Economics 520, Homework 8

Due Tuesday, November 8 at beginning of class

1. X has a binomial distribution with parameters $N = 1$ and $p = 1/2$. Y, which is independent of X, has a normal distribution with mean μ and variance 1. Consider the estimator for μ of the form $W_1 = Y + 2X - 1$.
   (a) Is W_1 unbiased?
   (b) What is the variance of W_1?
   (c) Consider the estimator $W_2 = E[W_1|Y]$. Is W_2 unbiased? How does its variance compare to that of W_1?

2. CB 7.9

3. CB 7.11

4. Suppose that $Y_1, \ldots, Y_T$ are generated in the following way: Let $Y_0 = 0$, and let the distribution of Y_t given $Y_0 = y_0, \ldots, Y_{t-1} = y_{t-1}$ be $N(\beta y_{t-1}, 1)$. This is a type of AR(1) model, since it can also be written
$$Y_t = \beta Y_{t-1} + \epsilon_t,$$
where ϵ_t are IID $N(0, 1)$.
   (a) Write down the likelihood function for β given $Y_1, \ldots, Y_T$, simplifying it as much as possible. (Note: we only have one observation on each Y_t. Despite the dependence across observations, the likelihood can be written in a product form.)
   (b) Obtain an explicit formula for the maximum likelihood estimator $\hat{\beta}$.

5. MATLAB exercise: Suppose that, for $i = 1, \ldots, 10$, $X_i = i$ (it is not random), and that Y_i are independent binary random variables with
$$Pr(Y_i = 1) = \frac{\exp(x_i \cdot \beta)}{1 + \exp(x_i \cdot \beta)}.$$
We observe the following values:

i	X	Y
1	1	0
2	2	0
3	3	1
4	4	0
5	5	1
6	6	1
7	7	1
8	8	1
9	9	1
10	10	0

(a) Write down the likelihood and log likelihood function (as a general function of $y_1, \ldots, y_{10}$).

(b) Plot the log likelihood function for β between 0 and 0.5.

(c) Find the maximum likelihood estimate of θ by starting at $\beta_0 = 0$ and using the Newton–Raphson algorithm for finding a maximum of a function

$$\beta_{k+1} = \beta_k - \frac{\partial^2 L}{\partial \beta^2}(\beta_k)^{-1} \cdot \frac{\partial L}{\partial \beta}(\beta_k)$$

where $L(\beta)$ is the log likelihood function. Report the sequence of values β_k and $L(\beta_k)$ for $k = 1, 2, 3, \ldots, 20$.