

## Economics 520, Homework 9

Due Thursday, November 16

1. Suppose  $Y_1, \dots, Y_n$  are IID discrete random variables with

$$\begin{aligned}P(Y_i = 0) &= \theta_0, \\P(Y_i = 1) &= \theta_1, \\P(Y_i = 2) &= \theta_2,\end{aligned}$$

where the parameter vector  $\theta = (\theta_0, \theta_1, \theta_2)$  satisfies:  $\theta_j \geq 0$  and  $\sum_{j=0}^2 \theta_j = 1$ .

- (a) Calculate  $E[Y_i]$  and  $E[Y_i^2]$ , and use the results to derive a method of moments estimator for the parameters  $(\theta_1, \theta_2)$ .
- (b) Show that the maximum likelihood estimator for  $\theta = (\theta_0, \theta_1, \theta_2)$  is

$$\begin{aligned}\hat{\theta}_0 &= \frac{1}{n} \sum_i 1(Y_i = 0), \\ \hat{\theta}_1 &= \frac{1}{n} \sum_i 1(Y_i = 1), \\ \hat{\theta}_2 &= \frac{1}{n} \sum_i 1(Y_i = 2).\end{aligned}$$

- (c) Prove that the maximum likelihood estimator of  $\theta_0$  converges in probability to the true value: as  $n \rightarrow \infty$ , we have  $\hat{\theta}_0 \xrightarrow{P} \theta_0$ .
- (d) Suppose that we take a Bayesian approach and use the following prior for  $\theta$ :

$$p(\theta) = c \cdot \theta_0^{\alpha_0 - 1} \cdot \theta_1^{\alpha_1 - 1} \cdot \theta_2^{\alpha_2 - 1},$$

for  $\theta_j \geq 0$  and  $\sum_{j=0}^2 \theta_j = 1$ . Here  $\alpha_0, \alpha_1, \alpha_2$  are some constants, and  $c$  is a normalizing constant. Derive the posterior distribution (up to a normalizing constant) and show that this choice of prior is a conjugate prior.

2. Let  $X_1, X_2, \dots, X_N$  represent a random sample from each of the distributions having the following probability density functions. In each case find the maximum likelihood estimator  $\hat{\theta}$  of  $\theta$ .

- (a)  $f(x; \theta) = \theta^x \exp(-\theta)/x!$ , for  $x = 0, 1, \dots$  and zero elsewhere, for  $\theta \geq 0$ .
- (b)  $f(x; \theta) = \theta x^{\theta-1}$ , for  $0 < x < 1$ , and zero elsewhere, for  $0 < \theta < \infty$ .
- (c)  $f(x; \theta) = (1/\theta) \exp(-x/\theta)$ , for  $0 < x < \infty$ , and zero elsewhere, for  $0 < \theta < \infty$ .
- (d)  $f(x; \theta) = (1/2) \exp(-|x - \theta|)$ , for  $-\infty < x < \infty$ , and zero elsewhere, for  $-\infty < \theta < \infty$ .
- (e)  $f(x; \theta) = \exp(-(x - \theta))$ , for  $\theta < x < \infty$ , and zero elsewhere, for  $-\infty < \theta < \infty$ .

3. The pareto distribution is sometimes used to model incomes and has cumulative distribution function

$$F_X(x; \theta_1, \theta_2) = 1 - (\theta_1/x)^{\theta_2}$$

for  $\theta_1 < x < \infty$  with  $\theta_1, \theta_2 > 0$ . If  $X_1, X_2, \dots, X_N$  is a random sample from this distribution, find the maximum likelihood estimator for  $\theta_1$  and  $\theta_2$ .

4. The following numbers were drawn independently from a Poisson distribution:

2 2 2 7 2 2 4 2 1 1 3 2 5 4 0 3 2 4 3 2

Calculate the maximum likelihood estimate of the Poisson parameter  $\lambda$ .

5. Suppose conditional on  $p$  the random variables  $X$  and  $Y$  are independent with binomial distributions with parameters  $N$  and  $p$  and  $M$  and  $p$  respectively. The prior distribution of  $p$  is Beta with parameters  $\alpha$  and  $\beta$ .
- (a) Find the posterior distribution of  $p$  given  $X$  and  $Y$ .
  - (b) What is the posterior mean and variance of  $p$ ?
  - (c) Suppose that conditional on  $p$  the random variable  $Z$  has a binomial distribution with parameters  $N + M$  and  $p$ . With the same Beta prior distribution with parameters  $\alpha$  and  $\beta$ , what is the posterior distribution of  $p$  given  $Z$ ? How does it differ from the posterior distribution in (a)?
6. Suppose that conditional on  $\tau$  the random variable  $X$  has normal distribution with mean zero and variance  $1/\tau$ . The prior distribution for  $\tau$  is Gamma with parameters  $\alpha = 1$  and  $\beta = 1$ .
- (a) Show that the posterior distribution of  $\tau$  is also a Gamma distribution.
  - (b) Find the parameters of the posterior distribution of  $\tau$ .