Economics 520, Homework 11

Due Tuesday, December 5

1. Check whether the following distributions are members of the exponential family, and find sufficient statistics for random samples of size N if they are.
   (a) A Gamma distribution with parameters $\alpha = \theta$ and $\beta = 1$.
   (b) A Gamma distribution with parameters $\alpha = 1$ and $\beta = \theta$.
   (c) A Chi–square distribution with θ degrees of freedom.
   (d) A binomial distribution with parameters $N = 5$ and $p = \theta$.
   (e) A uniform distribution on the interval 0 to θ.

2. Let $Y_1, Y_2, \ldots, Y_N$ be a random sample from a distribution with probability density function
$$f_Y(y, \theta) = 2y/\theta^2 \quad 0 < y < \theta, \quad 0 \text{ elsewhere}$$
   (a) Show that $W = 3\bar{Y}/2$ is an unbiased estimator for θ.
   (b) Find the Cramér–Rao bound.
   (c) Show that the variance of W is less than the Cramér–Rao bound.

3. CB Exercise 7.40.

4. Let X_1, X_2, …, X_N represent a random sample from a normal distribution with mean θ^* and variance 1.
   (a) Show that $\hat{\theta}_{ml} = \bar{x} = \sum_{i=1}^N x_i/N$ is the maximum likelihood estimator and show that the variance of $\sqrt{N} \cdot (\hat{\theta}_{ml} - \theta^*)$ is equal to $\mathcal{I}(\theta^*)^{-1} = 1$.
   (b) Calculate the probability α of the event
$$\hat{\theta}_{ml} - 1.96 \cdot \sqrt{\mathcal{I}(\theta^*)^{-1}}/\sqrt{N} < \theta^* < \hat{\theta}_{ml} + 1.96 \cdot \sqrt{\mathcal{I}(\theta^*)^{-1}}/\sqrt{N}.$$
   For this reason we call the interval $(\hat{\theta}_{ml} - 1.96 \cdot \sqrt{\mathcal{I}(\theta^*)^{-1}}/\sqrt{N}, \hat{\theta}_{ml} + 1.96 \cdot \sqrt{\mathcal{I}(\theta^*)^{-1}}/\sqrt{N})$ a $100 \times \alpha\%$ confidence interval.

5. Let $X_1, X_2, \ldots, X_N$ be independent random variables, all with a binomial $\mathcal{B}(2, p)$ distribution.
   (a) Find the maximum likelihood estimator for p.
   (b) Is the maximum likelihood estimator the minimum variance unbiased estimator?
   (c) Let $N = 100$, $\sum_{i=1}^N x_i = 40$, and $\sum_{i=1}^N x_i^2 = 48$. Find the MLE and its large sample variance.