Economics 520, Homework 10

Due Tuesday, November 28

1. Let Y_1, Y_2 be a pair of independent, identically distributed random variables with common density
$$f_Y(y|\theta) = \frac{1}{\theta}\exp(-y/\theta)$$
for $y > 0$ and zero elsewhere, and for some $\theta > 0$. Consider the estimator $W_{\lambda,\delta} = \lambda Y_1 + \delta Y_2$ for fixed values of λ and δ. For which values of λ and δ is $W_{\lambda,\delta}$ unbiased? Within the set of unbiased estimators $W_{\lambda,\delta}$, which is minimum variance?

2. X_1 and X_2 are independent normally distributed random variables with mean μ and variances σ_1^2 and σ_2^2 respectively. The variances are known and we are interested in estimating the means. Consider estimators of the form $W_{\lambda,\delta} = \lambda X_1 + \delta X_2$. Find the minimum variance unbiased estimator in this class of estimators.

3. X has a binomial distribution with parameters $N = 1$ and $p = 1/2$. Y, which is independent of X, has a normal distribution with mean μ and variance 1. Consider the estimator for μ of the form $W_1 = Y + 2X - 1$.

   (a) Is W_1 unbiased?

   (b) What is the variance of W_1?

   (c) Consider the estimator $W_2 = E[W_1|Y]$. Is W_2 unbiased? How does its variance compare to that of W_1?

4. Let $Y_1, Y_2, \ldots, Y_N$ be independent random variables with identical pdf $f_Y(y, \theta)$. For each of the following densities find a minimal sufficient statistic for θ. (Note: remember that a one–dimensional sufficient statistic does not always exist).

   (a) $f_Y(y, \theta) = 1/(1 - \theta)$, $\theta < y < 1$, 0 elsewhere

   (b) $f_Y(y, \theta) = \exp(-y + \theta)$, $\theta < y < \infty$, 0 elsewhere.

5. MATLAB exercise: Suppose that $X_1, \ldots, X_5$ are IID draws from a normal distribution with mean μ and variance σ^2. Let $\hat{\sigma}^2$ be the maximum likelihood estimator of σ^2. Also, consider the alternative estimator
$$\tilde{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2,$$
where $\bar{X}_n$ is the sample average of the X_i. Suppose that the true value of μ is 1, and the true value of σ^2 is 2. Conduct a simulation exercise to evaluate the distribution of $\hat{\sigma}^2$ and $\tilde{\sigma}^2$: randomly generate 5 draws from the $N(1, 2)$ distribution, and calculate both $\hat{\sigma}^2$ and $\tilde{\sigma}^2$. Repeat this 1000 times, saving each estimate $\hat{\sigma}^2$ and $\tilde{\sigma}^2$. Calculate the approximate mean and variance of each estimator, and the approximate mean squared error. Which estimator appears to be "better"?

6. MATLAB exercise: Let (X_i, Y_i) be the age and unemployment duration for person i, for $i = 1, \ldots, N$. Assume that age has a normal distribution with mean 40 and standard deviation 10. Given age X_i, Y_i, the duration of unemployment in weeks is assumed to have an exponential distribution with mean $\exp(\theta \cdot X_i)$. Assume that the observations for different individuals are independent. The observations are (25,10), (43,29), (40,25), (30,9), (55,40), (30,24), (24,18), (24,20), (42,30), (45,43).

   (a) Plot the log likelihood function for θ between -0.4 and 0.

   (b) Show that the log likelihood function is concave.

   (c) Find the maximum likelihood estimate of θ by starting at $\theta_0 = 0$ and using the Newton–Raphson algorithm for finding a maximum of a concave function, described next. The Newton-Raphson algorithm successively modifies the value of θ in a way that increases the log likelihood function. Eventually it should converge to the maximizer of the log likelihood function.

$$\theta_{k+1} = \theta_k - \frac{\partial^2 L}{\partial \theta^2}(\theta_k)^{-1} \cdot \frac{\partial L}{\partial \theta}(\theta_k)$$

   where $L(\theta)$ is the log likelihood function. Report the sequence of values θ_k and $L(\theta_k)$ for $k = 1, 2, 3, \ldots, 20$.