Economics 522A, Homework 8 Suggested Solutions

1. Ruud, 18.1, part (a): We can write

$$\hat{\beta} = \frac{\sum_{i=1}^{2} w_i^2 x_i y_i}{\sum_{i=1}^{2} w_i^2 x_i^2} = \frac{w_1^2 x_1 y_1 + w_2^2 x_2 y_2}{w_1^2 x_1^2 + w_2^2 x_2^2}.$$

So

$$V[\hat{\beta}|x_1, x_2] = \left(\frac{1}{w_1^2 x_1^2 + w_2^2 x_2^2}\right)^2 \left[(w_1^2 x_1)^2 \sigma_1^2 + (w_2^2 x_2)^2 \sigma_2^2\right].$$

We want to solve

$$\min_{w_1^2, w_2^2} V[\hat{\beta}|x_1, x_2].$$

We take the first order conditions with respect to w_1^2 and w_2^2. For example, setting the derivative with respect to w_1^2 equal to zero, we must have

$$w_1^2 x_1^2 \sigma_1^2 - \frac{(w_1^2 x_1)^2 \sigma_1^2 + (w_2^2 x_2)^2 \sigma_2^2}{w_1^2 x_1^2 + w_2^2 x_2^2} x_1^2 = 0$$

Consider the potential solution

$$w_1^2 = \frac{1}{\sigma_1^2}, \quad w_2^2 = \frac{1}{\sigma_2^2}.$$

It is easy to verify that these satisfy the first order conditions for a minimum.

2. We assume that $E[y|X] = X\beta$, and $V[y|X] = \Omega$, and let

$$\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y.$$

(a)

$$\begin{aligned} E[\hat{\beta}|X] &= E[(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y|X] \\ &= (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}X\beta \\ &= \beta. \end{aligned}$$

So $\hat{\beta}$ is unbiased.

(b) The variance is

$$\begin{aligned} V[\hat{\beta}|X] &= V[(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y|X] \\ &= (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}V[y|X]\Sigma^{-1}X(X'\Sigma^{-1}X)^{-1} \\ &= (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\Omega\Sigma^{-1}X(X'\Sigma^{-1}X)^{-1} \end{aligned}$$

3. Let C be the Cholesky factorization of Ω, so that $\Omega = CC'$. Let

$$\tilde{y} = C^{-1}y, \quad \tilde{X} = C^{-1}X.$$

Then

$$\tilde{y}|\tilde{X} \sim N(\tilde{X}\beta, I_n).$$

Also,

$$\hat{\beta}_{gls} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{y},$$

which is the OLS estimator using the transformed versions of y and X.

Consider

$$\begin{aligned} s^2 &= (\tilde{y} - \tilde{X}\hat{\beta})'(\tilde{y} - \tilde{X}\hat{\beta})/(n-k) \\ &= (y - X\hat{\beta})'C^{-1\prime}C^{-1}(y - X\hat{\beta})/(n-k) \\ &= (y - X\hat{\beta})'\Omega^{-1}(y - X\hat{\beta})/(n-k). \end{aligned}$$

So, from the results in LN9, we know that s^2 is independent of $\hat{\beta}$, and distributed as

$$\chi^2_{n-k}/(n-k).$$

(Here, since the variance of $\tilde{y}$ is I_n, we use $\sigma^2 = 1$.) By similarly using the results of LN9, we conclude that

$$R\hat{\beta}|X \sim N(R\beta, R(X'\Omega^{-1}X)^{-1}R'),$$

so

$$(R\hat{\beta} - R\beta)'[R(X'\Omega^{-1}X)^{-1}R']^{-1}(R\hat{\beta} - R\beta) \sim \chi^2_m.$$

So it follows that

$$\frac{(R\hat{\beta} - R\beta)'\left[R(X'\Omega^{-1}X)^{-1}R'\right]^{-1}(R\hat{\beta} - R\beta)/m}{(y - X\hat{\beta})'\Omega^{-1}(y - X\hat{\beta})/(n-k)}$$

has an F distribution with m and $n-k$ degrees of freedom.

We can use this result to test hypotheses about $R\beta$. Suppose we hypothesize that $R\beta = r$, where r is a vector of constants. Then we can replace $R\beta$ by r in the expression above, and calculate this test statistic, and compare it to the tables for the F distribution.

4. (a) All the code for this problem is in `hw8_3.m`, and in the Matlab function files `ols.m` and `ols_het.m`.

   Below are the OLS estimates with both conventional and Eicker-White standard errors. We see that the two sets of standard errors are fairly similar, so there is not obvious evidence of heteroskedasticity.

	b	se	se_robust
const	0.8289431	0.0776095	0.0752593
fe	-0.2491540	0.0266250	0.0266033
nw	-0.1335351	0.0371819	0.0347868
un	0.1802035	0.0369549	0.0304585
ed	0.0871102	0.0047333	0.0051529
age	0.0127601	0.0011718	0.0012338

(b) The F-statistic s_m^2/s_f^2 was equal to 0.976. Under the null, this should be distributed as F with 642 and 635 degrees of freedom. This distribution has a 0.025 quantile of 0.86, and a .975 quantile of 1.17. Thus, a two-sided test would reject the null hypothesis if the statistic was less than 0.86, or greater than 1.17. We do not reject the null hypothesis of equal variances.

(c) FGLS estimates and standard errors:

	b	se
const	0.8291232	0.0774374
fe	-0.2491516	0.0265931
nw	-0.1341320	0.0371522
un	0.1798848	0.0368714
ed	0.0870321	0.0047248
age	0.0127861	0.0011705