Economics 522A, Homework 1 Suggested Solutions

1. We have a sample of size 2, so the joint likelihood is

$$f(x_1, x_2; \theta) = \frac{1}{\theta^2} \exp\left(-\frac{x_1 + x_2}{\theta}\right).$$

By the Neyman-Pearson lemma, the most powerful test uses a critical region of the form

$$\begin{aligned} C_x &= \left\{(x_1, x_2) : \frac{f(x_1, x_2; \theta_1)}{f(x_1, x_2; \theta_0)} \geq k\right\} \\ &= \left\{(x_1, x_2) : \frac{4^2 \exp(-(x_1 + x_2)/4)}{2^2 \exp(-(x_1 + x_2)/2)} \geq k\right\} \\ &= \{(x_1, x_2) : \exp((x_1 + x_2)/4) \geq k'\} \\ &= \{(x_1, x_2) : x_1 + x_2 \geq k''\} \end{aligned}$$

So, we want to use a test statistic of the form $T(x_1, x_2) = x_1 + x_2$ and compare it to some cutoff k''.

For $\alpha = 0.1$, we want to set k'' such that

$$Pr(X_1 + X_2 \geq k'' | H_0) = 0.1.$$

Under $H_0 : \theta = 2$, $X_1 + X_2$ is the sum of two exponential distributions and is distributed as $Gamma(2, 2)$. Thus, we want k such that

$$Pr(Gamma(2, 2) \geq k) = 0.1$$

or equivalently,

$$Pr(Gamma(2, 2) < k) = 0.9$$

$$0.9 = \int_0^k \frac{1}{\Gamma(2)2^2} x e^{-x/2} dx.$$

Using integration by parts, we can get

$$0.9 = -\frac{1}{2} e^{-k/2} k - e^{k/2} + 1.$$

$$k \approx 7.78.$$

2.

$$f(x_1, \ldots, x_{10}; \sigma_0^2) = \left(\frac{1}{2\pi}\right)^5 \exp\left(-\frac{1}{2} \sum_i x_i^2\right),$$

$$f(x_1, \ldots, x_{10}; \sigma_1^2) = \left(\frac{1}{4\pi}\right)^5 \exp\left(-\frac{1}{4}\sum_i x_i^2\right),$$

So by the Neyman-Pearson lemma, we should use a critical region of the form

$$\begin{aligned} C_x &= \left\{(x_1, \ldots, x_{10} : \frac{\left(\frac{1}{4\pi}\right)^5 \exp\left(-\frac{1}{4}\sum_i x_i^2\right)}{\left(\frac{1}{2\pi}\right)^5 \exp\left(-\frac{1}{2}\sum_i x_i^2\right)} \geq k\right\} \\ &= \left\{(x_1, \ldots, x_{10}) : \exp\left(\sum_i x_i^2\right) \geq k'\right\} \\ &= \left\{(x_1, \ldots, x_{10}) : \sum_i x_i^2 \geq k''\right\}. \end{aligned}$$

For a test at the $\alpha = 0.05$ level, we need to find k such that

$$Pr(\sum_i x_i^2 \geq k | \sigma^2 = 1) = 0.05.$$

Note that since X_i are IID $N(0,1)$ under $H_0 : \sigma^2 = 1$,

$$\sum_i x_i^2 \sim \chi^2(10).$$

Using a table of the chi-squared distribution, we get

$$Pr(\chi^2(10) \geq 18.31) = 0.05.$$

So we should use

$$C_x = \left\{(x_1, \ldots, x_10) : \sum_i x_i^2 \geq 18.31\right\}.$$

For $H_a : \sigma^2 = 3$, we can repeat the analysis, and we get exactly the same critical region as being most powerful.

In fact, we get the same critical region for all $\sigma_1^2 > 1$. Therefore, if we are testing with the one-sided composite alternative $H_a : \sigma^2 > 1$, this test is uniformly most powerful.

3. CB 8.20:

   By the NP Lemma, the UMP test should reject for large values of $LR(x) := f(x|H_1)/f(x|H_0)$. We can tabulate this ratio for every possible x value:

x	1	2	3	4	5	6	7
$LR(x)$	6	5	4	3	2	1	.84

   Notice that the likelihood ratio is decreasing in x. So we should reject for *small values of* x:

$$C_x = x : x \leq k.$$

   Then we need to choose k so that the probability of rejecting under the null is equal to (or less than) α. Setting $k = 4$ gives the UMP test of size $\alpha = 0.04$. The Type II error probability is

$$P(X \geq 5 | H_1) = 0.82.$$

4. CB 8.22:

   (a) First, we can write

$$\begin{aligned} f(x_1, \ldots, x_{10}|p) &= \prod_{i=1}^{10} f(x_i|p) && (1) \\ &= \prod_{i=1}^{10} p^{x_i}(1-p)^{1-x_i} && (2) \\ &= p^{\sum_i x_i}(1-p)^{n-x_i} && (3) \end{aligned}$$

   So, letting $Y := \sum_{i=1}^{10} x_i$, the NP Lemma suggests to use a test of the form:

$$\frac{f(x_1, \ldots, x_{10}|p = 1/4)}{f(x_1, \ldots, x_{10}|p = 1/2)} = \frac{(1/4)^Y (3/4)^{n-Y}}{(1/2)^Y (1/2)^{n-Y}} \geq k$$

   Taking logs, this is equivalent to:

$$Y \log(1/4) + (n-Y)\log(3/4) - Y\log(1/2) - (n-Y)\log(1/2) \geq k'$$

$$\Leftrightarrow Y[\log(1/2) - \log(1/4)] + Y[\log(3/4) - \log(1/2)] \leq k''$$

$$\Leftrightarrow Y \leq k'''.$$

   So the UMP test rejects for small values of Y. Note that

$$Y \sim Bin(10, p).$$

   So the size of the test is:

$$\alpha = P(Y \leq c|p = 1/2) = \sum_{j=1}^{c} P(Y = j|p = 1/2) = \sum_{j=1}^{c} \binom{10}{j} (1/2)^j (1/2)^{10-j}.$$

   We try some different values of c, and find that for $c = 2$, the size is .0547. As for the power, the only other parameter value being considered is $p = 1/4$. By similar calculations as before, the power at $p = 1/4$ is

$$P(Y \leq 2|p = 1/4) \approx .526.$$

   (b) The size of this test is

$$P(Y \geq 6|p = 1/2) = \sum_{j=6}^{10} \binom{10}{j} (1/2)^j (1/2)^{10-j} \approx .377.$$

   The power function is:

$$\beta(p) = \sum_{j=6}^{10} \binom{10}{j} (p)^j (1-p)^{10-j}.$$

   (c) There is a UMP test for all α levels corresponding to the probabilities $P(Y \leq j|p = 1/2)$, where j is an integer. For $n = 10$, we can calculate these one by one, to get α values:

$$0, \frac{1}{1024}, \frac{11}{1024}, \frac{56}{1024}, \frac{176}{1024}, \frac{176}{1024}, \frac{386}{1024}, \frac{638}{1024}, \frac{848}{1024}, \frac{968}{1024}, \frac{1013}{1024}, \frac{1023}{1024}, 1.$$