Economics 520, Fall 2007

Lecture Note 17: Most Powerful Tests, CB 8.3.2-8.3.3

Example 1

Let us consider some examples of applications of the Neyman-Pearson Lemma. Suppose X has an exponential distribution with arrival rate λ. We wish to test the hypothesis that $\lambda = 1$ against the alternative that $\lambda = 2$:

$$\begin{aligned} H_0 : &\quad \lambda = 1; \\ H_a : &\quad \lambda = 2. \end{aligned}$$

By the Neyman-Pearson lemma, we should use a critical region of the form

$$\begin{aligned} C_X &= \{x : f_X(x;2) \geq k \cdot f_X(x;1)\} \\ &= \{x : 2 \cdot \exp(-2x) \geq k \cdot \exp(-x)\} \\ &= \{x : \exp(-2x) \geq \exp(k') \exp(-x)\} \\ &= \{x : -2x \geq k' - x\} \\ &= \{x : x \leq k''\}. \end{aligned}$$

All that is left to determine is k''. Suppose we wish to test at the 0.05 level. Then we choose k'' to satisfy

$$\begin{aligned} 0.05 &= Pr(X \leq k'' | H_0) \\ &= \int_0^{k''} \exp(-x) dx \\ &= 1 - \exp(-k''), \end{aligned}$$

or

$$k'' = -\ln(0.95) \approx 0.0513,$$

and the critical region is

$$C_X = [0, -\ln(0.95)].$$

□

Example 2

Suppose $X_1, \ldots, X_N$ are iid normal with mean μ and unit variance. We wish to test the null hypothesis $\mu = \mu_0$ against the alternative hypothesis that $\mu = \mu_1$, for some μ_1 and μ_0 with $\mu_1 > \mu_0$:

$$\begin{aligned} H_0 : &\quad \mu = \mu_0; \\ H_a : &\quad \mu = \mu_1. \end{aligned}$$

By Neyman-Pearson, we want the test to reject the null if

$$f(x_1, \ldots, x_N; \mu_1) \geq k \cdot f(x_1, \ldots, x_N; \mu_0)$$

or equivalently:

$$\frac{f(x_1,\ldots,x_N;\mu_1)}{f(x_1,\ldots,x_N;\mu_0)} \geq k.$$

This <u>ratio of likelihood functions</u> is

$$\begin{aligned}\frac{\mathcal{L}(\mu_1)}{\mathcal{L}(\mu_0)} &= \frac{\exp\left(-\frac{1}{2}\sum_i (x_i-\mu_1)^2\right)}{\exp\left(-\frac{1}{2}\sum_i (x_i-\mu_0)^2\right)} \\ &= \frac{\exp\left(-\frac{1}{2}\sum_i [x_i^2-2x_i\mu_1+\mu_1^2]\right)}{\exp\left(-\frac{1}{2}\sum_i [x_i^2-2x_i\mu_0+\mu_0^2]\right)} \\ &= \exp\left((\mu_1-\mu_0)\sum_i x_i\right)\cdot C,\end{aligned}$$

where C is a constant which does not depend on x. Since $\mu_1-\mu_0>0$, this ratio is larger than k if and only if

$$\sum_i x_i \geq k',$$

or equivalently,

$$\bar{x} \equiv \frac{1}{N}\sum_i x_i \geq k''.$$

The critical region is therefore of the form

$$C_X = \{(x_1,\ldots,x_N) : \bar{x} \geq k''\}.$$

Suppose we wish to test at the 0.05 level. Then

$$0.05 = Pr(\bar{x} \geq k'' | \mu = \mu_0).$$

Under the null the distribution of $\bar{x}$ is normal with mean μ_0 and variance $1/N$:

$$\bar{x} \sim N(\mu_0, \frac{1}{N}),$$

so

$$\frac{\bar{x}-\mu_0}{\sqrt{1/N}} \sim N(0,1).$$

Using a table for the standard normal distribution, we can determine that

$$Pr\left(\frac{\bar{x}-\mu_0}{\sqrt{1/N}} \geq 1.645\right) = 0.05.$$

So

$$Pr\left(\bar{x}-\mu_0 \geq \frac{1.645}{\sqrt{N}}\right) = 0.05,$$

and

$$Pr\left(\bar{x} \geq \mu_0 + \frac{1.645}{\sqrt{N}}\right) = 0.05.$$

Hence the critical region should be

$$C_X = \{(x_1, \ldots, x_N) : \bar{x} > \mu_0 + 1.645/\sqrt{N}\}.$$

□

Example 2 also illustrates an important phenomenon. There, *the critical region does not depend on the value of the parameter under the alternative hypothesis*, μ_1. Whether the alternative is $\mu_1 = \mu_0 + 1$ or $\mu_1 = \mu_0 + 4$ leads to exactly the same critical region. Thus, we can use the same test if we are testing the composite alternative hypothesis $H_a : \mu > \mu_0$. Moreover, since the test is most powerful for each specific point in the alternative, the test is uniformly most powerful against the composite alternative.

Uniformly most powerful tests do not always exist. They exist for some special models like the normal model, when the alternative is "one-sided" (i.e. $H_a : \mu > \mu_0$ or $H_a : \mu < \mu_0$). What if we consider the same normal model, and test

$$H_0 : \qquad \mu = \mu_0,$$

against the two-sided alternative

$$H_1 : \qquad \mu \neq \mu_0.$$

If the alternative is $\mu = \mu_1 > \mu_0$ the critical region for the most powerful test is of the form

$$C_X = \{(x_1, \ldots, x_N) : \bar{x} \geq k\}.$$

If the alternative is $\mu = \mu_1 < \mu_0$ the critical region of the most powerful test is of the form

$$C_X = \{(x_1, \ldots, x_N) : \bar{x} \leq k\}.$$

There is therefore no test that is most powerful for all values under the alternative. In other words, there is no uniformly most powerful test. One way to get around this problem, is to impose some additional restrictions on the test, and look for uniformly most powerful tests within the restricted set of tests.

A test is unbiased if the power function $\beta(\theta_1) \geq \beta(\theta_0)$ for all $\theta_1 \in \Theta_0^c$ and all $\theta_0 \in \Theta_0$. That is, the probability of rejecting the null hypothesis, or of an observation in the critical region, is at least as large for values of the parameters consistent with the alternative ($\theta \in \Theta_0^c$) as for values of the parameters consistent with the null hypothesis ($\theta \in \Theta_0$).

Let us consider this approach in detail for the case with a normal distribution with unknown mean and known variance. Let $X_1, \ldots, X_N$ be independent and normally distributed with unknown mean μ and known variance σ^2. We are interested in testing the null hypothesis

$$H_0 : \ \mu = \mu_0,$$

against the alternative

$$H_1 : \ \mu \neq \mu_0.$$

Let us consider the ratio of density functions to determine the critical region:

$$\frac{f(x_1,\dots,x_N|\mu_1)}{f(x_1,\dots,x_N|\mu_0)} = \frac{(2\pi\sigma^2)^{N/2}\exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^N x_i^2 - 2\mu_1\sum_{i=1}^N x_i + N\mu_1^2\right)\right)}{(2\pi\sigma^2)^{N/2}\exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^N x_i^2 - 2\mu_0\sum_{i=1}^N x_i + N\mu_0^2\right)\right)}$$

$$= \exp\Big(\frac{1}{\sigma^2}\cdot(\mu_1-\mu_0)\sum x_i\Big)\cdot\exp(-(\mu_1^2-\mu_0^2)N/(2\sigma^2)).$$

Hence if we are looking for a uniformly most powerful test against the alternative hypothesis $H_1:\ \mu > \mu_0$, the critical region ought to be of the form

$$C_X = \Big\{(x_1,\dots,x_N) : \bar{x} \geq k\Big\}.$$

If we were to test against the alternative hypothesis $H_1:\ \mu < \mu_0$, the critical region ought to be of the form

$$C_X = \Big\{(x_1,\dots,x_N) : \bar{x} \leq k\Big\}.$$

It therefore appears sensible to base a test on the value of $\bar{x}$, the sample average, which is a sufficient statistic for μ. It seems fairly clear that the critical region should be of the form

$$C_X = \Big\{(x_1,\dots,x_N) : \bar{x} \leq a \text{ or } \bar{x} \geq b\Big\}.$$

Unbiasedness of the test implies that

$$1-\beta(\mu) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2/N}}\exp\left(-\frac{1}{2\sigma^2/N}(\bar{x}-\mu)^2\right)d\bar{x},$$

is maximized at μ_0. The function is maximized at $\mu = (a+b)/2$, so that for unbiasedness we must have $b-\mu_0 = \mu_0 - a$. Hence the critical region is

$$C_X = \Big\{(x_1,\dots,x_N) : \bar{x} \leq \mu_0 - c \text{ or } \bar{x} \geq \mu_0 + c\Big\},$$

with the value of c determined by the size of the test. Under the null hypothesis the distribution of $\bar{x}$ is normal with mean μ_0 and variance σ^2/N. Hence, if we wish to test at the 10% level, recalling that for a standard normal random variable Z

$$Pr(-1.645 < Z < 1.645) = 0.90,$$

the critical region is

$$C_X = \Big\{(x_1,\dots,x_N) : \bar{x} \leq \mu_0 - 1.645\cdot\sigma/\sqrt{N}, \bar{x} \geq \mu_0 + 1.645\cdot\sigma/\sqrt{N}\Big\}.$$

This is the uniformly most powerful unbiased test.

If we wish to test at the 5% level, the critical region is

$$C_X = \Big\{(x_1,\dots,x_N) : \bar{x} \leq \mu_0 - 1.96\cdot\sigma/\sqrt{N}, \bar{x} \geq \mu_0 + 1.96\cdot\sigma/\sqrt{N}\Big\}.$$

Equivalently we can use the critical region

$$C_X = \left\{ (x_1, \ldots, x_N) : N \cdot (\bar{x} - \mu_0)^2 / \sigma^2 \geq 3.84 \right\},$$

which uses the Chi–squared distribution for the square of a standard normal random variable. In fact a common way of doing the test is to calculate the <u>test statistic</u>, here $N \cdot (\bar{x} - \mu_0)^2/\sigma^2$ which under the null hypothesis has a known distribution, in this case a $\chi^2(1)$ distribution. We reject the null hypothesis if the test statistic exceeds the critical value, in this case 3.84 at the 5% level or 2.706 at the 10% level.