

## Final Review Questions

Note: Many of these questions are drawn from previous years' finals. I will not provide solutions to these questions. The final will cover material from the entire semester, but with more weight on the second half of the course.

- Suppose that  $X_1$  is uniformly distributed on  $(0, 1)$  and  $X_2$  is uniformly distributed on  $(0, 2)$ , and  $X_1$  and  $X_2$  are independent. Let  $Y \equiv \max(X_1, X_2)$ .
  - Derive the CDF and PDF of  $Y$ .
  - Calculate  $E[Y]$ .
  - Suppose that instead of being independent,  $X_1$  and  $X_2$  are related in the following way:  $X_1$  is uniform  $(0, 1)$ , and  $X_2 = 2 \cdot X_1$ . Let  $Y$  be defined as before. Now calculate the CDF and PDF of  $Y$ .
- Suppose that for each  $n$ ,  $X_n \sim N(1/n, 1/n)$ .
  - Does  $X_n \xrightarrow{p} 0$ ? Prove your answer.
  - Does  $X_n \xrightarrow{qm} 0$ ? Prove your answer.
- Suppose  $X_n \sim \text{Bin}(n, \frac{\alpha}{n})$ , where  $\alpha > 0$ .
  - $X_n \xrightarrow{d} Y$ . What's the probability mass function for  $Y$ ?
  - Does  $\frac{X_n}{\sqrt{n}} \xrightarrow{p} 0$ ? Prove your answer.
- The conditional distribution of  $Y$  given  $X = x$  is Poisson with mean  $\theta^* \cdot x$ . The distribution of  $X$  is uniform on the interval  $[1, 2]$ . You have a random sample of size  $N$  from this distribution for  $(X_i, Y_i)$ . Give the asymptotic distribution of the maximum likelihood estimator and show how you would calculate the variance of this distribution.
- Let  $X$  have a binomial  $B(1, 1/2)$  distribution with  $\Pr(X = 1) = 1/2$ . Conditional on  $X = x$ , the random variable  $Y$  has a Poisson distribution with parameter  $\lambda(1 + x)$ :

$$\Pr(Y = y|X = x) = \frac{(\lambda(1 + x))^y \exp(-\lambda(1 + x))}{y!},$$

for  $x = 0, 1$ , and  $y = 0, 1, \dots$

- Calculate  $\Pr(Y = 2)$ .
  - Calculate  $\Pr(X = 1|Y = 2)$ .
  - Find an unbiased estimator for  $\lambda$  that is a function of  $Y$  alone.
- Let  $X$  be a random variable with probability density function

$$f_X(x; \mu) = \frac{1}{\mu} \exp(-x/\mu),$$

for  $x > 0$  and zero elsewhere.

- Calculate the mean and variance of  $X$ .

- (b) Calculate the mean and variance of  $X$  conditional on  $X < 8$ .
- (c) Let  $x_1, x_2, \dots, x_N$  be a random sample from this distribution, with  $N = 20$ ,  $\sum x = 95$ , and  $\sum x^2 = 590$ . Calculate the maximum likelihood estimate.
- (d) Test the hypothesis that  $\mu = 4$  at the 10% level using a likelihood ratio test.
- (e) Test the same hypothesis using a Lagrange multiplier (score) test.
7. Let the marginal distribution of  $X$  be binomial with  $N = 1$  and  $p = 1/4$ . Conditional on  $X$ , the random variable  $Y$  has a normal distribution with mean  $\mu \cdot (X + 1)$  and variance 1.
- (a) Find the marginal density of  $Y$ .
- (b) Suppose you have a random sample of size  $N$  from this joint distribution. What is the maximum likelihood estimator and its large sample variance?
- (c) Suppose you only observe  $y_1, \dots, y_N$ . Find an unbiased estimator for  $\mu$ . What is its large sample variance and how does that compare to that of the maximum likelihood estimator derived before?
8. Let the distribution of  $X$  have probability density function

$$f_X(x; \lambda) = \frac{1}{2} \lambda \exp(-|x|\lambda),$$

for  $-\infty < x < \infty$ . Let  $x_1, \dots, x_N$  be a random sample from this distribution.

- (a) Find a one-dimensional sufficient statistic for  $\lambda$ .
- (b) Let  $N = 20$ ,  $\sum_{i=1}^N x_i = 20$ ,  $\sum_{i=1}^N |x_i| = 100$ ,  $\sum_{i=1}^N x_i^2 = 200$ . Find the maximum likelihood estimator.
- (c) Test the null hypothesis
- $$H_0 : \quad \lambda = 0.125,$$
- against the alternative hypothesis
- $$H_1 : \quad \lambda \neq 0.125,$$
- at the 5% level using a likelihood ratio test.
- (d) Test the same null hypothesis using a Wald test.
- (e) Provide an approximate 95% confidence interval for  $\lambda$  based on MLE.
9. Suppose there is a random sample of size 10 ( $X_1, \dots, X_{10}$ ) from a Poisson distribution (so  $f_{X_i}(x_i|\theta) = \frac{\theta^{x_i} e^{-\theta}}{x_i!}$ , when  $x_i$  is a nonnegative integer). The sample mean of the random sample is 4.
- (a) Using the sample, what is the maximum likelihood estimate for  $\theta$ ?
- (b) Provide an approximate 95% confidence interval for  $\theta$ .
- (c) Using a large-sample LR test, test the hypothesis that  $\theta = 3$  at the 0.05 level. (Note that if  $Z$  is a chi-squared random variable with 1 degree of freedom,  $Pr(Z > 3.84) = 0.05$ .)

(d) Now suppose that the prior distribution for  $\theta$  is  $Gamma(3, 5)$ :

$$\theta \sim Gamma(3, 5).$$

What is the posterior distribution for  $\theta$  given  $X_1, \dots, X_{10}$ ? (Hint:  $Z \sim Gamma(\alpha, \beta)$  means  $f_Z(z|\alpha, \beta) = \frac{z^{\alpha-1}e^{-z/\beta}}{\Gamma(\alpha)\beta^\alpha}$ )

10. A firm samples machine parts until it finds a defective part; let  $X_i$  be the number of samples until a defective part. Assume that each sample is independent with probability  $p$  of being defective.

(a) Derive the probability mass function of  $X_i$ .

(b) Suppose that the firm obtains independent observations  $X_1, \dots, X_n$ . The firm has a Beta(1,1) prior distribution for  $p$ . What is the posterior distribution for  $p$ ?

Note: the Beta distribution with parameters  $\alpha$  and  $\beta$  has PDF

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}.$$

11. Suppose that  $X_1, \dots, X_n$  are IID with the following distribution: for  $\theta > 0$ ,

$$Pr(X_i = 0) = .5,$$

$$Pr(X_i = \theta) = .5.$$

(a) Derive  $E[X_i]$  and  $V[X_i]$ .

(b) Consider the estimator  $\hat{\theta} = b\bar{X}_n$ , where  $\bar{X}_n$  is the sample average and  $b > 0$  is a constant. For what value(s) of  $b$  is this estimator unbiased? What is the variance of this estimator?

(c) Let  $U_n = \max\{X_1, \dots, X_n\}$ . Derive  $Pr(U_n = 0)$  and  $Pr(U_n = \theta)$ .

(d) Consider the estimator  $\tilde{\theta} = cU_n$ . Assume that  $n = 3$ . For what value of  $c$  is this estimator unbiased? Calculate the variance of this estimator and compare it to  $\hat{\theta}$  (again assuming  $n = 3$ ). Which estimator has lower mean squared error?

(e) Show that  $U_n \xrightarrow{p} \theta$ . If  $\tilde{\theta}_n = c_n U_n$ , where  $c_n$  is such that the estimator is unbiased for each  $n$ , does  $\tilde{\theta}_n \xrightarrow{p} \theta$ ?