Economics 520, Fall 2007
Homework 9
Due Tuesday, November 13 at beginning of class

1. Suppose that a random sample $X_1, \ldots, X_n$ is taken from a $N(\mu, 1)$ distribution with unknown mean μ. However, we only observe Y_i, an indicator equal to 1 if X_i is less than 0. Derive the likelihood and log-likelihood function based on $(Y_1, \ldots, Y_n)$, and solve for the maximum likelihood estimator of μ.

2. Suppose that conditional on τ the random variable X has a normal distribution with mean zero and variance $1/\tau$. The prior distribution for τ is Gamma with parameters α and β.

   (a) Find the parameters of the posterior distribution of τ.

   (b) What is the prior mean of τ?

   (c) What is the posterior mean of τ?

3. Show that the priors in the following cases are conjugate priors:

   (a) $X_1, \ldots, X_n$ is a random sample from the Binomial(p, k) distribution with probability p and size k. Assume that k is known. The prior for p is a Beta distribution with parameters α and β.

   (b) $X_1, \ldots, X_n$ is a random sample from the uniform distribution on $(0, \theta)$. The prior for θ is
   $$f(\theta) = ba^b\theta^{-(b+1)} \cdot 1(\theta > a).$$

   (c) $X_1, \ldots, X_n$ is a random sample from the exponential distribution with density $f(x; \lambda) = \lambda \exp(-\lambda x)$ for $x > 0$. The prior for λ is a Gamma distribution with parameters α and γ.

4. **(Corrected 11/8/07)**. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. Let $\tau = \sigma^{-2}$, so we can write the distribution as $N(\mu, \tau^{-1})$.

   Suppose the prior for (μ, τ) has density
   $$f(\mu, \tau) \propto \frac{1}{\tau}, \quad -\infty < \mu < \infty, 0 < \tau < \infty.$$

   Note that this is an improper prior density.

   Show that the posterior density of (μ, τ) is equal to
   $$f(\mu, \tau | x_1, \ldots, x_n) = f(\mu | \tau, x_1, \ldots, x_n) \cdot f(\tau | x_1, \ldots, x_n),$$
   where $f(\mu | \tau, x_1, \ldots, x_n)$ is a normal density with mean $\bar{x}$ and variance $1/(\tau n) = \sigma^2/n$, and $f(\tau | x_1, \ldots, x_n)$ is a gamma density with parameters
   $$\frac{n-1}{2}, \quad \text{and} \quad \left[\sum_{i=1}^{n}(x_i - \bar{x})^2/2\right]^{-1}.$$