Economics 520, Fall 2008

Lecture Note 11: Limit Laws for Random Vectors

(Based on Van der Vaart, *Asymptotic Statistics*, Cambridge University Press.)

Most of the results discussed in LN10 extend fairly easily to random vectors in $\mathbb{R}^k$. As before, if X is a k-dimensional random vector, then $E[X]$ is the k-vector of expected values of the components of X, and $V[X]$ is the $k \times k$ symmetric positive definite matrix of variances and covariances.

For vectors $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k)$, let $a \leq b$ mean that each component of a is less than or equal to each corresponding component of b:

$$a_j \leq b_j \quad j = 1, 2, \ldots, k.$$

Then we can define the CDF of the random vector X as:

$$F_X(x) = Pr(X \leq x), \quad x \in \mathbb{R}^k.$$

Also, for a vector $a = (a_1, \ldots, a_k)$, let

$$||a|| = \left(\sum_{j=1}^{k} a_j^2 \right)^{1/2}$$

be the usual Euclidean norm. Note that $||a|| = \sqrt{a'a}$.

With this notation, we can define convergence in probability, convergence almost surely, and convergence in distribution in a similar way to the scalar case:

Convergence in Probability: a sequence of random vectors $\{X_n\}$ converges in probability to vector a if for all $\epsilon > 0$,

$$P\left(||X_n - a|| > \epsilon\right) \to 0,$$

as $n \to \infty$. We usually write this as $X_n \xrightarrow{p} a$ or $\text{plim}_{n\to\infty} X_n = a$.

Convergence Almost Surely: a sequence of random vectors $\{X_n\}$ converges almost surely to a if

$$P\left(\lim_{n\to\infty} ||X_n - a|| = 0\right) = 1.$$

We usually write this as $X_n \xrightarrow{as} a$.

As with the scalar case, convergence almost surely implies convergence in probability, but the converse does not hold.

Convergence in Distribution: a sequence of random variables $\{X_n\}$ is said to converge in distribution to a random variable X if

$$P(X_n \leq x) \longrightarrow P(X \leq x)$$

at every point x at which the limit distribution function $P(X \leq x)$ is continuous. This will be denoted by $X_n \xrightarrow{d} X$.

Basic Asymptotic Results:

Weak Law of Large Numbers (WLLN): Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that $E(||X_1||) < \infty$. Then

$$\bar{X}_n \equiv \frac{1}{n}\sum_i X_i \xrightarrow{p} E(X_1).$$

Strong Law of Large Numbers (SLLN): Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that $E(||X_1||) < \infty$. Then

$$\bar{X}_n \equiv \frac{1}{n}\sum_i X_i \xrightarrow{as} E(X_1).$$

Multivariate Central Limit Theorem (CLT): Let $X_1, X_2, \ldots$ be i.i.d. random vectors in $\mathbb{R}^k$ with $E[||X_1||^2] < \infty$. Let $\mu = E[X_1]$ and $\Sigma = V[X_1]$. Then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu) = \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \Sigma).$$

Continuous Mapping Theorem (CMT): Let $g(\cdot)$ be a function from $\mathbb{R}^k$ to $\mathbb{R}^m$, and suppose g is continuous at every point in a set C s.t. $P(X \in C) = 1$. Then

(i) If $X_n \xrightarrow{p} X$, then $g(X_n) \xrightarrow{p} g(X)$;

(ii) If $X_n \xrightarrow{d} X$, then $g(X_n) \xrightarrow{d} g(X)$.

Note that matrix addition and multiplication are continuous functions.

There are various useful facts about convergence in probability and convergence in distribution:

Result:

(i) Convergence in probability implies convergence in distribution:

$$X_i \xrightarrow{p} X \Rightarrow X_i \xrightarrow{d} X.$$

(ii) Convergence in distribution to a constant, implies convergence in probability:

$$X_i \xrightarrow{d} a \text{ (a constant) } \Rightarrow X_i \xrightarrow{p} a.$$

(iii) If $X_n \xrightarrow{d} X$ and $||X_n - Y_n|| \xrightarrow{p} 0$, then $Y_n \xrightarrow{d} X$.
(iv) If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} a$ where a is a constant, then the vector $(X_n, Y_n) \xrightarrow{d} (X, c)$.
(v) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, then $(X_n, Y_n) \xrightarrow{p} (X, Y)$. (This is not true for convergence in distribution.)

A useful corollary of the previous result:

Slutsky's Lemma: Let X_n, X, and Y_n be random vectors or matrices. If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} c$, where c is a conformable constant vector or matrix, then

(i) $X_n + Y_n \xrightarrow{d} X + c$;

(ii)$Y_n X_n \xrightarrow{d} cX$;

(iii) $Y_n^{-1} X_n \xrightarrow{d} c^{-1} X$, provided c is invertible.

Delta Method: Let X_n be a sequence of $d-$dimensional random vectors such that

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} N(0, \Sigma),$$

where Σ is positive definite and finite. Let g denote a continuously differentiable function from $\mathbb{R}^d$ into $\mathbb{R}^k$, and let $G(x) = \partial g / \partial x$ denote the $k \times d$ matrix of partial derivatives. Then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} N(0, G(\mu) \Sigma G(\mu)').$$