

Economics 520, Homework 9

Due Thursday, November 16

1. Suppose Y_1, \dots, Y_n are IID discrete random variables with

$$\begin{aligned}P(Y_i = 0) &= \theta_0, \\P(Y_i = 1) &= \theta_1, \\P(Y_i = 2) &= \theta_2,\end{aligned}$$

where the parameter vector $\theta = (\theta_0, \theta_1, \theta_2)$ satisfies: $\theta_j \geq 0$ and $\sum_{j=0}^2 \theta_j = 1$.

- (a) Calculate $E[Y_i]$ and $E[Y_i^2]$, and use the results to derive a method of moments estimator for the parameters (θ_1, θ_2) .
- (b) Show that the maximum likelihood estimator for $\theta = (\theta_0, \theta_1, \theta_2)$ is

$$\begin{aligned}\hat{\theta}_0 &= \frac{1}{n} \sum_i 1(Y_i = 0), \\ \hat{\theta}_1 &= \frac{1}{n} \sum_i 1(Y_i = 1), \\ \hat{\theta}_2 &= \frac{1}{n} \sum_i 1(Y_i = 2).\end{aligned}$$

- (c) Prove that the maximum likelihood estimator of θ_0 converges in probability to the true value: as $n \rightarrow \infty$, we have $\hat{\theta}_0 \xrightarrow{P} \theta_0$.
- (d) Suppose that we take a Bayesian approach and use the following prior for θ :

$$p(\theta) = c \cdot \theta_0^{\alpha_0 - 1} \cdot \theta_1^{\alpha_1 - 1} \cdot \theta_2^{\alpha_2 - 1},$$

for $\theta_j \geq 0$ and $\sum_{j=0}^2 \theta_j = 1$. Here $\alpha_0, \alpha_1, \alpha_2$ are some constants, and c is a normalizing constant. Derive the posterior distribution (up to a normalizing constant) and show that this choice of prior is a conjugate prior.

2. Let X_1, X_2, \dots, X_N represent a random sample from each of the distributions having the following probability density functions. In each case find the maximum likelihood estimator $\hat{\theta}$ of θ .

- (a) $f(x; \theta) = \theta^x \exp(-\theta)/x!$, for $x = 0, 1, \dots$ and zero elsewhere, for $\theta \geq 0$.
- (b) $f(x; \theta) = \theta x^{\theta-1}$, for $0 < x < 1$, and zero elsewhere, for $0 < \theta < \infty$.
- (c) $f(x; \theta) = (1/\theta) \exp(-x/\theta)$, for $0 < x < \infty$, and zero elsewhere, for $0 < \theta < \infty$.
- (d) $f(x; \theta) = (1/2) \exp(-|x - \theta|)$, for $-\infty < x < \infty$, and zero elsewhere, for $-\infty < \theta < \infty$.
- (e) $f(x; \theta) = \exp(-(x - \theta))$, for $\theta < x < \infty$, and zero elsewhere, for $-\infty < \theta < \infty$.

3. The pareto distribution is sometimes used to model incomes and has cumulative distribution function

$$F_X(x; \theta_1, \theta_2) = 1 - (\theta_1/x)^{\theta_2}$$

for $\theta_1 < x < \infty$ with $\theta_1, \theta_2 > 0$. If X_1, X_2, \dots, X_N is a random sample from this distribution, find the maximum likelihood estimator for θ_1 and θ_2 .

4. The following numbers were drawn independently from a Poisson distribution:

2 2 2 7 2 2 4 2 1 1 3 2 5 4 0 3 2 4 3 2

Calculate the maximum likelihood estimate of the Poisson parameter λ .

5. Suppose conditional on p the random variables X and Y are independent with binomial distributions with parameters N and p and M and p respectively. The prior distribution of p is Beta with parameters α and β .
- (a) Find the posterior distribution of p given X and Y .
 - (b) What is the posterior mean and variance of p ?
 - (c) Suppose that conditional on p the random variable Z has a binomial distribution with parameters $N + M$ and p . With the same Beta prior distribution with parameters α and β , what is the posterior distribution of p given Z ? How does it differ from the posterior distribution in (a)?
6. Suppose that conditional on τ the random variable X has normal distribution with mean zero and variance $1/\tau$. The prior distribution for τ is Gamma with parameters $\alpha = 1$ and $\beta = 1$.
- (a) Show that the posterior distribution of τ is also a Gamma distribution.
 - (b) Find the parameters of the posterior distribution of τ .