Midterm Review Questions

Note: I will not provide solutions to these questions. Some of these questions are drawn from previous years' exams.

1. Events M and N are said to be mutually exclusive provided that $M \cap N = \emptyset$.

   Suppose that $P(A) > 0$ and $P(B) > 0$. Show that if A and B are independent, then they cannot be mutually exclusive.

2. Consider two events A and B such that $P(A) \neq 0$, and $P(B) \neq 0$. Show that

$$P(A|B) > P(A) \iff P(B|A) > P(B).$$

   (Here, $\iff$ means "if and only if".)

3. Suppose that the continuous random variable X has PDF

$$f(x) = \frac{4}{3}(1 - x^3), \qquad 0 < x < 1,$$

   and 0 otherwise. Determine the values of the following probabilities:

   (a) $Pr(X < 1/2)$.
   (b) $Pr(1/4 < X < 3/4)$.
   (c) $Pr(X > 1/3)$.

4. Suppose that Y has PDF $f(y) = 3y^2$ for $0 < y < 1$ and 0 otherwise. Find the median of Y.

5. Consider the following pdf:

$$f_X(x) = \frac{1}{6\theta^4} x^3 \exp\left\{-\frac{x}{\theta}\right\}, \qquad 0 < x < \infty, \qquad 0 < \theta < \infty$$

   a) Obtain the moment generating function of this pdf.

   b) Use the moment generating function you obtained in (a) to find $E(X)$ and $V(X)$.

   c) Check if this pdf is a member of the exponential family of densities.

6. Suppose that a random variable has mean 12 and variance 2. Using the Chebyshev inequality, provide a bound on the probability that the variable is between 10 and 14.

7. Consider the following experiment. A fair coin is tossed. If a head appears one point is recorded. If a tail appears two points are recorded. The coin is tossed repeatedly (independently) each time recording the points. The experiment stops as soon as the total number of points (from all tosses) is greater than or equal to three. Let X be the random variable denoting the number of times the coin is tossed.

   (a) What is the PMF for the random variable X? What is the CDF of X?
   (b) What is $Pr(X \geq 2)$? What is $Pr(X \geq 3)$?
   (c) What is $E(X)$?

(d) What is the moment generating function $M_X(t)$ for the random variable X? Check that $M'_X(0)$ is equal to your answer in (c).

8. Suppose that Y has a gamma distribution with parameters α and β, and suppose that the conditional distribution of X given $Y = y$ is exponential with parameter y (i.e. the conditional mean of X given $Y = y$ is $1/y$). What is the conditional distribution of Y given $X = x$?

9. Suppose that X and Y have a continuous joint distribution with joint PDF

$$f(x, y) = x + y \qquad \text{for } 0 \le x \le 1, 0 \le y \le 1,$$

and 0 otherwise. Find $E[Y|X]$.

10. Let X_1, X_2 and X_3 have joint pdf:

$$\begin{aligned} f_{X_1X_2X_3}(x_1, x_2, x_3) &= \exp\{-(x_1 + x_2 + x_3)\} \\ & \quad 0 < x_1 < \infty, 0 < x_2 < \infty, 0 < x_3 < \infty \end{aligned}$$

Are X_1, X_2 and X_3 independent? Show carefully how you reached your conclusion.

11. A miner is trapped in a mine containing two doors. The first door leads to a tunnel that will take him to safety after 2 hours. The second door leads to a tunnel that will return him to the mine after 1 hour of travel. Assume that the miner is equally likely to choose either door each time. Find the expected length of time until he reaches safety.

(Hint: Use the law of iterated expectations.)

12. Suppose that Y has PDF $f_Y(y) = \frac{192}{y^4}\mathbf{1}(y \ge 4)$, and suppose that the conditional distribution of X given $Y = y$ is Uniform on $[0, y]$. Find the conditional PDF of Y given $X = 5$.