*Statement 3: Potential bidders with  $v_i > r$  who arrive during the bidding phase place a sealed bid equal to  $v_i$  at some point before the end of the auction.*

This is also straightforward given that Assumption 2 rules out weakly dominated strategies. Suppose  $\sigma$  prescribes that a bidder with some  $v_i > r$ , arriving at  $a_i$  during the bidding phase, never takes any action. This strategy is weakly dominated by the strategy  $\sigma^*$  that is identical to  $\sigma$  except that it prescribes that a bidder with  $(a_i, v_i)$  places a bid equal to  $v_i$  before the end of the auction:  $\sigma$  results in a 0 utility realization,  $\sigma^*$  cannot result in a strictly negative realization, and there are feasible opponent strategy profiles under which  $\sigma^*$  leads to a strictly positive utility outcome (e.g. suppose all other bidders follow a strategy of bidding  $r$  immediately upon arrival).

Suppose  $\sigma$  prescribes that a bidder with some  $v_i > r$ , arriving at  $a_i$  during the bidding phase, places a bid  $b_i < v_i$  at some point before the end of the auction. This strategy is weakly dominated by the strategy  $\sigma^*$  that is identical to  $\sigma$  except that it prescribes that a bidder with  $(a_i, v_i)$  places a bid equal to  $v_i$  before the end of the auction. There is no set of opponent strategies (i.e.,

realization of  $\Sigma^{-i}$ ) such that  $\sigma$  leads to a strictly greater utility realization than does  $\sigma^*$ , and there are realizations of  $\Sigma^{-i}$  such that  $\sigma^*$  leads to a strictly greater utility realization than does  $\sigma$  (e.g. suppose all other bidders follow a strategy of bidding their valuation immediately upon arrival, and consider realizations from  $\lambda$  and  $F_V$  such that only one bidder arrives prior to  $a_i$  and this bidder has a valuation in the interval  $(b_i, v_i)$ ).

Suppose  $\sigma$  prescribes that a bidder with some  $v_i > r$ , arriving at  $a_i$  during the bidding phase, places a bid  $b_i > v_i$  at some point before the end of the auction. This strategy is weakly dominated by the strategy  $\sigma^*$  that is identical to  $\sigma$  except that it prescribes that a bidder with  $(a_i, v_i)$  places a bid equal to  $v_i$  before the end of the auction. There is no set of opponent strategies (realization of  $\Sigma^{-i}$ ) such that  $\sigma$  leads to a strictly greater utility realization than does  $\sigma^*$ , and there are realizations of  $\Sigma^{-i}$  such that  $\sigma^*$  leads to a strictly greater utility realization than does  $\sigma$  (e.g. suppose all other bidders follow a strategy of bidding their valuation immediately upon arrival, and consider realizations from  $\lambda$  and  $F_V$  such that only one bidder arrives prior to  $a_i$  and this bidder has a valuation in the interval  $(v_i, b_i)$ ).

Hence, any strategy that involves any bidder arriving during the auction phase with  $v > r$  doing anything other than bidding their valuation before the end of the auction is weakly dominated.

*Statement 2: Potential bidders with  $v_i > r$  who arrive during the buy price phase immediately either a) "Accept the BP," i.e. purchase the good at  $p$ ; or b) "Reject the BP" by placing a sealed bid equal to  $v_i$ .*

The proof of this statement is a bit more involved. Note that it involves decisions prior to the auction entering the bidding phase. Hence, we enforce the implications of Statements 1 and 3 on behavior in the bidding phase. In particular, if the auction ever enters the bidding phase, all bidders arriving before the end of the bidding phase with  $v > r$  bid their valuations.

Consider a bidder arriving at time  $a_i$  while the BP is still available. Define

$$S_1 = \{s \in S \mid s \text{ prescribes accepting/rejecting the BP prior to } a_i\}.$$

Then, at time  $a_i$ , this bidder's expected utility is:

$$E[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i] = \int U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) dP(\Sigma^{-i} \mid \Sigma^{-i}(S_1) = 0),$$

where  $P(\cdot \mid \cdot)$  is the conditional probability measure of  $\Sigma^{-i}$  implied by  $P_{\Pi\sigma}$ . This conditions on  $\Sigma^{-i}(S_1) = 0$ , because bidder  $i$  knows that the BP is still available, and hence the realized point measure  $\Sigma^{-i}$  has no points in the set  $S_1$  of state-contingent action functions under which the BP would have been accepted or rejected prior to  $a_i$ . Other than this, we do not need to fully specify anything about  $P(\Sigma^{-i} \mid \Sigma^{-i}(S_1) = 0)$ , except that the opponent strategies must satisfy the implications of Statements 1 and 3 with probability one.

Now suppose that  $\sigma$  prescribes that a bidder arriving while the BP is still available at  $a_i$  with valuation  $v_i > r$  waits upon arrival at  $a_i$  (i.e. they do not immediately reject or accept the BP). We want to show that this contradicts  $\sigma$  being an equilibrium. We can consider three possibilities for  $\sigma$ , depending on what it prescribes in the *hypothetical* scenario where no other bidders take actions after time  $a_i$ :

1. if no other bidders were to take action after  $a_i$ , there is some finite  $t_R^* > a_i$  such that the bidder waits until  $t_R^*$  and then rejects the BP at  $t_R^*$ ;
2. if no other bidders were to take action after  $a_i$ , the bidder waits indefinitely (i.e. never accepting or rejecting the BP);
3. if no other bidders were to take action after  $a_i$ , there is some finite  $t_A^* > a_i$  such that the bidder waits until  $t_A^*$  and then accepts the BP at  $t_A^*$ .

We consider these alternatives one by one, in each case contradicting  $\sigma$  being an optimal strategy.

(Case 1) Consider the alternative strategy  $\sigma^*(a_i, v_i)$  that is identical to  $\sigma(a_i, v_i)$  except that it prescribes that a bidder with  $(a_i, v_i)$  immediately rejects the BP. For any realization of  $\Sigma^{-i}$  (satisfying Statements 1 and 3), it must be the case that

$$U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \geq U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}).$$

This is because under  $\sigma^*(a_i, v_i)$  the auction enters the bidding phase with probability 1 at  $a_i$ , and therefore, the bidder will compete in the bidding phase with only bidders that arrive prior to  $a_i + \tau$ . Even if the auction ends up in the bidding phase under  $\sigma(a_i, v_i)$ , the bidder competes with a weakly larger set of bidders.

It is also the case that for any opponent strategies satisfying Statements 1 and 3, there are realizations of  $\Sigma^{-i}$  with positive measure given  $\Sigma^{-i}(S_1) = 0$  such that

$$U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) > U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}).$$

To find such a set, consider realizations of  $\Sigma^{-i}$  corresponding to cases where no other bidders arrive through time  $a_i + \tau$ , and one other bidder (with  $v_i > v_j > r$ ) enters in the time interval  $(a_i + \tau, t_R^* + \tau]$ . Under strategy  $\sigma^*(a_i, v_i)$ , the bidder wins the item at  $r$ , while under strategy  $\sigma(a_i, v_i)$ , the bidder wins the item at  $v_j$ .

Hence, as long as opponents strategies satisfy Statements 1 and 3, it must be the case that

$$E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i] > E[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i],$$

which contradicts any strategy involving waiting to reject the BP being an equilibrium.

(Case 2) Consider the alternative strategy  $\sigma^*(a_i, v_i)$  that is identical to  $\sigma(a_i, v_i)$  except that it prescribes that a bidder with  $(a_i, v_i)$  immediately rejects the BP (rather than waiting indefinitely, as prescribed by  $\sigma(a_i, v_i)$ ). For any opponent strategies (satisfying Statements 1 and 3) and any realization of the point measure  $\Sigma^{-i}$ , it must be the case that

$$U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \geq U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}),$$

by arguments identical to Case 1.

It is also the case that for any opponent strategies satisfying Statements 1 and 3, there are realizations of  $\Sigma^{-i}$  with positive measure given  $\Sigma^{-i}(S_1) = 0$  such that

$$U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) > U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}).$$

To find such a set, consider realizations of  $\Sigma^{-i}$  corresponding to cases where no other bidders arrive through time  $a_i + \tau$ , and one other bidder (with  $v_j > v_i > r$ ) enters (and takes an action) after time  $a_i + \tau$ . Under strategy  $\sigma^*(a_i, v_i)$ , the bidder wins the item at  $r$ , while under strategy  $\sigma(a_i, v_i)$ , the bidder never wins the item.

Hence, as long as opponents strategies satisfy Statements 1 and 3, it must be the case that

$$E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i] > E[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i],$$

which contradicts any strategy involving waiting indefinitely being an equilibrium.

(Case 3) Now consider two alternative strategies:  $\sigma^*(a_i, v_i)$ , where a realized bidder with  $(a_i, v_i)$  immediately rejects the BP, and  $\sigma^{**}(a_i, v_i)$ , where a realized bidder with  $(a_i, v_i)$  immediately accepts the BP. We will show that either

$$E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0] > E[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0],$$

or

$$E[U(a_i, v_i, \sigma^{**}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0] > E[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0],$$

which contradicts  $\sigma$  being an equilibrium.

Define

$$S_2 = \{s \in S \mid s \text{ prescribes accepting/rejecting the BP at or prior to } t_A^*\}.$$

We can then write

$$\begin{aligned}
& E [U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0] \\
&= \Pr(\Sigma^{-i}(S_2) = 0 \mid \Sigma^{-i}(S_1) = 0) E [U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0] \\
&+ \Pr(\Sigma^{-i}(S_2) > 0 \mid \Sigma^{-i}(S_1) = 0) E [U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) > 0, \Sigma^{-i}(S_1) = 0]. \quad (16)
\end{aligned}$$

First, note that

$$\begin{aligned}
& E [U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0] \\
&= \delta(t_A^* - a_i)U(v_i - p) \\
&< U(v_i - p) \\
&= E [U(a_i, v_i, \sigma^{**}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0]. \quad (17)
\end{aligned}$$

The strict inequality follows from strict discounting. The last equality follows because according to  $\sigma^{**}(a_i, v_i)$ , the bidder immediately accepts the BP and obtains  $U(v_i - p)$  (this assumes either that 1) there is a tiebreaking rule that if more than one bidder accepts/rejects at exactly  $t$ , the action of the most recent arrival takes precedence, or 2) competitor strategies are such that another bidder accepting or rejecting the BP at exactly  $a_i$  is a zero probability event).

Next, note that

$$\begin{aligned}
& E [U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) > 0, \Sigma^{-i}(S_1) = 0] \\
&\leq E [U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) > 0, \Sigma^{-i}(S_1) = 0] \\
&\leq E [U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0]. \quad (18)
\end{aligned}$$

The first inequality follows because for any opponent strategies satisfying Statements 1 and 3, for any realization of  $\Sigma^{-i}$  such that  $\Sigma^{-i}(S_2) > 0$  and  $\Sigma^{-i}(S_1) = 0$ , it must be the case that

$$U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \leq U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}).$$

This is because for any realization of  $\Sigma^{-i}$  under which bidder  $i$  wins the auction using strategy  $\sigma(a_i, v_i)$ , the bidder would also win the auction (at a weakly lower price) under  $\sigma^*(a_i, v_i)$  (since under  $\sigma^*(a_i, v_i)$  the auction enters the bidding phase immediately). The second inequality of (18) follows from Lemma 10 below.

Combining (16), (17), and (18), we have

$$\begin{aligned}
& E [U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0] \\
&< \Pr(\Sigma^{-i}(S_2) = 0 \mid \Sigma^{-i}(S_1) = 0) E [U(a_i, v_i, \sigma^{**}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0] \\
&+ \Pr(\Sigma^{-i}(S_2) > 0 \mid \Sigma^{-i}(S_1) = 0) E [U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0].
\end{aligned}$$

Since  $\Pr(\Sigma^{-i}(S_2) = 0 \mid \Sigma^{-i}(S_1) = 0) + \Pr(\Sigma^{-i}(S_2) > 0 \mid \Sigma^{-i}(S_1) = 0) = 1$ , it must be the case that either

$$E[U(a_i, v_i, \sigma^{**}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0] > E[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0],$$

or

$$E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0] > E[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0].$$

Hence,  $\sigma$  cannot be an equilibrium.  $\square$

**Lemma 10** *Under the conditions of Proposition 1,*

$$E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) > 0, \Sigma^{-i}(S_1) = 0] \leq E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0].$$

**Proof:** It suffices to show that

$$E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0] \geq E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0]$$

Intuitively, the expected utility from immediately rejecting the BP is higher when we condition on the realized  $\Sigma^{-i}$  not having opponents who would have accepted or rejected the BP prior to  $t_A^*$ . Divide the random point measure  $\Sigma^{-i}$  into two components,  $\Sigma_{S_2}^{-i}$  and  $\Sigma_{-S_2}^{-i}$ .  $\Sigma_{S_2}^{-i}$  is the restriction of  $\Sigma^{-i}$  to  $S_2$ , and  $\Sigma_{-S_2}^{-i}$  is the restriction of  $\Sigma^{-i}$  to the complement of  $S_2$ . By the Poisson property, realizations of these point measures are independent of each other, so

$$\begin{aligned} & E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0] \\ &= E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0] \\ &= \int U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma_{S_2}^{-i}, \Sigma_{-S_2}^{-i}) dP(\Sigma_{-S_2}^{-i}) dP(\Sigma_{S_2}^{-i} \mid \Sigma_{S_2}^{-i}(S_2) = 0) \\ &= \int U(a_i, v_i, \sigma^*(a_i, v_i), \{0\}, \Sigma_{-S_2}^{-i}) dP(\Sigma_{-S_2}^{-i}). \end{aligned} \tag{19}$$

The first equality follows because  $S_1 \subseteq S_2$ . The second equality follows from the independence of the two random point measures. The third equality follows because  $\Sigma_{S_2}^{-i}(S_2) = 0$  implies that  $\Sigma_{S_2}^{-i}(S) = 0$  for any set  $S \subseteq S_2$ . Therefore, conditioning on  $\Sigma_{S_2}^{-i}(S_2) = 0$  makes  $\Sigma_{S_2}^{-i}$  non-random. (Our notation denotes this realization of  $\Sigma_{S_2}^{-i}$  as  $\{0\}$ .) Similarly,

$$\begin{aligned} & E[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0] \\ &= \int U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma_{S_2}^{-i}, \Sigma_{-S_2}^{-i}) dP(\Sigma_{-S_2}^{-i}) dP(\Sigma_{S_2}^{-i} \mid \Sigma_{S_1}^{-i}(S_1) = 0) \\ &\leq \int U(a_i, v_i, \sigma^*(a_i, v_i), \{0\}, \Sigma_{-S_2}^{-i}) dP(\Sigma_{-S_2}^{-i}). \end{aligned} \tag{20}$$

The last inequality follows because as long as opponents strategies satisfy Statements 1 and 3,

$$U\left(a_i, v_i, \sigma^*(a_i, v_i), \{0\}, \Sigma_{S_2}^{-i}\right) \geq U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma_{S_2}^{-i}, \Sigma_{S_2}^{-i}\right),$$

for any realization of  $\Sigma_{S_2}^{-i}$  and  $\Sigma_{S_2}^{-i}$  (i.e. the bidder is weakly better off with weakly fewer competitors in the auction). Combining (19) and (20), we get our desired result:

$$E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0\right] \geq E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right].$$

□

## A.2 Proof of Proposition 2

Consider the term

$$e^{-\gamma}U(v-r) + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} F_V^n(v) E_n[U(v - \max\{r, Y\}) \mid Y \leq v]. \quad (21)$$

Note that

$$\begin{aligned} & F_V^n(v) E_n[U(v - \max(r, Y)) \mid Y \leq v] \\ &= \int_0^v U(v - \max\{r, y\}) n F_V^{n-1}(y) f_V(y) dy \\ &= \int_0^r U(v-r) n F_V^{n-1}(y) f_V(y) dy + \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy \\ &= U(v-r) F_V^n(r) + \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy. \end{aligned}$$

So we can write (21) as:

$$\begin{aligned} & e^{-\gamma}U(v-r) + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} U(v-r) F_V^n(r) + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy \\ &= e^{-\gamma}U(v-r) \left[1 + \sum_{n=1}^{\infty} \frac{\gamma^n F_V^n(r)}{n!}\right] + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy \\ &= e^{-\gamma}U(v-r) \exp(\gamma F_V(r)) + e^{-\gamma} \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy \\ &= U(v-r) \exp[\gamma F_V(r) - \gamma] + e^{-\gamma} \int_r^v U(v-y) f_V(y) \left[ \sum_{n=1}^{\infty} \frac{\gamma^n n F_V^{n-1}(y)}{n!} \right] dy, \end{aligned}$$

where the last equality follows from the dominated convergence theorem. We also have

$$\sum_{n=1}^{\infty} \frac{\gamma^n n F_V^{n-1}(y)}{n!} = \sum_{n=1}^{\infty} \frac{\gamma \gamma^{n-1} F_V^{n-1}(y)}{(n-1)!} = \gamma \exp(\gamma F_V(y)),$$

so

$$U^R(v, t) = \delta(T - t) \left\{ U(v - r) \exp[\gamma F(r) - \gamma] + \int_r^v U(v - y) \exp(\gamma F_V(y) - \gamma) \gamma f_V(y) dy \right\}.$$

The other parts of the Proposition are straightforward to verify.

□

### A.3 Proof of Proposition 3

The certainty equivalent function  $M(v, r, \tau, t)$  is given by

$$M(v, r, \tau, t) = U^{-1} \left( \delta(\tau) \left( \alpha(r, \tau, t) U(v - r) + \int_r^v U(v - y) h(y, \tau, t) dy \right) \right).$$

We want to show that given Assumptions 1, 2, and 3, there exists a cutoff function  $c(p, r, \tau, t)$  such that when  $v > r$ :

1. For  $v > c(p, r, \tau, t)$ ,  $U^A(v, p) > U^R(v, r, \tau, t)$ ;
2. For  $v < c(p, r, \tau, t)$ ,  $U^A(v, p) < U^R(v, r, \tau, t)$ ;
3. For  $v = c(p, r, \tau, t)$ ,  $U^A(v, p) = U^R(v, r, \tau, t)$ .

First, suppose that  $p = r$ . In this case,

$$\begin{aligned} U^R(v, r, \tau, t) &= \delta(\tau) \left( \alpha(p, \tau, t) U(v - p) + \int_p^v U(v - y) h(y, \tau, t) dy \right) \\ &< \delta(\tau) U(v - p) \\ &< U(v - p) \\ &= U^A(v, p), \end{aligned}$$

for all  $v > r$ . The strict inequalities follow from the conditions on  $F_V$ ,  $\lambda$ ,  $U$ , and  $\delta$  in Assumption 1. Hence, in this case we can define the cutoff function  $c(p, r, \tau, t) = r$ , since all bidders with  $v > r$  strictly prefer accepting the BP to rejecting it (and those with  $v = r$  are indifferent).

Now, consider a buy price  $p > r$  and define  $c(p, r, \tau, t)$  to be the  $c$  that solves

$$G(c, p, r, \tau, t) := c - p - M(c, r, \tau, t) = 0. \tag{22}$$

We show that such a  $c$  exists and is unique and finite. First note that if  $c = r$ ,  $G(c, p, r, \tau, t) = r - p < 0$  (since  $M(r, r, \tau, t) = 0$  and  $p > r$ ).

Next, Assumption 3 implies that

$$\frac{\partial G(c, p, r, \tau, t)}{\partial c} = 1 - M_v(c, r, \tau, t) > \epsilon > 0.$$

Hence, at  $c = r + \frac{p-r}{\epsilon}$ ,  $c - p - M(c, r, \tau, t) > 0$ . Since  $M(c, r, \tau, t)$  can be shown to be continuous and strictly increasing in its first argument, there is a unique  $c \in (r, r + \frac{p-r}{\epsilon})$  s.t.  $c - p - M(c, r, \tau, t) = 0$ . By the global implicit function theorem (e.g. Ge and Wang (2002), Lemma 1), the function  $c(p, r, \tau, t)$  exists.

Since  $G(c, p, r, \tau, t)$  is strictly increasing in  $c$ , it must be the case that for  $v > c(p, r, \tau, t)$ ,

$$\begin{aligned} v - p - M(v, r, \tau, t) &> 0; \\ v - p &> M(v, r, \tau, t); \\ U(v - p) &> \delta(\tau) \left( \alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)dy \right); \end{aligned}$$

so potential bidders with  $v > c(p, r, \tau, t)$  will optimally accept the BP. Similarly, when  $v < c(p, r, \tau, t)$ ,

$$\begin{aligned} v - p - M(v, r, \tau, t) &< 0; \\ v - p &< M(v, r, \tau, t); \\ U(v - p) &< \delta(\tau) \left( \alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)dy \right); \end{aligned}$$

so the potential bidder will optimally reject the BP. Potential bidders with  $v = c(p, r, \tau, t)$  are indifferent between accepting and rejecting the BP. Hence, under our assumptions, the BP decision follows a cutoff strategy.

Since  $c(p, r, \tau, t) - M(c(p, r, \tau, t), r, \tau, t) = p$  for  $p > r$ , we also have

$$\begin{aligned} c(p, r, \tau, t) - p &= M(c(p, r, \tau, t), r, \tau, t); \\ U(c(p, r, \tau, t) - p) &= U(M(c(p, r, \tau, t), r, \tau, t)); \\ U(c(p, r, \tau, t) - p) &= \delta(\tau) \left( \alpha(r, \tau, t)U(c(p, r, \tau, t) - r) + \int_r^{c(p, r, \tau, t)} U(c(p, r, \tau, t) - y)h(y, \tau, t)dy \right). \end{aligned}$$

Note that this equation is also satisfied when  $p = r$  (and thus  $c(p, r, \tau, t) = r = p$ ), since both sides of the equation equal 0.

To derive the properties of the derivatives of the cutoff function  $c(p, r, \tau, t)$ , recall that the cutoff satisfies Equation (22). Under our assumptions,  $G(c, p, r, \tau, t)$  is continuously differentiable in all its arguments, and as previously shown,  $\frac{\partial G(c, p, r, \tau, t)}{\partial c} = 1 - M_v(c, r, \tau, t) > \epsilon > 0$ . Hence, by the implicit function theorem,

$$\begin{aligned} c_p(p, r, \tau, t) &= -\frac{\frac{\partial G(c, p, r, \tau, t)}{\partial p}}{\frac{\partial G(c, p, r, \tau, t)}{\partial c}} = -\frac{-1}{1 - M_v(c, r, \tau, t)} > 0 \\ c_r(p, r, \tau, t) &= -\frac{\frac{\partial G(c, p, r, \tau, t)}{\partial r}}{\frac{\partial G(c, p, r, \tau, t)}{\partial c}} = -\frac{-M_r(c, r, \tau, t)}{1 - M_v(c, r, \tau, t)} < 0 \\ c_\tau(p, r, \tau, t) &= -\frac{\frac{\partial G(c, p, r, \tau, t)}{\partial \tau}}{\frac{\partial G(c, p, r, \tau, t)}{\partial c}} = -\frac{-M_\tau(c, r, \tau, t)}{1 - M_v(c, r, \tau, t)} \leq 0. \end{aligned}$$

Note that  $M_r(c, r, \tau, t) < 0$ , since increasing the reserve price strictly decreases the certainty equivalent of participating in the auction. Similarly,  $M_\tau(c, r, \tau, t) \leq 0$ , because increasing the length of the bidding phase  $\tau$  increases the level of competition in the auction, hence decreasing the certainty equivalent. (The inequality is weak because  $M_\tau(c, r, \tau, t) = 0$  when  $c = r$ .) Since  $p \geq r$ , the derivatives w.r.t.  $p$  and  $r$  should be interpreted as one-sided derivatives when  $p = r$ . Lastly,  $c(p, r, \tau, t) = r$  when  $p = r$  by our construction of  $c(p, r, \tau, t)$ . (When  $p = r$ , all bidders with  $v > r$  prefer to accept the BP.) Moreover  $c(p, r, \tau, t) > p$  when  $p > r$ , since we have just shown that  $c_p(p, r, \tau, t) > 0$ .

□

#### A.4 Proof of Proposition 4

We show properties of the inverse cutoff function defined by

$$\begin{aligned} p(c, r, \tau, t) &= c - U^{-1} \left( \delta(\tau) \left( \alpha(r, \tau, t)U(c - r) + \int_r^c U(c - y)h(y, \tau, t)dy \right) \right) \\ &= c - M(c, r, \tau, t) \end{aligned}$$

over the support  $r \in [0, \infty)$ ,  $c \in [r, \infty)$ ,  $\tau \in (0, \infty)$ , and  $t \in (0, \infty)$ .

We start by deriving some useful properties of  $U^{-1'}(x)$  and  $U^{-1''}(x)$  given Assumption 1. Starting with the identity

$$z = U^{-1}(U(z)),$$

differentiate w.r.t.  $z$  to get

$$1 = U^{-1'}(U(z))U'(z).$$

Evaluating this expression at  $z = U^{-1}(x)$  obtain

$$U^{-1'}(x) = \frac{1}{U'(U^{-1}(x))} = (U'(U^{-1}(x)))^{-1}.$$

Differentiating this results in

$$\begin{aligned} U^{-1''}(x) &= - (U'(U^{-1}(x)))^{-2} U''(U^{-1}(x)) U^{-1'}(x) \\ &= - (U'(U^{-1}(x)))^{-2} U''(U^{-1}(x)) (U'(U^{-1}(x)))^{-1} \\ &= - (U'(U^{-1}(x)))^{-3} U''(U^{-1}(x)). \end{aligned}$$

Given our assumptions on  $U(x)$ , these results imply that:

1.  $U^{-1'}(0) = 1$ ;
2.  $U^{-1''}(0) = -U''(0)$ ;
3.  $U^{-1'}(\cdot) > 0$  and is bounded away from 0 and  $\infty$ ;
4.  $U^{-1''}(\cdot) \geq 0$  and is bounded away from  $\infty$ .

With these results in hand, consider the statements in the Proposition one by one. First,  $p_c(c, r, \tau, t) > 0$  because by Assumption 3, the derivative of  $M$  w.r.t. its first argument is strictly less than 1.

$p_c(c, r, \tau, t) < 1$ , since

$$\begin{aligned} p_c(c, r, \tau, t) &= 1 - U^{-1'} \left( \delta(\tau) \left( \alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t) dy \right) \right) \\ &\quad \cdot \delta(\tau) \left( \alpha(r, \tau, t) U'(c-r) + \int_r^c U'(c-y) h(y, \tau, t) dy \right), \end{aligned}$$

and because under our assumptions,  $U^{-1'}(\cdot) > 0$ ,  $\delta(\cdot) > 0$ ,  $U'(\cdot) > 0$ ,  $\alpha(y, \tau, t) > 0$ , and  $h(y, \tau, t) > 0$  for  $y > r$ .

$p_r(c, r, \tau, t) > 0$ , since

$$\begin{aligned} p_r(c, r, \tau, t) &= -U^{-1'} \left( \delta(\tau) \left( \alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t) dy \right) \right) \\ &\quad \cdot \delta(\tau) \left( \frac{\partial \alpha(r, \tau, t)}{\partial r} U(c-r) - \alpha(r, \tau, t) U'(c-r) + U(c-r) h(r, \tau, t) \right) \\ &= U^{-1'} \left( \delta(\tau) \left( \alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t) dy \right) \right) \delta(\tau) \alpha(r, \tau, t) U'(c-r). \end{aligned}$$

The second line follows since  $\frac{\partial \alpha(r, \tau, t)}{\partial r} = h(r, \tau, t)$ , and the term is strictly positive since under our assumptions,  $U^{-1}(\cdot) > 0$ ,  $\delta(\cdot) > 0$ ,  $U'(\cdot) > 0$ , and  $\alpha(y, \tau, t) > 0$ .

$p_\tau(c, r, \tau, t) \geq 0$ , since

$$p_\tau(c, r, \tau, t) = -U^{-1'} \left( \delta(\tau) \left( \alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t)dy \right) \right) \\ \cdot \left[ \begin{array}{l} \delta'(\tau) \left( \alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t)dy \right) \\ + \delta(\tau) \frac{\partial \left( \alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t)dy \right)}{\partial \tau} \end{array} \right].$$

The first term in the square brackets is weakly negative since Assumption 1 implies  $\delta'(\cdot) < 0$ ,  $\alpha(\cdot, \cdot, \cdot) > 0$ ,  $h(\cdot, \cdot, \cdot) > 0$ , and  $U(\cdot) \geq 0$ . The second term in the square brackets is weakly negative since  $\delta(\tau) > 0$  and the derivative of the expected utility from rejecting the BP w.r.t.  $\tau$  is weakly negative (since the distribution of the highest competitor valuation is stochastically increasing in the length of the bidding phase  $\tau$  (this derivative is zero when  $c = r$ )). Since  $U^{-1'}(x) > 0$ , this implies  $p_\tau(c, r, \tau, t) \geq 0$ .

$p(c, r, \tau, t) = r$  iff  $c = r$ , because

$$p(c, c, \tau, t) = c - U^{-1} \left( \delta(\tau) \left( \alpha(c, \tau, t)U(c-c) + \int_c^c U(c-y)h(y, \tau, t)dy \right) \right) \\ = c - U^{-1}(0) = c = r.$$

The “only if” follows because  $p_c(c, r, \tau, t) > 0$ , and because  $p(c, r, \tau, t)$  is only defined for  $c \geq r$ .

$p(c, r, \tau, t) \geq r$  from a similar argument, since  $p(c, r, \tau, t) = r$  when  $c = r$  and  $p_c(c, r, \tau, t) > 0$ .

$p(c, r, \tau, t) \leq c$ , since

$$p(c, r, \tau, t) = c - U^{-1} \left( \delta(\tau) \left( \alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t)dy \right) \right)$$

and  $U^{-1}(\cdot) \geq 0$ .

$p_c(z, z, \tau, t) = 1 - \delta(\tau)\alpha(z, \tau, t)$ , since

$$p_c(z, z, \tau, t) = 1 - U^{-1'} \left( \delta(\tau) \left( \alpha(z, \tau, t)U(z-z) + \int_z^z U(z-y)h(y, \tau, t)dy \right) \right) \\ \cdot \delta(\tau) \left( \alpha(z, \tau, t)U'(z-z) + \int_z^z U'(z-y)h(y, \tau, t)dy \right) \\ = 1 - U^{-1'}(0) \delta(\tau)\alpha(z, \tau, t)U'(0) \\ = 1 - \delta(\tau)\alpha(z, \tau, t).$$

$p_r(z, z, \tau, t) = \delta(\tau)\alpha(z, \tau, t)$ , since

$$\begin{aligned}
p_r(z, z, \tau, t) &= -U^{-1'} \left( \delta(\tau) \left( \alpha(z, \tau, t)U(z-z) + \int_z^z U(z-y)h(y, \tau, t)dy \right) \right) \\
&\quad \cdot \delta(\tau) \left( \frac{\partial \alpha(z, \tau, t)}{\partial z} U(z-z) - \alpha(z, \tau, t)U'(z-z) + U(z-z)h(z, \tau, t) \right) \\
&= U^{-1'} \left( \delta(\tau) \left( \alpha(z, \tau, t)U(z-z) + \int_z^z U(z-y)h(y, \tau, t)dy \right) \right) \delta(\tau)\alpha(z, \tau, t)U'(z-z) \\
&= U^{-1'}(0) \delta(\tau)\alpha(z, \tau, t)U'(0) \\
&= \delta(\tau)\alpha(z, \tau, t),
\end{aligned}$$

where the last line follows because  $U^{-1'}(0) = U'(0) = 1$ .

Next, we consider the second derivatives of the inverse cutoff function w.r.t.  $c$  and  $r$ , i.e.  $p_{cc}(c, r)$ ,  $p_{rr}(c, r)$ , and  $p_{cr}(c, r)$ . We drop the  $\tau$  and  $t$  arguments for compactness.

For  $p_{cc}(c, r)$ , we have

$$p_{cc}(c, r) = 1 - U^{-1'} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta \left[ \alpha(r)U'(c-r) + \int_r^c U'(c-y)h(y)dy \right],$$

so

$$\begin{aligned}
p_{cc}(c, r) &= -U^{-1''} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta^2 \left[ \alpha(r)U'(c-r) + \int_r^c U'(c-y)h(y)dy \right]^2 \\
&\quad - U^{-1'} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta \left[ \alpha(r)U''(c-r) + \int_r^c U''(c-y)h(y)dy + h(c) \right].
\end{aligned}$$

Under our assumptions, all these terms are bounded away from  $\infty$  and  $-\infty$ , so  $p_{cc}(c, r)$  is bounded away from  $\infty$  and  $-\infty$ . Moreover, if we evaluate this expression at  $c = r = z$ , we get

$$\begin{aligned}
p_{cc}(z, z) &= -U^{-1''} \left( \delta \left( \alpha(z)U(z-z) + \int_z^z U(z-y)h(y)dy \right) \right) \delta^2 \left[ \alpha(z)U'(z-z) + \int_z^z U'(z-y)h(y)dy \right]^2 \\
&\quad - U^{-1'} \left( \delta \left( \alpha(z)U(z-z) + \int_z^z U(z-y)h(y)dy \right) \right) \delta \left[ \alpha(z)U''(z-z) + \int_z^z U''(z-y)h(y)dy + h(z) \right] \\
&= -U^{-1''}(0) \delta^2 \alpha(z)^2 - U^{-1'}(0) \delta [\alpha(z)U''(0) + h(z)] \\
&= -U''(0) \delta \alpha(z) (1 - \delta \alpha(z)) - \delta h(z).
\end{aligned}$$

For  $p_{rr}(c, r)$ , we have

$$p_r(c, r) = U^{-1'} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta\alpha(r)U'(c-r),$$

so

$$\begin{aligned} p_{rr}(c, r) &= -U^{-1''} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta^2\alpha(r)^2U'(c-r)^2 \\ &\quad + U^{-1'} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta [\alpha'(r)U'(c-r) - \alpha(r)U''(c-r)]. \end{aligned}$$

Again, under our assumptions, all the terms in this expression are bounded away from  $\infty$  and  $-\infty$ , so  $p_{rr}(c, r)$  is bounded away from  $\infty$  and  $-\infty$ . If we evaluate this expression at  $c = r = z$ , we get

$$\begin{aligned} p_{rr}(z, z) &= -U^{-1''} \left( \delta \left( \alpha(z)U(z-z) + \int_z^z U(z-y)h(y)dy \right) \right) \delta^2\alpha(z)^2U'(z-z)^2 \\ &\quad + U^{-1'} \left( \delta \left( \alpha(z)U(z-z) + \int_z^z U(z-y)h(y)dy \right) \right) \delta [\alpha'(z)U'(z-z) - \alpha(z)U''(z-z)] \\ &= -U^{-1''}(0) \delta^2\alpha(z)^2U'(0)^2 + U^{-1'}(0) \delta [\alpha'(z)U'(0) - \alpha(z)U''(0)] \\ &= U''(0) \delta^2\alpha(z)^2 + \delta [\alpha'(z) - \alpha(z)U''(0)] \\ &= -U''(0) \delta\alpha(z) (1 - \delta\alpha(z)) + \delta\alpha'(z). \end{aligned}$$

For  $p_{rc}(c, r) = p_{cr}(c, r)$ , we have

$$p_r(c, r) = U^{-1'} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta\alpha(r)U'(c-r),$$

so

$$\begin{aligned} p_{rc}(c, r) &= U^{-1'} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta\alpha(r)U''(c-r) \\ &\quad + U^{-1''} \left( \delta \left( \alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy \right) \right) \delta\alpha(r)U'(c-r) \\ &\quad \cdot \left[ \delta \left( \alpha(r)U'(c-r) + \int_r^c U'(c-y)h(y)dy \right) \right]. \end{aligned}$$

Again, all the terms are bounded away from  $\infty$  and  $-\infty$ , so  $p_{rc}(c, r)$  is bounded away from  $\infty$  and  $-\infty$ . Evaluated at  $c = r = z$ , we get

$$\begin{aligned}
p_{rc}(z, z) &= U^{-1'} \left( \delta \left( \alpha(z)U(z-z) + \int_z^z U(z-y)h(y)dy \right) \right) \delta\alpha(z)U''(z-z) \\
&\quad + U^{-1''} \left( \delta \left( \alpha(z)U(z-z) + \int_z^z U(z-y)h(y)dy \right) \right) \delta\alpha(z)U'(z-z) \\
&\quad \cdot \left[ \delta \left( \alpha(z)U'(z-z) + \int_z^z U'(z-y)h(y)dy \right) \right] \\
&= U^{-1'}(0) \delta\alpha(z)U''(0) + U^{-1''}(0) \delta\alpha(z)U'(0)\delta\alpha(z)U'(0) \\
&= \delta\alpha(z)U''(0) - U''(0) \delta\alpha(z)\delta\alpha(z) \\
&= U''(0)\delta\alpha(z)(1 - \delta\alpha(z)).
\end{aligned}$$

□

## A.5 Proof of Proposition 5

We have the differential equation

$$U''(c-r) = \frac{(\Phi_r(c, r, \tau, t_1) + h(r, \tau, t_1))}{\Phi(c, r, \tau, t_1)} U'(c-r), \quad (23)$$

where

$$\Phi(c, r, \tau, t_1) = \alpha(r, \tau, t_1) \left[ \frac{(1 - p_c(c, r, \tau, t_1))}{p_r(c, r, \tau, t_1)} - 1 \right].$$

We wish to prove that this differential equation has a unique solution  $U(\cdot)$  under our assumptions, which include:

1.  $U(\cdot)$  is twice continuously differentiable;
2.  $U'(\cdot) > \epsilon$  for some  $\epsilon > 0$ ;
3.  $0 \geq U''(\cdot) > -B$ , for some  $0 < B < \infty$ ;
4.  $U(0) = 0$ ,  $U'(0) = 1$ .

We can show that (9) has a unique solution even when  $r$ ,  $\tau$ , and  $t_1$  are fixed. Hence, we assume  $r = 0$ ,  $\tau = \tau^*$ , and  $t_1 = t_1^*$ . This results in a simple first-order linear ODE with a variable coefficient:

$$U''(c) = \Psi(c)U'(c), \quad (24)$$

where

$$\Psi(c) = \frac{(\Phi_r(c, 0, \tau^*, t_1^*) + h(0, \tau^*, t_1^*))}{\Phi(c, 0, \tau^*, t_1^*)}.$$

has already been shown to be identified.

Since (24) implies that  $\Psi(c) = \frac{U''(c)}{U'(c)}$ , our assumed properties of  $U$  imply that the coefficient  $\Psi(c)$  is integrable. It is well known that when  $\Psi(c)$  is integrable, (24) has solutions in  $C^2$  given by

$$|U'(c)| = k \exp\left(\int_0^c \Psi(c) dz\right), \quad (25)$$

where  $k$  is a constant that can be determined by the initial condition on  $U'(\cdot)$ . Since  $U'(\cdot) > 0$  and we have the initial condition  $U'(0) = 1$ , it follows that the solution to (24) is unique and given by

$$U'(c) = \exp\left(\int_0^c \Psi(c) dz\right). \quad (26)$$

Since  $U'(c)$  is identified, the initial condition  $U(0) = 0$  identifies  $U(\cdot)$ .  $\square$

## A.6 Proof of Proposition 8

Since the hazard rate of the first action (accept or reject the BP) is observed in the data and satisfies

$$\theta(t_1|p, r, \tau_0) = \lambda(t_1)(1 - F_V(r)), \quad (27)$$

it is clear that  $\lambda(t_1)(1 - F_V(r))$  is identified on  $r \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T)$ .

We next show that this implies that  $\alpha(r, \tau_0, t_1)$  is identified on  $r \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T - \tau_0)$ . By definition

$$\alpha(r, \tau, t_1) = \exp(\gamma F_V(r) - \gamma)$$

where

$$\gamma = \int_t^{t+\tau} \lambda(s) ds.$$

Therefore

$$\begin{aligned} \alpha(r, \tau, t_1) &= \exp\left(- (1 - F_V(r)) \int_t^{t+\tau_0} \lambda(s) ds\right) \\ &= \exp\left(- \int_t^{t+\tau_0} \lambda(s)(1 - F_V(r)) ds\right). \end{aligned}$$

Since  $\lambda(t_1)(1 - F_V(r))$  is identified on  $r \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T)$ , this implies that  $\alpha(r, \tau, t_1)$  is identified on  $r \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T - \tau_0)$ .

Next we show that  $h(y, \tau_0, t_1)$  is identified on  $y \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T - \tau_0]$ . Again, by definition

$$\begin{aligned} h(y, \tau_0, t_1) &= \exp(\gamma F_V(y) - \gamma) \gamma f_V(y) \\ &= \alpha(r, \tau_0, t_1) \int_t^{t+\tau_0} \lambda(s) f_V(y) ds. \end{aligned}$$

Since  $\lambda(t_1)(1 - F_V(y))$  is identified on  $y \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T]$ , its derivative  $-\lambda(t_1)f_V(y)$ , is also identified on  $y \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T]$ . This implies  $\int_t^{t+\tau_0} \lambda(s)f_V(y)ds$  is identified on  $y \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T - \tau_0]$ . Therefore,  $h(y, \tau_0, t)$  is identified on  $y \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T - \tau_0]$ .

Next, we consider identification of  $c(p, r, \tau_0, t_1)$ . From Section 3.3, we know

$$\Pr(B = 1|p, r, \tau_0, t_1) = \frac{1 - F_V(c(p, r, \tau_0, t_1))}{1 - F_V(r)},$$

where  $\Pr(B = 1|p, r, \tau_0, t_1)$  is observed on the support  $r \in [\underline{r}, \bar{r}]$ ,  $p \in [r, \bar{p}]$  and  $t_1 \in [0, T]$  (at  $\tau_0$ ). Therefore,

$$\begin{aligned} \Pr(B = 1|p, r, \tau_0, t_1) &= \frac{\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))}{\lambda(t_1)(1 - F_V(r))} \\ &= \frac{\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))}{\theta(t_1|p, r, \tau_0)}, \end{aligned}$$

and therefore  $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$  is identified on the same support. Note that this term is the hazard rate of the BP being accepted.

Since we have already identified  $\lambda(t_1)(1 - F_V(r))$  on  $r \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T]$ , this implies that

$$c(p, r, \tau_0, t_1) = z,$$

where  $z$  satisfies

$$\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1))) = \lambda(t_1)(1 - F_V(z)). \quad (28)$$

Intuitively, this says that the cutoff at  $(p, r, \tau_0, t_1)$  is equal to the hypothetical reserve price that would imply that the hazard rate of first action is equal  $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$ .

It remains to be verified that we can identify the  $z$  that satisfies (28). Note that the r.h.s. of (28) is strictly decreasing in  $z$ . Since  $c(p, r, \tau_0, t_1) \geq \underline{r}$ , the l.h.s. is  $\leq$  the r.h.s. at  $z = \underline{r}$ . Hence, we want to increase  $z$  above  $\underline{r}$  to satisfy (28). The problem is that we only observe the r.h.s. for  $z \in [\underline{r}, \bar{r}]$ . However, as long as  $c(p, r, \tau_0, t_1) \leq \bar{r}$ , we can find a  $z \in [\underline{r}, \bar{r}]$  that satisfies (28). This implies that  $c(p, r, \tau_0, t_1)$  is identified on the set  $(p, r, t_1)$  such that  $c(p, r, \tau_0, t_1) \leq \bar{r}$ .<sup>21</sup> This immediately implies that inverse cutoff function  $p(c, r, \tau_0, t_1)$  is identified on the set  $r \in [\underline{r}, \bar{r}]$ ,  $t_1 \in [0, T - \tau_0]$ , and

<sup>21</sup>This set exists. To show this, consider a situation where  $r = \underline{r}$  and  $p = \underline{r} + \epsilon$  for some arbitrarily small  $\epsilon$ . For small enough  $\epsilon$ ,  $c(p, r, \tau_0, t_1)$  will be below  $\bar{r}$  (since  $c$  is continuous and  $c(\bar{r}, \bar{r}, \tau_0, t_1) = \bar{r}$ ). Obviously the size of this set will depend on the range  $[\underline{r}, \bar{r}]$ .

$c \in [\underline{r}, \bar{r}]$ .

Thus, we have shown that:

1.  $\alpha(r, \tau, t_1)$  is identified on  $r \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T - \tau_0)$ ;
2.  $h(y, \tau_0, t_1)$  is identified on  $y \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, T - \tau_0)$ ;
3.  $p(c, r, \tau_0, t_1)$  is identified on the set  $r \in [\underline{r}, \bar{r}]$ ,  $t_1 \in [0, T - \tau_0)$ , and  $c \in [\underline{r}, \bar{r}]$ .

Recall that our integral equation

$$U(c - p(c, r, \tau_0, t_1)) = \delta(\tau_0) \left( \alpha(r, \tau_0, t_1)U(c - r) + \int_r^c U(c - y)h(y, \tau_0, t_1)dy \right) \quad (29)$$

can be reduced to

$$U''(c - r) = \frac{(\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1))}{\Phi(c, r, \tau_0, t_1)} U'(c - r), \quad (30)$$

where

$$\Phi(c, r, \tau, t_1) = \alpha(r, \tau, t_1) \left[ \frac{(1 - p_c(c, r, \tau, t_1))}{p_r(c, r, \tau, t_1)} - 1 \right].$$

Identification of  $\alpha(r, \tau, t_1)$ ,  $h(y, \tau_0, t_1)$ , and  $p(c, r, \tau_0, t_1)$  implies that we can identify  $\frac{(\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1))}{\Phi(c, r, \tau_0, t_1)}$  on  $r \in [\underline{r}, \bar{r}]$ ,  $t_1 \in [0, T - \tau_0)$ , and  $c \in [\underline{r}, \bar{r}]$ . Hence, by arguments similar to Proposition 3, Equation (30) identifies  $U(\cdot)$  on  $[0, \underline{r} - \bar{r}]$ . By the same arguments as Section 3.5,  $\delta(\cdot)$  is identified at  $\tau_0$ .

□

## A.7 Proof of Proposition 9

Assumption 8 further restricts the support of  $p$  to  $[p_0 - \epsilon, p_0 + \epsilon]$ . We also assume that  $p_0$  is such that there exists a  $r^* \in (\underline{r}, \bar{r})$  and a  $t_1^*$  such that  $c(p_0, r^*, \tau_0, t_1^*) \in (\underline{r}, \bar{r})$ . By the same arguments as in the proof of Corollary 3, we know:

1.  $\alpha(r, \tau, t_1)$  is identified on  $r \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, \bar{T} - \tau_0)$ ;
2.  $h(y, \tau_0, t_1)$  is identified on  $y \in [\underline{r}, \bar{r}]$  and  $t_1 \in [0, \bar{T} - \tau_0)$ .

By the same arguments as above (and the condition that  $c \in (\underline{r}, \bar{r})$ ), one can see that  $c(p, r, \tau_0, t_1)$  will be identified for  $p \in (p_0 - \epsilon, p_0 + \epsilon)$ ,  $r \in (r - \eta, r + \eta)$ ,  $t_1 = t_1^*$ , and  $\tau = \tau_0$ , for  $\eta$  sufficiently small. Therefore, the inverse cutoff function  $p(c, r, \tau_0, t_1)$  will be identified at  $t_1 = t_1^*$ , and  $\tau = \tau_0$  in a ball centered at  $(c(p_0, r^*, \tau_0, t_1^*), r^*)$ . This implies that  $p_r(c, r, \tau_0, t_1)$  and  $p_c(c, r, \tau_0, t_1)$  are identified over that same region, as are  $\Phi(c, r, \tau_0, t_1)$  and  $\Phi_r(c, r, \tau_0, t_1)$ . We have

$$\frac{U''(c - r)}{U'(c - r)} = \frac{(\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1))}{\Phi(c, r, \tau_0, t_1)} \quad (31)$$

Hence, the Arrow-Pratt measure of risk aversion  $\frac{U''}{U'}$  is identified at the point  $c(p_0, r^*, \tau_0, t_1^*) - r^*$ . Again, by the same arguments as Section 3.5,  $\delta(\cdot)$  is identified at  $\tau_0$ .

□

## B Appendix: Proof that $U''' \leq 0$ is a sufficient condition for Assumption 3

We have

$$M(v, r, \tau, t) = U^{-1} \left( \delta(\tau) \left( \alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)dy \right) \right)$$

so

$$\begin{aligned} M_v(v, r, \tau, t) &= U^{-1'} \left( \delta(\tau) \left( \alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)dy \right) \right) \\ &\quad \cdot \delta(\tau) \left( \alpha(r, \tau, t)U'(v - r) + \int_r^v U'(v - y)h(y, \tau, t)dy \right) \\ &= \frac{\delta(\tau) \left( \alpha(r, \tau, t)U'(v - r) + \int_r^v U'(v - y)h(y, \tau, t)dy \right)}{U' \left( U^{-1} \left( \delta(\tau) \left( \alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)dy \right) \right) \right)} \\ &< \frac{\delta(\tau) \left( \alpha(\cdot)U'(v - r) + \int_r^v U'(v - y)h(\cdot)dy + \left( 1 - \alpha(\cdot) - \int_r^v h(\cdot)dy \right) U'(0) \right)}{U' \left( U^{-1} \left( \delta(\tau) \left( \alpha(\cdot)U(v - r) + \int_r^v U(v - y)h(\cdot)dy + \left( 1 - \alpha(\cdot) - \int_r^v h(\cdot)dy \right) U(0) \right) \right) \right)}. \end{aligned}$$

The strict inequality holds because of our normalizations that  $U(0) = 0$  and  $U'(0) = 1$ , and because  $1 - \alpha(r, \tau, t) - \int_r^v h(y, \tau, t)dy > 0$  for any finite  $v$ .

Therefore, we have

$$M_v(v, r, \tau, t) < \frac{\delta(\tau)EU'(x)}{U'(U^{-1}(\delta(\tau)EU(x)))},$$

where the random variable  $x$  has a mixed-continuous distribution, taking the value 0 with probability  $1 - \alpha(r, \tau, t) - \int_r^v h(y, \tau, t)dy$ , the value  $v - r$  with probability  $\alpha(r, \tau, t)$ , and having density  $h(y, \tau, t)$  over the interval  $(0, v - r)$ . Because  $U'' \leq 0$  and  $\delta(\tau) < 1$ , Jensen's Inequality implies that

$\delta(\tau)EU(x) < U(Ex)$ . Therefore

$$\begin{aligned} M_v(v, r, \tau, t) &< \frac{\delta(\tau)EU'(x)}{U'(U^{-1}(U(Ex)))} \\ &= \frac{\delta(\tau)EU'(x)}{U'(Ex)}. \end{aligned}$$

Since  $U''' \leq 0$ , Jensen's inequality implies  $EU'(x) \leq U'(Ex)$ . Hence,

$$M_v(v, r, \tau, t) < \delta(\tau) < 1.$$

□

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