Econ 520, Fall 2008
Midterm Review Questions

Note: I will not provide solutions to these questions. Some of these questions are drawn from previous years' exams.

1. Events M and N are said to be mutually exclusive provided that $M \cap N = \emptyset$.

   Suppose that $P(A) > 0$ and $P(B) > 0$. Show that if A and B are independent, then they cannot be mutually exclusive.

2. Suppose that the continuous random variable X has PDF

$$f(x) = \frac{4}{3}(1 - x^3), \qquad 0 < x < 1,$$

   and 0 otherwise. Determine the values of the following probabilities:

   (a) $Pr(X < 1/2)$.
   (b) $Pr(1/4 < X < 3/4)$.
   (c) $Pr(X > 1/3)$.

3. Consider the following experiment. A fair coin is tossed. If a head appears one point is recorded. If a tail appears two points are recorded. The coin is tossed repeatedly (independently) each time recording the points. The experiment stops as soon as the total number of points (from all tosses) is greater than or equal to three. Let X be the random variable denoting the number of times the coin is tossed.

   (a) What is the PMF for the random variable X? What is the CDF of X?
   (b) What is $Pr(X \geq 2)$? What is $Pr(X \geq 3)$?
   (c) What is $E(X)$?
   (d) What is the moment generating function $M_X(t)$ for the random variable X? Check that $M'_X(0)$ is equal to your answer in (c).

4. Suppose that X has PDF

$$f(x) = \frac{1}{2} \exp\left(-|x|\right), \quad -\infty < x < \infty.$$

   (a) What is the median of X?
   (b) Calculate the CDF of X.
   (c) Suppose we extend this distribution to a parametric family, with parameter ν, as follows:

$$f(x; \nu) = \frac{1}{2\nu} \exp\left(-\frac{|x|}{\nu}\right).$$

   Is this an exponential family? Explain your reasoning.

5. Let X_1, X_2 and X_3 have joint pdf:

$$\begin{aligned} f_{X_1 X_2 X_3}(x_1, x_2, x_3) &= \exp\{-(x_1 + x_2 + x_3)\} \\ & \quad 0 < x_1 < \infty, 0 < x_2 < \infty, 0 < x_3 < \infty \end{aligned}$$

   Are X_1, X_2 and X_3 independent? Show carefully how you reached your conclusion.

6. Suppose that Y has PDF $f_Y(y) = \frac{192}{y^4}\mathbf{1}(y \geq 4)$, and suppose that the conditional distribution of X given $Y = y$ is Uniform on $[0, y]$. Find the conditional PDF of Y given $X = 5$.

7. Suppose that the joint PDF of X and Y is

$$f(x, y) = \frac{3x + y}{7} \qquad \text{for } 0 < x < 2, 0 < y < 1.$$

   (a) Find the marginal density of X.

   (b) Find the conditional density of Y given X.

8. Suppose that X is distributed Uniform on $[0, 1]$ and that Y is a random variable with

$$E[Y|X = x] = \alpha + \beta x^2.$$

   (a) Calculate $E[Y]$.

   (b) Let $U = Y - \alpha - \beta X^2$. Calculate the covariance between U and X.

9. Suppose that you have a fair coin, with sides labeled "1" and "2", and a (fairly weighted) 4-sided die, with sides labeled 1,2,3,4. You toss the coin, toss the die, and add together the two numbers. Call the result X.

   (a) Write down the probability mass function for X.

   (b) Calculate the expected value of X.

   (c) Use Markov's inequality to calculate a bound on $P(X \geq 4)$, and compare this to the actual value of $P(X \geq 4)$.

10. Suppose that (X, Y) are continuously jointly distributed, with the following joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ and } -x < y < x \\ 0 & \text{otherwise .} \end{cases}$$

   (a) What is the conditional density of Y given $X = x$? Are X and Y independent?

   (b) Calculate $E[Y|X = x]$.

   (c) Show that $Cov(X, Y) = 0$. (Hint: one way is to use iterated expectations.)

11. (a) Suppose that X and Y are independent random variables, and consider functions $g(X)$ and $h(Y)$. Show that

$$E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)].$$

   (b) Suppose that X and Y are independent random variables with moment generating functions $M_X(t)$ and $M_Y(t)$. Let $Z = X + Y$. Show that its MGF is

$$M_Z(t) = M_X(t) \cdot M_Y(t).$$

   (c) The moment generating function for a normal random variable with mean μ and variance σ^2 is

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).$$

   Show that if X and Y are independent and normally distributed (not necessarily with the same mean or variance), then $X + Y$ is normally distributed.

12. Suppose that X and Y are jointly continuously distributed on the disk $\{(x, y) : x^2 + y^2 \leq 1\}$:

$$f(x, y) = C \cdot 1(x^2 + y^2 \leq 1).$$

(a) What is the value of C? (Hint: the area enclosed by a circle of radius r is πr^2, and its circumference is $2\pi r$.)

(b) Calculate the conditional density of X given Y: $f(x|y)$. Are X and Y independent? Be sure to explain why or why not.