Final Review Questions

Note: Many of these questions are drawn from previous years' finals. I will not provide solutions to these questions. The final will cover material from the entire semester, but with more weight on the second half of the course.

1. Consider the following sequence of random variables $W_1, W_2, \ldots$ with

$$W_n = \begin{cases} \frac{1}{n} & \text{with probability } .5 \\ 0 & \text{with probability } .5 \end{cases}$$

   Using the definition of convergence in probability, show that $W_n \xrightarrow{p} 0$.

2. Suppose that for each n, $X_n \sim N(1/n, 1/n)$.
   (a) Does $X_n \xrightarrow{p} 0$? Prove your answer.
   (b) Does $X_n \xrightarrow{qm} 0$? Prove your answer.

3. Suppose $X_n \sim Bin(n, \frac{\alpha}{n})$, where $\alpha > 0$.
   (a) $X_n \xrightarrow{d} Y$. What's the probability mass function for Y?
   (b) Does $\frac{X_n}{\sqrt{n}} \xrightarrow{p} 0$? Prove your answer.

4. Suppose that X is Binomial with parameters $N = 3$ and p. Suppose we have a prior distribution for p that is discrete:

$$\begin{aligned} Pr(p = 0) &= .25 \\ Pr(p = .5) &= .5 \\ Pr(p = 1) &= .25. \end{aligned}$$

   (a) What is the posterior distribution of p if we observe $X = 1$?
   (b) What is the posterior distribution of p if we observe $X = 0$?
   (c) What value of a solves

$$\min_{a \in [0,1]} E[(p - a)^2 | X = 1].$$

5. Consider a random sample $X_1, X_2, \ldots, X_n$ from the density $f(x|\theta) = \frac{k}{\sqrt{\theta}} 5^{-x^2/\theta}$, where $-\infty < x < \infty$, $\theta > 0$, and k is a constant. (Hint: $b^a = e^{a \ln b}$)
   (a) Calculate the value of k.
   (b) Give a one-dimensional sufficient statistic for θ.
   (c) Derive the maximum likelihood estimator for θ.
   (d) Calculate the Cramer-Rao Lower Bound.
   (e) Is the MLE a minimum variance unbiased estimator?

6. The conditional distribution of Y given $X = x$ is Poisson with mean $\theta^* \cdot x$. The distribution of X is uniform on the interval $[1, 2]$. You have a random sample of size N from this distribution for (X_i, Y_i). Give the asymptotic distribution of the maximum likelihood estimator and show how you would calculate the variance of this distribution.

7. Let $Y_1, Y_2, \ldots, Y_N$ be independent random variables with common cdf

$$F_Y(y) = 1 - \exp(-\lambda e^y),$$

for $-\infty < y < \infty$, for some parameter $\lambda > 0$.

(a) Find a one-dimensional sufficient statistic.

(b) Find the maximum likelihood estimator

(c) How would you estimate the variance of the maximum likelihood estimator?

8. Consider the following probability mass function:

$$f(x|\theta) = \begin{cases} .4 - \theta & x = 1 \\ .3 & x = 2 \\ .1 + \theta & x = 3 \\ .2 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $-.1 \leq \theta \leq .4$.

Suppose X is a random variable with this pmf.

(a) What's $E(X)$? What's $V(X)$?

(b) Suppose we have a random sample of size 1 from $f(x|\theta)$. Give the critical region for a uniformly most powerful test (with level .1) of $H_0 : \theta = 0$ vs. $H_1 : \theta = .2$.

(c) Would the critical region be the same for a uniformly most powerful test (with level .1) of $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$? Why?

9. Consider the following probability mass function:

$$f(x|p) = \begin{cases} p & \text{if } x = 0 \\ p^2 & \text{if } x = 1 \\ 1 - p - p^2 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X_1, \ldots, X_n$ are IID with PMF $f(x|p)$.

(a) Suppose $W_1 = (2X_1 - 1)^2$. What is the PMF for W_1?

Now suppose $\{3, 0, 0, 3, 0, 3\}$ is a random sample of size $n = 6$ from the PMF $f(x|p)$.

(b) Calculate the MLE of p given this sample of data.

(c) What is the asymptotic distribution of $\sqrt{n}(\hat{p}_{ML} - p)$? (Use $\hat{p}_{ML}$ to provide an estimate of the asymptotic variance.)

(d) Perform a Wald test (with size .05) of $H_0 : p = .2$ vs $H_1 : p \neq .2$, using your estimate of the variance matrix from the previous part.

(e) Suppose $n = 1$. What is the critical region for the uniformly most powerful test with size .24 of $H_0 : p = .2$ vs $H_1 : p = .4$?

10. Let X have a binomial $B(1, 1/2)$ distribution with $Pr(X = 1) = 1/2$. Conditional on $X = x$, the random variable Y has a Poisson distribution with parameter $\lambda(1 + x)$:

$$Pr(Y = y|X = x) = \frac{(\lambda(1 + x))^y \exp(-\lambda(1 + x))}{y!},$$

for $x = 0, 1$, and $y = 0, 1, \ldots$.

(a) Calculate $Pr(Y = 2)$.

(b) Calculate $Pr(X = 1|Y = 2)$.

(c) Find an unbiased estimator for λ that is a function of Y alone.

11. Let X be a random variable with probability density function

$$f_X(x; \mu) = \frac{1}{\mu} \exp(-x/\mu),$$

for $x > 0$ and zero elsewhere.

(a) Calculate the mean and variance of X.

(b) Calculate the mean and variance of X conditional on $X < 8$.

(c) Let $x_1, x_2, \ldots, x_N$ be a random sample from this distribution, with $N = 20$, $\sum x = 95$, and $\sum x^2 = 590$. Calculate the maximum likelihood estimate.

(d) Estimate the probability that $X < 8$ and its asymptotic variance.

(e) Test the hypothesis that $\mu = 4$ at the 10% level using a likelihood ratio test.

(f) Test the same hypothesis using a Lagrange multiplier (score) test.

12. Let the marginal distribution of X be binomial with $N = 1$ and $p = 1/4$. Conditional on X, the random variable Y has a normal distribution with mean $\mu \cdot (X + 1)$ and variance 1.

(a) Find the marginal density of Y.

(b) Suppose you have a random sample of size N from this joint distribution. What is the maximum likelihood estimator and its large sample variance?

(c) Suppose you only observe $y_1, \ldots, y_N$. Find an unbiased estimator for μ. What is its large sample variance and how does that compare to that of the maximum likelihood estimator derived before?

(d) Suppose that $N = 100$, $\sum_{i=1}^{N} x_i = 25$, $\sum_{i=1}^{N} x_i y_i = 60$, and $\sum_{i=1}^{N}(1 - x_i)y_i = 80$. Test the null hypothesis that $\mu = 1$ against the alternative that $\mu \neq 1$ at the 5% level using a likelihood ratio test.

(e) Repeat the test using a Wald test.

13. Let the distribution of X have probability density function

$$f_X(x; \lambda) = \frac{1}{2} \lambda \exp(-|x|\lambda),$$

for $-\infty < x < \infty$. Let $x_1, \ldots, x_N$ be a random sample from this distribution.

(a) Find a one-dimensional sufficient statistic for λ.

(b) Let $N = 20$, $\sum_{i=1}^N x_i = 20$, $\sum_{i=1}^N |x_i| = 100$, $\sum_{i=1}^N x_i^2 = 200$. Find the maximum likelihood estimator.

(c) Test the null hypothesis

$$H_0 : \qquad \lambda = 0.125,$$

against the alternative hypothesis

$$H_1 : \qquad \lambda \neq 0.125,$$

at the 5% level using a likelihood ratio test.

(d) Test the same null hypothesis using a Wald test.

(e) Provide an approximate 95% confidence interval for λ based on the MLE.

14. Suppose there is a random sample of size 10 ($X_1, \ldots, X_{10}$) from a Poisson distribution (so $f_{X_i}(x_i|\theta) = \frac{\theta^{x_i} e^{-\theta}}{x_i!}$, when x_i is a nonnegative integer). The sample mean of the random sample is 4.

(a) Using the sample, what is the maximum likelihood estimate for θ?

(b) Provide an approximate 95% confidence interval for θ.

(c) Using a large-sample LR test, test the hypothesis that $\theta = 3$ at the 0.05 level. (Note that if Z is a chi-squared random variable with 1 degree of freedom, $Pr(Z > 2.718) = 0.10$ and $Pr(Z > 3.84) = 0.05$.)

(d) Now suppose that the prior distribution for θ is $Gamma(3, 5)$:

$$\theta \sim Gamma(3, 5).$$

What is the posterior distribution for θ given $X_1, \ldots, X_{10}$? (Hint: $Z \sim$Gamma(α, β) means $f_Z(z|\alpha, \beta) = \frac{z^{\alpha-1} e^{-z/\beta}}{\Gamma(\alpha)\beta^\alpha}$)