Economics 520, Homework 9

Due Tuesday, November 15 at beginning of class

1. Suppose that a random sample $X_1, \ldots, X_n$ is taken from a $N(\mu, 1)$ distribution with unknown mean μ. However, we only observe Y_i, an indicator equal to 1 if X_i is less than 0. Derive the likelihood and log-likelihood function based on $(Y_1, \ldots, Y_n)$, and solve for the maximum likelihood estimator of μ.

2. Suppose that conditional on τ the random variable X has a normal distribution with mean zero and variance $1/\tau$. The prior distribution for τ is Gamma with parameters α and β.

   (a) Find the parameters of the posterior distribution of τ.

   (b) What is the prior mean of τ?

   (c) What is the posterior mean of τ?

3. Show that the priors in the following cases are conjugate priors:

   (a) $X_1, \ldots, X_n$ is a random sample from the Binomial(p, k) distribution with probability p and size k. Assume that k is known. The prior for p is a Beta distribution with parameters α and β.

   (b) $X_1, \ldots, X_n$ is a random sample from the uniform distribution on $(0, \theta)$. The prior for θ is
   $$f(\theta) = ba^b \theta^{-(b+1)} \cdot 1(\theta > a).$$

   (c) $X_1, \ldots, X_n$ is a random sample from the exponential distribution with density $f(x; \lambda) = \lambda \exp(-\lambda x)$ for $x > 0$. The prior for λ is a Gamma distribution with parameters α and γ.

4. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. Suppose the prior for (μ, σ^2) has density
   $$f(\mu, \sigma^2) = \frac{1}{\sigma^2}, \quad -\infty < \mu < \infty, 0 < \sigma^2 < \infty.$$
   Note that this is an improper prior density.

   Show that the posterior density of (μ, σ^2) is equal to
   $$f(\mu, \sigma^2 | x_1, \ldots, x_n) = f(\mu | \sigma^2, x_1, \ldots, x_n) \cdot f(\sigma^2 | x_1, \ldots, x_n),$$
   where $f(\mu | \sigma^2, x_1, \ldots, x_n)$ is a normal density with mean $\bar{x}$ and variance σ^2/n, and $f(\sigma^2 | x_1, \ldots, x_n)$ is a gamma density with parameters
   $$\frac{n-1}{2}, \quad \text{and} \quad \left[\sum_{i=1}^{n} (x_i - \bar{x})^2 / 2 \right]^{-1}.$$

5. Let Y_1, Y_2 be a pair of independent, identically distributed random variables with common density

$$f_Y(y|\theta) = \frac{1}{\theta}\exp(-y/\theta)$$

for $y > 0$ and zero elsewhere, and for some $\theta > 0$. Consider the estimator $W_{\lambda,\delta} = \lambda Y_1 + \delta Y_2$ for fixed values of λ and δ. For which values of λ and δ is $W_{\lambda,\delta}$ unbiased? Within the set of unbiased estimators $W_{\lambda,\delta}$, which is minimum variance?

6. X_1 and X_2 are independent normally distributed random variables with mean μ and variances σ_1^2 and σ_2^2 respectively. The variances are known and we are interested in estimating the means. Consider estimators of the form $W_{\lambda,\delta} = \lambda X_1 + \delta X_2$. Find the minimum variance unbiased estimator in this class of estimators.