Econ 520, Fall 2007
Midterm Review Questions

Note: I will not provide solutions to these questions. Some of these questions are drawn from previous years' exams.

1. Events M and N are said to be mutually exclusive provided that $M \cap N = \emptyset$.

   Suppose that $P(A) > 0$ and $P(B) > 0$. Show that if A and B are independent, then they cannot be mutually exclusive.

2. Consider two events A and B such that $P(A) \neq 0$, and $P(B) \neq 0$. Show that

$$P(A|B) > P(A) \iff P(B|A) > P(B).$$

   (Here, $\iff$ means "if and only if".)

3. Suppose that the continuous random variable X has PDF

$$f(x) = \frac{4}{3}(1 - x^3), \qquad 0 < x < 1,$$

   and 0 otherwise. Determine the values of the following probabilities:

   (a) $Pr(X < 1/2)$.
   (b) $Pr(1/4 < X < 3/4)$.
   (c) $Pr(X > 1/3)$.

4. Suppose that Y has PDF $f(y) = 3y^2$ for $0 < y < 1$ and 0 otherwise. Find the median of Y.

5. Consider the following pdf:

$$f_X(x) = \frac{1}{6\theta^4} x^3 \exp\left\{-\frac{x}{\theta}\right\}, \qquad 0 < x < \infty, \qquad 0 < \theta < \infty$$

   a) Obtain the moment generating function of this pdf.

   b) Use the moment generating function you obtained in (a) to find $E(X)$ and $V(X)$.

   c) Check if this pdf is a member of the exponential family of densities.

6. Suppose that a random variable has mean 12 and variance 2. Using the Chebyshev inequality, provide a bound on the probability that the variable is between 10 and 14.

7. Consider the following experiment. A fair coin is tossed. If a head appears one point is recorded. If a tail appears two points are recorded. The coin is tossed repeatedly (independently) each time recording the points. The experiment stops as soon as the total number of points (from all tosses) is greater than or equal to three. Let X be the random variable denoting the number of times the coin is tossed.

   (a) What is the PMF for the random variable X? What is the CDF of X?
   (b) What is $Pr(X \geq 2)$? What is $Pr(X \geq 3)$?
   (c) What is $E(X)$?

(d) What is the moment generating function $M_X(t)$ for the random variable X? Check that $M'_X(0)$ is equal to your answer in (c).

8. Suppose that Y has a gamma distribution with parameters α and β, and suppose that the conditional distribution of X given $Y = y$ is exponential with parameter y (i.e. the conditional mean of X given $Y = y$ is $1/y$). What is the conditional distribution of Y given $X = x$?

9. Suppose that X and Y have a continuous joint distribution with joint PDF

$$f(x, y) = x + y \qquad \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1,$$

and 0 otherwise. Find $E[Y|X]$.

10. Let X_1, X_2 and X_3 have joint pdf:

$$\begin{aligned} f_{X_1X_2X_3}(x_1, x_2, x_3) &= \exp\{-(x_1 + x_2 + x_3)\} \\ &\quad 0 < x_1 < \infty, 0 < x_2 < \infty, 0 < x_3 < \infty \end{aligned}$$

Are X_1, X_2 and X_3 independent? Show carefully how you reached your conclusion.

11. Suppose that Y has PDF $f_Y(y) = \frac{192}{y^4}\mathbf{1}(y \geq 4)$, and suppose that the conditional distribution of X given $Y = y$ is Uniform on $[0, y]$. Find the conditional PDF of Y given $X = 5$.

12. Suppose that the joint PDF of X and Y is

$$f(x, y) = \frac{3x + y}{7} \qquad \text{for } 0 < x < 2, 0 < y < 1.$$

(a) Find the marginal density of X.

(b) Find the conditional density of Y given X.

13. Suppose that X is distributed Uniform on $[0, 1]$ and that Y is a random variable with

$$E[Y|X = x] = \alpha + \beta x^2.$$

(a) Calculate $E[Y]$.

(b) Let $U = Y - \alpha - \beta X^2$. Calculate the covariance between U and X.

The following questions are from last year's midterm:

14. Suppose that you have a fair coin, with sides labeled "1" and "2", and a (fairly weighted) 4-sided die, with sides labeled 1,2,3,4. You toss the coin, toss the die, and add together the two numbers. Call the result X.

(a) Write down the probability mass function for X.

(b) Calculate the expected value of X.

(c) Use Markov's inequality to calculate a bound on $P(X \geq 4)$, and compare this to the actual value of $P(X \geq 4)$.

15. Suppose that X has PDF

$$f_X(x) = C \cdot (x^2 + x)$$

for $0 < x < 1$, and 0 otherwise.

(a) Calculate the value of C.

(b) Derive the density of $Y = X^2$.

(c) Which is greater, $E[X]$ or $E[Y]$? If X had some different density function, but with the same support $(0, 1)$, and $Y = X^2$, would the result still hold? Explain your reasoning.

16. Suppose that (X, Y) are continuously jointly distributed, with the following joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ and } -x < y < x \\ 0 & \text{otherwise .} \end{cases}$$

(a) What is the conditional density of Y given $X = x$? Are X and Y independent?

(b) Calculate $E[Y|X = x]$.

(c) Show that $Cov(X, Y) = 0$. (Hint: one way is to use iterated expectations.)