

Lecture Note 8: Multivariate Normal Distribution

Mean Vector and Variance Matrix Suppose we have a K -dimensional random vector

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_K \end{pmatrix}.$$

We can define its mean vector as

$$E[Y] = \begin{bmatrix} E[Y_1] \\ \vdots \\ E[Y_K] \end{bmatrix},$$

and we can define its joint variance matrix as

$$V[Y] = E[(Y - E[Y])(Y - E[Y])'].$$

A variance matrix is $K \times K$, symmetric, and positive semidefinite. There are a number of standard properties of variance matrices. For example, let $\mu = E[Y]$. Then

$$V(Y) = E[YY'] - \mu\mu'.$$

(Here $E[YY']$ is the element-by-element expectation of the $K \times K$ outer product of Y with itself.) Also, for a vector of constants c ,

$$V(c'Y) = c'[V(Y)]c.$$

As an example, consider a 2 dimensional random vector

$$Y = \begin{pmatrix} X \\ Z \end{pmatrix}.$$

(We can write this compactly as $Y = (X, Z)'$.) Then

$$E[Y] = \begin{bmatrix} E[X] \\ E[Z] \end{bmatrix},$$

and

$$V(Y) = \begin{bmatrix} \begin{pmatrix} X - E[X] \\ Z - E[Z] \end{pmatrix} \cdot (X - E[X] & Z - E[Z]) \end{bmatrix} = \begin{bmatrix} V(X) & Cov(X, Z) \\ Cov(X, Z) & V(Z) \end{bmatrix}.$$

Sometimes the variance matrix is called instead the covariance matrix, or the variance-covariance matrix.

Definition: (Multivariate Normal Distribution) a K -dimensional random vector Y has a multivariate normal distribution with mean vector μ and nonsingular variance matrix Σ if its joint PDF can be written as

$$f_Y(y) = \det(2\pi \cdot \Sigma)^{-1/2} \exp \left[-\frac{1}{2}(y - \mu)' \Sigma^{-1}(y - \mu) \right].$$

We denote the multivariate normal distribution by $N(\mu, \Sigma)$, or if we want to make the dimension explicit, $N_K(\mu, \Sigma)$.

The multivariate normal distribution is a generalization of the (scalar) normal distribution. It is very important in statistical and economic applications, and has some nice properties. Proofs of the following properties can be found in Ruud, P., *An Introduction to Classical Econometric Theory*.

Nice Property 1 (Linear Transform): Suppose that $Z \sim N_K(\mu, \Sigma)$. Let $Y = a + BZ$, where a is a K -dimensional vector of constants and B is a $K \times K$ nonsingular matrix. Then

$$Y \sim N_K(a + B\mu, B\Sigma B').$$

Nice Property 2 (Partitioning): Let $Y \sim N_K(\mu, \Sigma)$. Partition Y into two subvectors (of dimension K_1 and K_2):

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.$$

Analogously partition the mean vector and variance matrix:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma'_{12} & \Sigma_{22} \end{bmatrix}.$$

Then the marginal distributions of Y_1 and Y_2 are:

$$Y_1 \sim N_{K_1}(\mu_1, \Sigma_{11}),$$

$$Y_2 \sim N_{K_2}(\mu_2, \Sigma_{22}).$$

Also, the conditional distribution of Y_1 given $Y_2 = y_2$ is

$$Y_1|Y_2 = y_2 \sim N_{K_1}(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma'_{12}).$$

Nice Property 3 (Covariance and Independence): Let $Y \sim N_K(\mu, \Sigma)$. Partition Y into two subvectors (of dimension K_1 and K_2):

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.$$

Analogously partition the mean vector and variance matrix:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma'_{12} & \Sigma_{22} \end{bmatrix}.$$

Then Y_1 and Y_2 are independent if and only if $\Sigma_{12} = \mathbf{0}$ (a matrix of zeros).

By the last property, if the covariance between jointly multivariate normal random variables is zero, then they are independent. As we noted before, this is not true for all distributions.

Also note that it is possible for two random variables to have normal marginal distributions, but not be *jointly* normal.