

Final Review Questions

Note: Many of these questions are drawn from previous years' finals. I will not provide solutions to these questions. The final will cover material from the entire semester, but with more weight on the second half of the course.

1. Suppose that X_1 is uniformly distributed on $(0, 1)$ and X_2 is uniformly distributed on $(0, 2)$, and X_1 and X_2 are independent. Let $Y \equiv \max(X_1, X_2)$.
 - (a) Derive the CDF and PDF of Y .
 - (b) Calculate $E[Y]$.
 - (c) Suppose that instead of being independent, X_1 and X_2 are related in the following way: X_1 is uniform $(0, 1)$, and $X_2 = 2 \cdot X_1$. Let Y be defined as before. Now calculate the CDF and PDF of Y .
2. Consider the following sequence of random variables W_1, W_2, \dots with

$$W_n = \begin{cases} \frac{1}{n} & \text{with probability } .5 \\ 0 & \text{with probability } .5 \end{cases}$$

Using the definition of convergence in probability, show that $W_n \xrightarrow{p} 0$.

3. Suppose that for each n , $X_n \sim N(1/n, 1/n)$.
 - (a) Does $X_n \xrightarrow{p} 0$? Prove your answer.
 - (b) Does $X_n \xrightarrow{qm} 0$? Prove your answer.
4. Suppose $X_n \sim \text{Bin}(n, \frac{\alpha}{n})$, where $\alpha > 0$.
 - (a) $X_n \xrightarrow{d} Y$. What's the probability mass function for Y ?
 - (b) Does $\frac{X_n}{\sqrt{n}} \xrightarrow{p} 0$? Prove your answer.
5. Suppose that X is Binomial with parameters $N = 3$ and p . Suppose we have a prior distribution for p that is discrete:

$$\begin{aligned} Pr(p = 0) &= .25 \\ Pr(p = .5) &= .5 \\ Pr(p = 1) &= .25. \end{aligned}$$

- (a) What is the posterior distribution of p if we observe $X = 1$?
- (b) What is the posterior distribution of p if we observe $X = 0$?
- (c) What value of a solves

$$\min_{a \in [0, 1]} E[(p - a)^2 | X = 1].$$

6. Consider a random sample X_1, X_2, \dots, X_n from the density $f(x|\theta) = \frac{k}{\sqrt{\theta}} 5^{-x^2/\theta}$, where $-\infty < x < \infty$, $\theta > 0$, and k is a constant. (Hint: $b^a = e^{a \ln b}$)
 - (a) Calculate the value of k .

- (b) Give a one-dimensional sufficient statistic for θ .
 - (c) Derive the maximum likelihood estimator for θ .
 - (d) Calculate the Cramer-Rao Lower Bound.
 - (e) Is the MLE a minimum variance unbiased estimator?
7. The conditional distribution of Y given $X = x$ is Poisson with mean $\theta^* \cdot x$. The distribution of X is uniform on the interval $[1, 2]$. You have a random sample of size N from this distribution for (X_i, Y_i) . Give the asymptotic distribution of the maximum likelihood estimator and show how you would calculate the variance of this distribution.
8. Let Y_1, Y_2, \dots, Y_N be independent random variables with common cdf

$$F_Y(y) = 1 - \exp(-\lambda e^y),$$

for $-\infty < y < \infty$, for some parameter $\lambda > 0$.

- (a) Find a one-dimensional sufficient statistic.
 - (b) Find the maximum likelihood estimator
 - (c) How would you estimate the variance of the maximum likelihood estimator?
9. Consider the following probability mass function:

$$f(x|p) = \begin{cases} p & \text{if } x = 0 \\ p^2 & \text{if } x = 1 \\ 1 - p - p^2 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

Suppose X_1, \dots, X_n are IID with PMF $f(x|p)$.

- (a) Suppose $W_1 = (2X_1 - 1)^2$. What is the PMF for W_1 ?

Now suppose $\{3, 0, 0, 3, 0, 3\}$ is a random sample of size $n = 6$ from the PMF $f(x|p)$.

- (b) Calculate the MLE of p given this sample of data.
 - (c) What is the asymptotic distribution of $\sqrt{n}(\hat{p}_{ML} - p)$?
10. Let X have a binomial $B(1, 1/2)$ distribution with $Pr(X = 1) = 1/2$. Conditional on $X = x$, the random variable Y has a Poisson distribution with parameter $\lambda(1 + x)$:

$$Pr(Y = y|X = x) = \frac{(\lambda(1 + x))^y \exp(-\lambda(1 + x))}{y!},$$

for $x = 0, 1$, and $y = 0, 1, \dots$

- (a) Calculate $Pr(Y = 2)$.
- (b) Calculate $Pr(X = 1|Y = 2)$.
- (c) Find an unbiased estimator for λ that is a function of Y alone.

11. Let X be a random variable with probability density function

$$f_X(x; \mu) = \frac{1}{\mu} \exp(-x/\mu),$$

for $x > 0$ and zero elsewhere.

- (a) Calculate the mean and variance of X .
 - (b) Calculate the mean and variance of X conditional on $X < 8$.
 - (c) Let x_1, x_2, \dots, x_N be a random sample from this distribution, with $N = 20$, $\sum x = 95$, and $\sum x^2 = 590$. Calculate the maximum likelihood estimate.
 - (d) Estimate the probability that $X < 8$ and its asymptotic variance.
12. Let the marginal distribution of X be binomial with $N = 1$ and $p = 1/4$. Conditional on X , the random variable Y has a normal distribution with mean $\mu \cdot (X + 1)$ and variance 1.
- (a) Find the marginal density of Y .
 - (b) Suppose you have a random sample of size N from this joint distribution. What is the maximum likelihood estimator and its large sample variance?
 - (c) Suppose you only observe y_1, \dots, y_N . Find an unbiased estimator for μ . What is its large sample variance and how does that compare to that of the maximum likelihood estimator derived before?

13. Let the distribution of X have probability density function

$$f_X(x; \lambda) = \frac{1}{2} \lambda \exp(-|x|\lambda),$$

for $-\infty < x < \infty$. Let x_1, \dots, x_N be a random sample from this distribution.

- (a) Find a one-dimensional sufficient statistic for λ .
 - (b) Let $N = 20$, $\sum_{i=1}^N x_i = 20$, $\sum_{i=1}^N |x_i| = 100$, $\sum_{i=1}^N x_i^2 = 200$. Find the maximum likelihood estimator.
14. Suppose there is a random sample of size 10 (X_1, \dots, X_{10}) from a Poisson distribution (so $f_{X_i}(x_i|\theta) = \frac{\theta^{x_i} e^{-\theta}}{x_i!}$, when x_i is a nonnegative integer). The sample mean of the random sample is 4.
- (a) Using the sample, what is the maximum likelihood estimate for θ ?
 - (b) Now suppose that the prior distribution for θ is $Gamma(3, 5)$:

$$\theta \sim Gamma(3, 5).$$

What is the posterior distribution for θ given X_1, \dots, X_{10} ? (Hint: $Z \sim Gamma(\alpha, \beta)$ means $f_Z(z|\alpha, \beta) = \frac{z^{\alpha-1} e^{-z/\beta}}{\Gamma(\alpha)\beta^\alpha}$)