Economics 522A, Homework 2 Suggested Solutions

1. (a) Let $X_1, X_2, \ldots, X_{1000}$ denote indicator functions equal to 1 if the ith coin toss is heads and 0 if tails. Then we can write

$$T = \frac{\sum_{i=1}^{1000} X_i}{1000},$$

and note that $1000 \cdot T = \sum_i X_i$ is Binomial with parameters 1000 and p. Under the null hypothesis, $p = 1/2$. So we can write

$$\begin{aligned} P(T > c) &= P(\sum_i X_i > 1000c) \\ &= \sum_{j=1000c}^{1000} \binom{1000}{j} (1/2)^j (1/2)^{1000-j} \end{aligned}$$

Note: if $1000c$ is not an integer, then the sum should start at the first integer above $1000c$.

See file `hw2.m` for the Matlab function to calculate this. Here is a brief description of how the function works:

- First, set up a variable to hold $P(T > c)$.
- Check if $1000c$ is an integer. If not, calculate the first integer greater than $1000c$.
- Loop from $1000c$ (or first integer above) to 1000. At each step in the loop, calculate the inner term of the sum above, and add it to the variable holding $P(T > c)$.

(b) $T = \sum_i X_i/n$ is a sample average, and each X_i is IID Bernoulli, with expected value $E[X_i] = p$ and $V[X_i] = p(1-p)$. Therefore, by the central limit theorem,

$$\sqrt{n}(T - p) \xrightarrow{\text{d}} N(0, p(1-p)).$$

Under $H_0 : p = 1/2$,

$$\sqrt{n}(T - 1/2) \xrightarrow{\text{d}} N(0, 1/4).$$

Or we can write

$$2\sqrt{n}(T - 1/2) \xrightarrow{\text{d}} N(0, 1).$$

Now, consider $P(T > c)$. Rewrite this as

$$\begin{aligned} P(T > c) &= P(T - 1/2 > c - 1/2) \\ &= P(2\sqrt{n}(T - 1/2) > 2\sqrt{n}(c - 1/2)) \\ &\to 1 - \Phi(2\sqrt{n}(c - 1/2)). \end{aligned}$$

Here, $\Phi(\cdot)$ is the CDF of the standard normal distribution.

(c) Using the central limit approximation,

$$P(2\sqrt{n}(T-1/2) > 1.645) \approx 0.05.$$

Therefore,

$$P\left(T > 1/2 + \frac{1.645}{2\sqrt{1000}}\right) \approx 0.05.$$

So the cutoff $c = 1/2 + \frac{1.645}{2\sqrt{1000}} = 0.526$.

We calculate $T = 560/1000 = .56$. This is greater than c, so we reject the null hypothesis.

2. (a) Obtain the Cramer-Rao bound for estimation of θ.

$$f(x;\theta) = \theta x^{\theta-1}, \qquad \text{for } 0 \le x \le 1,$$

$$\log f(x;\theta) = \log\theta + (\theta-1)\log x.$$

$$\frac{\partial}{\partial\theta}\log f(x;\theta) = \frac{1}{\theta} + \log x.$$

$$\frac{\partial^2}{\partial\theta^2}\log f(x;\theta) = -\frac{1}{\theta^2}.$$

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right] = \frac{1}{\theta^2}.$$

$$CR = \frac{\theta^2}{n} \quad \text{(for a sample of size n).}$$

(b) Calculate a general formula for the maximum likelihood estimator of θ.

Log likelihood:

$$L(\theta) = \sum_{i=1}^{n}[\log\theta + (\theta-1)\log X_i] = N\log\theta + (\theta-1)\sum_i \log X_i.$$

First order condition:

$$\frac{\partial L(\theta)}{\partial\theta} = \frac{n}{\theta} + \sum_i \log X_i = 0.$$

$$\Rightarrow \hat{\theta} = -\frac{n}{\sum_i \log X_i}.$$

(c)

$$\hat{\theta} = \frac{n}{-\sum_i \log X_i} = \frac{100}{25} = 4.$$

$$\theta_0 = 3.$$

$$WALD = n(\hat{\theta}-\theta_0)^2 \cdot \hat{I} = \frac{100}{16}(4-3)^2 = 6.25.$$

Note you could use a different estimator of the information matrix and get a slightly different answer.

The critical value at the 5% level is 3.84, so we reject the null hypothesis that $\theta = 3$.

(d)

$$\begin{aligned} LR &= 2\left[L(\hat\theta) - L(\theta_0)\right] \\ &= 2\left[n\log\hat\theta + (\hat\theta - 1)\sum_i \log X_i - n\log\theta_0 - (\theta_0 - 1)\sum_i \log X_i\right] \\ &= 7.54. \end{aligned}$$

So we reject the null hypothesis.

$$\begin{aligned} LM &= \frac{1}{n}\left(\sum_i \frac{\partial}{\partial\theta}\log f(X_i;\theta_0)\right)^2 / \hat I \\ &= \frac{1}{n}\left(\frac{n}{\theta_0} + (\theta_0 - 1)\sum_i \log X_i\right)^2 / \hat I \\ &= \frac{1}{100}\left(\frac{100}{3} - 2\cdot 25\right)^2 / \frac{1}{9} \\ &= 25. \end{aligned}$$

So we reject the null hypothesis.

(e)

$$\sqrt{n}(\hat\theta - \theta) \to N(0, I(\theta)^{-1}) \sim N(0, \theta^2).$$

So

$$\frac{1}{\hat\theta}\sqrt{n}(\hat\theta - \theta) \to N(0,1).$$

Confidence interval is:

$$\left[\hat\theta - 1.96\sqrt{\frac{\hat\theta^2}{n}}, \hat\theta + 1.96\sqrt{\frac{\hat\theta^2}{n}}\right]$$

$$= \left[4 - 1.96\left(\frac{4}{10}\right), 4 + 1.96\left(\frac{4}{10}\right)\right]$$

$$= [3.22, 4.78]\,.$$