

Lecture Note 13: Bayesian Analysis Part 2

Previously, we considered an example where we observed  $X$  given the unknown mean  $\mu$ , and  $\mu$  had a normal prior distribution. The posterior distribution turned out to be normal as well. This was an example of a conjugate prior distribution, which has the property that the posterior distribution is of the same family as the prior.

To understand why conjugate priors are useful, suppose we use another distribution as the prior distribution. Say we use a Beta distribution with parameters 4 and 5 as the prior distribution. In that case the posterior is proportional to

$$f_{\mu|X}(\mu|x) \propto \exp\left(-\frac{1}{2}(x - \mu)^2\right) \cdot 1\{0 < \mu < 1\} \cdot \mu^3 \cdot (1 - \mu)^4.$$

Unfortunately, this is not the kernel of any standard density. Thus, in order to obtain the normalizing constant that makes the density integrate to 1, we would have to actually integrate the kernel. Moreover, if we want the posterior mean of  $\mu$  given  $X = x$ , this will again require very difficult calculations. Thus, using conjugate families greatly simplifies the task of finding posterior distributions.<sup>1</sup>

Let us consider the normal model in slightly more generality. Suppose that given  $\mu$  the random variable  $X$  has a normal distribution with mean  $\mu$  and known variance  $\sigma^2$ . The prior distribution for  $\mu$  is normal with mean  $\mu_0$  and variance  $\tau^2$ . The posterior distribution is proportional to:

$$\begin{aligned} f_{\mu|X}(\mu|x) &\propto \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right) \cdot \exp\left(-\frac{1}{2 \cdot \tau^2}(\mu - \mu_0)^2\right) \\ &\propto \exp -\frac{1}{2} \left( \frac{x^2}{\sigma^2} - \frac{2x\mu}{\sigma^2} + \frac{\mu^2}{\sigma^2} + \frac{\mu^2}{\tau^2} - \frac{2\mu\mu_0}{\tau^2} + \frac{\mu_0^2}{\tau^2} \right) \\ &\propto \exp -\frac{1}{2} \left( \mu^2 \frac{\sigma^2 + \tau^2}{\tau^2 \sigma^2} - \mu \frac{2x\tau^2 + 2\mu_0\sigma^2}{\tau^2 \cdot \sigma^2} \right) \\ &\propto \exp -\frac{1}{2(1/(1/\tau^2 + 1/\sigma^2))} \left( (\mu - (x/\sigma^2 + \mu_0/\tau^2)/(1/\sigma^2 + 1/\tau^2)) \right)^2. \end{aligned}$$

Hence the posterior distribution is normal with mean  $(x/\sigma^2 + \mu_0/\tau^2)/(1/\sigma^2 + 1/\tau^2)$  and variance  $(1/(1/\tau^2 + 1/\sigma^2))$ . The calculations are messy but the result is quite intuitive: the posterior mean is a weighted average of the prior mean  $\mu_0$  and the observation  $x$  with weights adding up to one and proportional to the precision (defined as one over the variance),  $1/\sigma^2$  for  $x$  and  $1/\tau^2$  for  $\mu_0$ . The posterior precision is obtained by adding

<sup>1</sup>More recently, a large literature has developed on using numerical integration and simulation methods (such as Markov chain Monte Carlo) to calculate posteriors when the prior is not conjugate. So there is less need to restrict attention to conjugate families than before the rise of fast, cheap computing power. Nevertheless, conjugate analysis still remains practically very important, because there are still limitations on what can be handled by computational algorithms.

up the precision for each component. So, what you expect ex post,  $E[\mu|X]$ , that is, after seeing the data, is a weighted average of what you expected before,  $E[\mu] = \mu_0$ , and the data,  $X$ , with the weights determined by their respective variances.

There are a number of insights obtained by studying this example more carefully. Suppose we are really sure about the value of  $\mu$  before we conduct the experiment. In that case we would set  $\tau^2$  small and the weight given to the observation would be small, and the posterior distribution would be close to the prior distribution. Suppose on the other hand we are very unsure about the value of  $\mu$ . What value for  $\tau$  should we choose? One possibility is to let  $\tau \rightarrow \infty$ . Loosely speaking, this prior is flat, i.e. proportional to a constant, and gives roughly equal weight to any possible value of  $\mu$ , and might be a reasonable choice if we want the prior to be “uninformative.”<sup>2</sup> In this case, the posterior distribution is essentially a normalized version of the likelihood function.

Note that the “flat prior” is not really a probability distribution – it does not integrate to one, and cannot be normalized. We call this an improper prior distribution. Despite the fact that the prior is not really a probability density, the posterior distribution (in this case) can still be calculated by Bayes’ rule, so we can think of this as a generalized posterior distribution, which is a convenient approximation to a situation where the prior information is very diffuse.

Now suppose we have two observations. Let us assume that conditional on  $\mu$  these two observations are independent and identically distributed with a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . We start with a prior distribution for  $\mu$  that is normal with mean  $\mu_0$  and variance  $\tau^2$ . We can factor the conditional distribution of  $\mu$  given  $X_1$  and  $X_2$  as

$$f(\mu|X_1, X_2) = \frac{f(\mu, X_1, X_2)}{f(X_1, X_2)} = \frac{f(X_1|\mu, X_2)f(X_2|\mu)f(\mu)}{f(X_1, X_2)}.$$

By independence of  $X_1$  and  $X_2$  conditional on  $\mu$  this is equal to

$$\frac{f(X_1|\mu)f(X_2|\mu)f(\mu)}{f(X_1, X_2)} \propto f(X_1|\mu)f(\mu)f(X_2|\mu).$$

The first two factors combined are proportional to  $f(\mu|X_1)$ . So, the total expression is the posterior of  $\mu$  given  $X_2$  with prior distribution for  $\mu$  equal to  $f(\mu|X_1)$ . In other words, we can update our information about  $\mu$  sequentially. First we calculate the prior distribution of  $\mu$  given  $X_1$ , taking  $f(\mu)$  as the prior distribution. Then we take the resulting posterior distribution as the prior distribution and add the information in  $X_2$ .

In the first step the prior is

$$\mu \sim \mathcal{N}(\mu_0, \tau^2).$$

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<sup>2</sup>Note that just because a prior is flat does not always mean it is uninformative. There is a large literature on choosing uninformative priors for various types of models.

Our previous calculations showed that the posterior distribution is

$$\mu|X_1 \sim \mathcal{N}\left(\frac{\mu_0/\tau^2 + x_1/\sigma^2}{1/\tau^2 + 1/\sigma^2}, \frac{1}{1/\tau^2 + 1/\sigma^2}\right).$$

Taking this as the prior distribution and  $X_2$  as the observation we get for the posterior distribution of  $\mu$  given  $x_1$  and  $x_2$ :

$$\mu|X_1, X_2 \sim \mathcal{N}\left(\frac{\mu_0/\tau^2 + x_1/\sigma^2}{1/\tau^2 + 1/\sigma^2}, \frac{1}{1/\tau^2 + 1/\sigma^2}\right) = \mathcal{N}\left(\frac{\mu_0/\tau^2 + (x_1 + x_2)/\sigma^2}{1/\tau^2 + 2/\sigma^2}, \frac{1}{1/\tau^2 + 2/\sigma^2}\right).$$

With  $N$  observations  $X_1, \dots, X_N$  we would get

$$\mu|X_1, \dots, X_N \sim \mathcal{N}\left(\frac{\mu_0/\tau^2 + \sum x_i/\sigma^2}{1/\tau^2 + N/\sigma^2}, \frac{1}{1/\tau^2 + N/\sigma^2}\right).$$

Note that when  $N$  is very large, the terms coming from the prior distribution get swamped by the terms coming from the likelihood. More formally, the distribution of  $\sqrt{N}(\mu - \bar{x})$  is approximately

$$\mathcal{N}(0, \sigma^2).$$

In other words, in large samples the influence of the prior distribution disappears, unless the prior distribution is chosen particularly badly, e.g., equal to zero on an important part of the parameter space. This turns out to be true more generally, i.e. for most models that satisfy some weak “regularity” conditions.

Let us return to the Bernoulli example. Suppose that conditional on  $P = p$ ,  $X_1, X_2, \dots, X_N$  are independent with Bernoulli distributions with probability  $p$ . Let the prior distribution of  $P$  be Beta with parameters  $\alpha$  and  $\beta$ . Now consider the conditional distribution of  $P$  given  $X_1, \dots, X_N$ :

$$f_{P|X_1, \dots, X_N}(p|x) \propto p^{\alpha-1+\sum_{i=1}^N X_i} \cdot (1-p)^{\beta-1+N-\sum_{i=1}^N X_i},$$

which is a Beta distribution with parameters  $\alpha + \sum_{i=1}^N X_i$  and  $\beta + N - \sum_{i=1}^N X_i$ . Let  $Y_N = \sum_{i=1}^N X_i$ . Then the mean and variance are

$$E[P|X_1, \dots, X_N] = \frac{\alpha + Y_N}{\alpha + \beta + N},$$

and

$$V(P) = \frac{(\alpha + Y_N)(\beta + N - Y_N)}{(\alpha + \beta + N)^2(\alpha + \beta + 1 + N)}.$$

What happens if  $n$  gets large? Let  $\hat{p} = Y_N/N$  be the relative frequency of success. Then the mean and variance converge to

$$E[P|X_1, \dots, X_N] \approx \hat{p},$$

and

$$V(P) \approx 0.$$

As the sample size gets larger, the posterior distribution becomes concentrated at a value that does not depend on the prior distribution. This in fact can be taken a step further. In this example, the limiting distribution of  $\sqrt{N} \cdot (P - \hat{p})$  conditional on the data, can be shown to be normal with mean zero and variance  $\hat{p}(1 - \hat{p})$ , again irrespective of the choice of  $\alpha$  and  $\beta$ . The interpretation of this result is very important: in large sample the choice of prior distribution is not very important in the sense that the information in the prior distribution gets dominated by the sample information. That is, unless your prior beliefs are so strong that they cannot be overturned by evidence (i.e., the prior distribution is zero over some important range of the parameter space), at some point the evidence in the data outweighs any prior beliefs you might have started out with.