Notes for
"Mergers and the Evolution of Industry Concentration: Results from the Dominant Firm Model"

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by

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Contents

1. Proof of Strict Inequalities in Lemma 3

Lemma 3. Consider the finite-horizon version of the model $t \in \{1, 2, ...T\}$, $T < \infty$. (i) For all t, $v_{f,t}(m,K) > v_{d,t}(m,K)$, if $m > 0$, and $v_{f,t}(m,K) = v_{d,t}(m,K)$ if $m = 0$. (ii) For all t, $v_{d,t}(1,K) > mv_{d,t}(m,K) + (1-m)v_{f,t}(m,K)$, if $m < 1$.

Proof of (i)

We show the more general claim that for all t,

$$v_{f,t}(m,K) > v_{d,t}(m,K), \text{ if } m > 0, \tag{a}$$

$$v_{f,t}(0,K) = v_{d,t}(0,K), \tag{b}$$

$$v_{f,t}(0,K) < v^{\circ}_{d,t}(m^{\circ},K), \text{ if } m^{\circ} > 0. \tag{c}$$

The results from the single-period case imply that the strict inequality (a) and the equality (b) hold in the final period $t = T$. To see that (c) also holds for $t = T$, observe that when $m^{\circ}_T > 0$, the dominant firm could always choose to set $m_T = 0$ and receive $v_{f,T}(0,K)$ but from Proposition 2 strictly prefers not to completely divest. We will show that if the conditions (a), (b), (c) hold for $t+1$ then they also hold for t, so by induction the proof will be complete.

Before proceeding further, it is useful to introduce some additional notation. Define $V_t(q_f, e_f \,|p, p_k)$ to be the value to a fringe firm at time period t in the output/investment stage given an output sequence $q_f = (q_{f,t}, q_{f,t+1}, ..., q_{f,T})$ and a per-unit capital acquisition sequence $e_f = (e_{f,t+1}, ..., e_{f,T})$, conditional on p and p_k. Let (m', K') be a particular state in the investment output stage in period t and $v'_{d,t}$ and $v'_{f,t}$ be the equilibrium values at that state. Then

$$\begin{aligned} v'_{f,t} &= V_t\left(q'_f, e'_f \,|p', p'_k\right) \\ &= \max_{(q_f, e_f)} V_t\left(q_f, e_f \,|p', p'_k\right) \end{aligned}$$

where $\left(q'_f, e'_f\right)$ is the equilibrium fringe sequence given this state and the equilibrium price sequences (p', p'_k) and $\left(q'_f, e'_f\right)$ maximizes $V_t\left(q'_f, e'_f\,|p', p'_k\right)$. It is straightforward to check that there is a unique quantity $q'_{f,t}$ in the initial period t in any solution to the above problem. This follows from the strict concavity of the cost function $c(q)$.[1]

We first prove that (a) holds for t. We know from the analysis in the text that the weak inequality version of (a) holds, $v_{f,t}(m,K) \geq v_{d,t}(m,K)$. Suppose for some $m' > 0$ and K' there is equality, $v'_{d,t} = v'_{f,t}$. Now $v'_{d,t} = V_t\left(q'_d, e'_d\,|p', p'_k\right)$. Since $v'_{d,t} = v'_{f,t}$, (q'_d, e'_d) solve the above problem. Since $q'_{f,t}$ is the unique period t output in any solution, $q'_{d,t} = q'_{f,t}$, so the current return in period t is the same for the dominant firm and the fringe. Since $v'_{d,t} = v'_{f,t}$ and since the current return is the same, it must be that $v'_{d,t+1} = v'_{f,t+1}$. Since $q'_{d,t} = q'_{f,t}$ and $m'_t > 0$, it follows that $m^{\circ\prime}_{t+1} = m'_t > 0$. Since (c) holds for $t+1$ and since $m^{\circ\prime}_{t+1} > 0$, we know that $m'_{t+1} > 0$, i.e., that the dominant firm will not completely divest. But $m'_{t+1} > 0$ and $v'_{d,t+1} = v'_{f,t+1}$ contradict that (a) holds for $t+1$. Thus (a) holds for t.

It is immediate that (b) holds for t since the fact that (a) and (b) hold in future periods implies the dominant firm's market share remains zero in all future periods.

To show that (c) holds for t, we consider the first-order necessary condition of the dominant firm's merger decision problem

$$\frac{1}{K_t}\frac{\partial w^{\circ}_{d,t}}{\partial m} = (v_{d,t} - v_{f,t}) + m\left(\frac{\partial v_{d,t}}{\partial m} - \frac{\partial v_{f,t}}{\partial m}\right) + m^{\circ}_t\frac{\partial v_{f,t}}{\partial m},$$

where for simplicity we have divided through by K_t. In Lemma A1 in below, we prove the intuitive result that in the limit where $m_t = 0$, $\frac{\partial v_{d,t}}{\partial m} = \frac{\partial v_{f,t}}{\partial m} = 0$ and $\frac{\partial^2 v_{f,t}}{\partial m^2} > 0$. This implies that when $m^{\circ}_t > 0$, the above is strictly postive for small enough m. Thus it is not optimal to completely divest at t, so (c) holds.

Proof of (ii)

To prove (ii), we need to show that that the unique maximum of industry profit is obtained at $m = 1$. From the analysis of the single-period model, the result is immediately

[1] The quantity solving this problem in subsequent periods is not necessarily unique because if the fringe firm chooses a complete sell-off at a later period $j > t$ so that $e_{f,j} = -1$ (which it is indifferent to doing) then subsequent choices of q are irrelevant

true for $t = T$. So consider some period $t < T$ and consider some $m'_t \in [0, 1)$. Suppose the discounted industry profit beginning at (m'_t, K_t) at this state is equal to the discounted industry profit at pure monopoly $(1, K_t)$. The pure monopoly problem is concave so there is a unique solution. Since industry profit starting at (m'_t, K_t) is the same as industry profit starting at $(1, K_t)$, it must be that at (m'_t, K_t) the fringe and dominant firm output at date t are identical to the monopoly output (per unit of capital) which implies that the future sequence of prices is equal to the monopoly sequence of prices. However, it is straightforward to show that the fringe output in a period given pure monopoly prices in every remaining period is strictly higher than the pure monopoly output in the period, which yields a contradiction.

■

2. Proof of Proposition 5(ii)

The proof involves calculating derivatives and evaluating them at the limit of perfect competition. The proof involves many calculations. We divide the proof into five parts. The first four parts prove Lemmas A1, A2, A3, and A4 which are intermediate steps. The fifth part proves the main result.

2.1 Lemma A1

Lemma A1. Consider the finite-horizon version with horizon T. For all $t \in \{1, 2, ...T\}$,

$$q_{d,t}(0, K) = q_{f,t}(0, K). \tag{i}$$

$$\frac{\partial q_{d,t}(0, K_t)}{\partial m_t} = \frac{K_t q_t \left(P_t' + \beta\sigma^2 \frac{\partial v_{f,2}}{\partial K}\right)}{c_t'' - P_t' K_t - \beta\sigma^2 \frac{\partial v_2}{\partial K} K_t} < 0 \tag{ii}$$

$$\frac{\partial q_{f,t}(0, K)}{\partial m} = 0. \tag{iii}$$

$$\frac{\partial^2 q_{f,t}(0, K_t)}{\partial m_t^2} = \frac{\left(P_t' + \beta\sigma^2 \frac{\partial v_{f,t+1}}{\partial K_{t+1}}\right) 2K_t \frac{\partial q_{d,t}}{\partial m_t} + \beta\sigma \frac{\partial^2 v_{f,t+1}}{\partial m_{t+1}^2} \left(\frac{\partial \tilde{m}_{t+1}}{\partial m_{t+1}^\circ}\right)^2}{c_t'' - \left(P' + \beta\sigma^2 \frac{\partial v_{f,2}}{\partial K_2}\right) K_t} > 0 \tag{iv}$$

$$\frac{\partial v_{d,t}(0, K)}{\partial m} = \frac{\partial v_{f,t}(0, K)}{\partial m} = 0 \tag{v}$$

$$\infty > \frac{\partial v_{f,t}^2(0, K)}{\partial m^2} > 0 \tag{vi}$$

$$\frac{\partial v_{f,t}^2(0, K)}{\partial m^2} > \frac{\partial v_{d,t}^2(0, K)}{\partial m^2}$$

$$\tilde{m}_t(0, K) = 0 \tag{vii}$$

$$\frac{\partial \tilde{m}_t(0,K)}{\partial m_t} = \frac{2\frac{\partial v_{f,t}^2}{\partial m_t^2}}{3\left[\frac{\partial v_{f,t}^2}{\partial m_t^2} - \frac{\partial v_{d,t}^2}{\partial m_t^2}\right]} \tag{viii}$$

$$m_{next,t}^{\circ}(0,K) = 0 \tag{ix}$$

$$\frac{\partial m_{next,t}^{\circ}(0,K)}{\partial m} = 1 \tag{x}$$

Proof.

Suppose the result is true for $t+1$. We will show the result holds for t. Defining $v_{f,T+1}(m,K) = 0$ and $v_{d,T+1}(m,K) = 0$, the same proof shows that the result holds for $t = T$. Thus by induction we get our result.

For notational simplicity, it is useful to let period t be period 1 and period $t+1$ be period 2.

1. The FONC from the Output/Investment Stage

We begin by deriving the FONC of the dominant firm and the fringe at the output/investment stage

Suppose that $\tilde{q}_{f,1}(q_{d,1})$ is the equilibrium fringe output and $\tilde{p}_1(q_{d,1})$ is the equilibrium price as a function of the dominant firm's choice of $q_{d,1}$(the dependence on m_1 and K_1 is implicit). The dominant firm's problem is then

$$\max_{q_{d,1}} \tilde{p}_1(q_{d,1})q_{d,1}m_1K_1 - m_1K_1c(q_{d,1}) + \beta w_2^{\circ}(m_2^{\circ}, K_2)$$

where

$$\begin{aligned} m_2^{\circ} &= \frac{m_1 q_{d,1}}{m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}} \\ K_2 &= \left(m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}\right)\sigma K_1 \end{aligned}$$

The dominant firm's FONC is

$$\frac{d\tilde{p}_1}{dq_d} q_{d,1} m_1 K_1 + \left(p_1 - c_{d,1}'\right) m_1 K_1 + \beta \frac{\partial w_2^{\circ}}{\partial m^{\circ}} \frac{dm_2^{\circ}}{dq_{d,1}} + \beta \frac{\partial w_2^{\circ}}{\partial K} \frac{dK_2}{dq_{d,1}} = 0 \tag{1}$$

Now

$$\begin{aligned} w_2^\circ(m^\circ, K) &= \max_m mKv_{d,2}(m,K) - (m - m^\circ)Kv_{f,2}(m,K) \\ \frac{\partial w_2^\circ}{\partial m^\circ} &= Kv_{f,2} \\ \frac{\partial w_2^\circ}{\partial K} &= mv_{d,2} - (m-m^\circ)v_{f,2} + mK\frac{\partial v_{d,2}}{\partial K} - (m-m^\circ)K\frac{\partial v_{f,2}}{\partial K} \\ &= m\left(v_{d,2} + K\frac{\partial v_{d,2}}{\partial K}\right) - (m-m^\circ)\left(v_{f,2} + K\frac{\partial v_{f,2}}{\partial K}\right) \end{aligned}$$

And

$$\begin{aligned} \frac{dm_2^\circ}{dq_{d,1}} &= \frac{m_1\left[m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}\right] - \left(m_1 + (1-m_1)\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right) m_1 q_{d,1}}{\left[m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}\right]^2} \\ \frac{dK_2}{dq_{d,1}} &= \left(m_1 + (1-m_1)\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right)\sigma K_1 \end{aligned}$$

Dividing the FONC (1) by $m_1 K_1$ yields

$$\begin{aligned} H &= \left[\frac{d\tilde{p}_1}{dq_{d,1}} q_{d,1} + \left(p_1 - c'_{d,1}\right)\right] + \beta v_{f,2}\frac{1}{m_1}\frac{K_2}{K_1}\frac{dm_2^\circ}{dq_{d,1}} + \beta\frac{1}{m_1 K_1}\frac{\partial w_2^\circ}{\partial K}\frac{dK_2}{dq_{d,1}} \qquad (2) \\ &= \left[\frac{d\tilde{p}_1}{dq_{d,1}} q_{d,1} + \left(p_1 - c'_{d,1}\right)\right] + \beta v_{f,2}\frac{K_2}{K_1}\frac{\left[m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}\right] - \left(m_1 + (1-m_1)\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right) q_{d,1}}{\left[m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}\right]^2} \\ &\quad +\beta\sigma\left[\frac{m_2}{m_1}\left(v_{d,2} + K_2\frac{\partial v_{d,2}}{\partial K_2}\right) - \frac{(m_2 - m_2^\circ)}{m_1}\left(v_{f,2} + K_2\frac{\partial v_{f,2}}{\partial K_2}\right)\right]\left(m_1 + (1-m_1)\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right) \end{aligned}$$

The fringe FONC is

$$P((m_1 K_1 q_{d,1} + (1-m_1)K_1\tilde{q}_{f,1}) + \beta\sigma v_{f,2}(\tilde{m}_2(m_2^\circ, K_2), K_2) = c'_{f,1} \qquad (3)$$

This sets discounted marginal revenue $p_1 + \beta\sigma v_{f,2}$ equal to marginal cost.

2. Output levels at the Limit

Here we begin our look at the limit $m_1 = 0$. It is immediate from (3) that at $m_1 = 0$, the dominant firm's choice of $q_{d,1}$ has no effect on the fringe output $q_{f,1}$ or the market price.

$$\begin{aligned} \lim_{m_1 \to 0} \frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}} &= 0 \qquad (4) \\ \lim_{m_1 \to 0} \frac{d\tilde{p}_1}{dq_{d,1}} &= 0 \end{aligned}$$

Taking the limit of (2) and substituting in (4), the third term of (2) drops out, yielding

$$\begin{aligned}\lim_{m_1\to 0} H &= p_1 - c'_{d,1} + \beta v_{f,2}\frac{K_2}{K_1}\frac{q_{f,1}}{q_{f,1}^2} \\ &= p_1 - c'_{d,1} + \beta v_{f,2}\frac{\sigma q_{f,1}K_1}{K_1}\frac{q_{f,1}}{q_{f,1}^2} \\ &= p_1 - c'_{d,1} + \beta\sigma v_{f,2}\end{aligned}$$

Thus for $m_1 = 0$, the dominant firm's FONC reduces to the FONC of the fringe, so that $q_{d,1}(0,K) = q_{f,1}(0,K)$ (proving (i)) and $v_{d,t}(0,K) = v_{f,t}(0,K)$, which proves state (i) of the lemma. Statement (ix) that $m^{\circ}_{next,t}(0,K) = 0$ is immediate. To simplify notation let $q_1 = q_{d,1}(0,K) = q_{f,1}(0,K)$ in what follows and let $c_1 = c'(q_1)$.

3. The Limiting Behavior of the Fringe Reaction Function

In order to look at the slopes in the limit we need to calculate the limiting behavior of the fringe reaction function. Differentiating the fringe FONC (3) with respect to $q_{d,1}$ yields

$$\begin{aligned}& P'\left(m_1K_1 + (1-m_1)K_1\frac{\partial\tilde{q}_{f,1}}{\partial q_{d,1}}\right) + \beta\sigma\frac{\partial v_{f,2}}{\partial m_2}\frac{\partial m_2}{\partial m_2^{\circ}}\left[\frac{\partial m_2^{\circ}}{\partial q_{d,1}} + \frac{\partial m_2^{\circ}}{\partial q_{f,1}}\frac{\partial\tilde{q}_{f,1}}{\partial q_{d,1}}\right] \\ & +\beta\sigma\left[\frac{\partial v_{f,2}}{\partial m_2}\frac{\partial m_2}{\partial K_2} + \frac{\partial v_{f,2}}{\partial K_2}\right]\left[\frac{\partial K_2}{\partial q_{d,1}} + \frac{\partial K_2}{\partial q_{f,1}}\frac{\partial\tilde{q}_{f,1}}{\partial q_{d,1}}\right] - c''_{f,1}\frac{\partial\tilde{q}_{f,1}}{\partial q_{d,1}} \\ = &\ 0\end{aligned} \tag{5}$$

We are interested in the limit of $\frac{\partial^2\tilde{q}_{f,1}}{\partial q_{d,1}\partial m_1}$. Differentiating (5) with respect to m_1, and evaluating at the limit of $m_1 = 0$ we have,

$$\begin{aligned}& P'K_1 + P'K_1\frac{\partial^2\tilde{q}_{f,1}}{\partial q_{d,1}\partial m_1} + 0 + \beta\sigma\frac{\partial v_{f,2}}{\partial K_2}\frac{d}{dm_1}\left[\frac{\partial K_2}{\partial q_{d,1}} + \frac{\partial K_2}{\partial q_{f,1}}\frac{\partial\tilde{q}_{f,1}}{\partial q_{d,1}}\right] \\ & -c''_{f,1}\frac{\partial^2\tilde{q}_{f,1}}{\partial q_{d,1}\partial m_1} \\ = &\ 0\end{aligned} \tag{6}$$

Note this result uses (1) the fact that $\frac{\partial v_{f,2}}{\partial m_2} = 0$ in the limit from the fact that (v) holds for $t = 2$ and (2) that $\frac{\partial m_2^{\circ}}{\partial q_{d,1}} = 0$, $\frac{\partial\tilde{q}_{f,1}}{\partial q_{d,1}} = 0$, $\frac{\partial K_2}{\partial q_{d,1}} = 0$ all obviously hold at the limit.

So we need to look at the next period capital stock,

$$K_2 = \left(m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}\right)\sigma K_1$$

The slope is

$$\begin{aligned}
\left[\frac{\partial K_2}{\partial q_{d,1}} + \frac{\partial K_2}{\partial q_{f,1}}\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right] &= m_1\sigma K_1 + (1-m_1)\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\sigma K_1 \\
\lim_{m_1\to 0}\frac{d}{dm_1}\left[\frac{\partial K_2}{\partial q_{d,1}} + \frac{\partial K_2}{\partial q_{f,1}}\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right] &= \sigma K_1\left[1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial m_1 \partial q_{d,1}}\right]
\end{aligned}$$

We can substitute this into (6) to get

$$\begin{aligned}
&P_1' K_1 + P' K_1 \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m_1} + \beta\sigma\frac{\partial v_{f,2}}{\partial K_2}\sigma K_1\left[1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial m_1 \partial q_{d,1}}\right] \\
&-c_{f,1}''\frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m_1} = 0
\end{aligned}$$

So solving we get

$$\begin{aligned}
\lim_{m_1\to 0}\frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m_1} &= \frac{p' K_1 + \beta\sigma\frac{\partial v_{f,2}}{\partial K_2}\sigma K_1}{\left[-P' K_1 - \beta\sigma^2\frac{\partial v_{f,2}}{\partial K_2}K_1 + c_{f,1}''\right]} \qquad (7) \\
\lim_{m_1\to 0}\frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m_1} + 1 &= \frac{c_1''}{c_1'' - P_1' K_1 - \beta\sigma^2\frac{\partial v_{f,2}}{\partial K_2}K_1}
\end{aligned}$$

4. The Slopes of the Equilibrium Output Levels

Here we calculate the limiting values of $\frac{\partial q_{d,1}}{\partial m}$ and $\frac{\partial q_{f,1}}{\partial m}$. To calculate $\frac{\partial q_{d,1}}{\partial m}$, we need to differentiate the dominant firm FONC (2). Note first that

$$\lim_{m_1\to 0}\frac{\partial H}{\partial q_{d,1}} = -c_1''$$

Next differentiate H with respect to m_1 and evaluate at $m_1 = 0$, holding $q_{d,1}$ fixed.

$$\begin{aligned}
\frac{\partial H}{\partial m_1} &= \frac{d^2\tilde{p}}{dq_{d,1}dm}q_{d,1} + \beta v_{f,2}\frac{K_2}{K_1}\frac{dG_1}{dm_1} + \beta\sigma\frac{dG_2}{dm_1} \qquad (8) \\
&= \frac{d^2\tilde{p}}{dq_{d,1}dm}q_{d,1} - \beta v_{f,2}\frac{K_2}{K_1}q_{f,1}^{-1}\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right) \\
&\quad + \beta\sigma\frac{m_2^\circ}{m_1}\left(v_{f,2} + K_2\frac{\partial v_{f,2}}{\partial K_2}\right)\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right)
\end{aligned}$$

where

$$
\begin{aligned}
G_1 &= \frac{[m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}] - \left(m_1 + (1-m_1)\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right) q_{d,1}}{[m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}]^2} \\
G_2 &= \left[\frac{m_2}{m_1}\left(v_{d,2} + K_2 \frac{\partial v_{d,2}}{\partial K_2}\right) - \frac{(m_2 - m_2^\circ)}{m_1}\left(v_{f,2} + K_2 \frac{\partial v_{f,2}}{\partial K_2}\right)\right]\left(m_1 + (1-m_1)\frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right)
\end{aligned}
$$

$$
\frac{dG_1}{dm_1} = -q_{f,1}^{-1}\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right)
$$

$$
\frac{dG_2}{dm_1} = \frac{m_2^\circ}{m_1}\left(v_{f,2} + K_2 \frac{\partial v_{f,2}}{\partial K_2}\right)\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right)
$$

and

$$
\begin{aligned}
\tilde{p}_1 &= P((m_1 K_1 q_{d,1} + (1-m_1)K_1 \tilde{q}_{f,1}) \qquad (9) \\
\frac{\partial \tilde{p}_1}{\partial q_{d,1}} &= P'\left(m_1 K_1 + (1-m_1)K_1 \frac{\partial \tilde{q}_{f,1}}{\partial q_{d,1}}\right) \\
\lim \frac{\partial^2 \tilde{p}_1}{\partial q_{d,1}\partial m_1} &= P'\left(K_1 + K_1 \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m_1}\right)
\end{aligned}
$$

So

$$
\begin{aligned}
\lim_{m_1 \to 0} \frac{\partial H}{\partial m_1} &= \frac{d^2 \tilde{p}}{dq_{d,1} dm} q_{d,1} - \beta v_{f,2} \frac{K_2}{K_1} q_{f,1}^{-1}\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right) \\
&\quad + \beta\sigma \frac{m_2^\circ}{m_1}\left(v_{f,2} + K_2 \frac{\partial v_{f,2}}{\partial K_2}\right)\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right) \\
&= \left(P' K_1 q_{f,1} - \beta v_{f,2} \frac{K_2}{K_1} q_{f,1}^{-1} + \beta\sigma \frac{m_2^\circ}{m_1}\left(v_{f,2} + K_2 \frac{\partial v_{f,2}}{\partial K_2}\right)\right)\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right) \qquad (10) \\
&= \left(P' K_1 q_{f,1} - \beta v_{f,2} \frac{K_2}{K_1} q_{f,1}^{-1} + \beta\sigma\left(v_{f,2} + K_2 \frac{\partial v_{f,2}}{\partial K_2}\right)\right)\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right) \qquad (11) \\
&= \left(P' K_1 q_{f,1} - \beta\sigma v_{f,2} + \beta\sigma\left(v_{f,2} + K_2 \frac{\partial v_{f,2}}{\partial K_2}\right)\right)\left(1 + \frac{\partial^2 \tilde{q}_{f,1}}{\partial q_{d,1}\partial m}\right) \qquad (12) \\
&= K_1 q_1 \left(P_1' + \beta\sigma^2 \frac{\partial v_{f,2}}{\partial K}\right)\frac{c_1''}{c_1'' - P_1' K_1 - \beta\sigma^2 \frac{\partial v_2}{\partial K} K_1} \qquad (13)
\end{aligned}
$$

The above derivation uses (7), (9), and the fact that $\lim_{m_1 \to 0} \frac{m_2^\circ}{m_1} = 1$, which follows from,

$$
m_2^\circ = \frac{m_1 q_{d,1}}{m_1 q_{d,1} + (1-m_1)\tilde{q}_{f,1}}
$$

$$\lim_{m_1 \to 0} \frac{\partial m_2^\circ}{\partial m_1} = \frac{1}{q_{f,1}^2} \left[q_{d,1} q_{f,1} - \left(q_{d,1} - q_{f,1} + (1-m) \frac{\partial \tilde{q}_{f,1}}{\partial m_1}) \right) 0 \right] = 1$$

and an application of l'Hôpital's rule (note that $q_{d,1}$ is held fixed as we take the limit $\lim_{m_1 \to 0} \frac{m_2^\circ}{m_1}$ here).

The slope of the dominant firm output function at the limit is then

$$\begin{aligned} \frac{\partial q_{d,1}(0, K_1)}{\partial m_1} &= -\frac{\lim_{m_1 \to 0} \frac{\partial H}{dm_1}}{\lim_{m_1 \to 0} \frac{\partial H}{dq_{d,1}}} \\ &= \frac{K_1 q_1 \left(P_1' + \beta \sigma^2 \frac{\partial v_{f,2}}{\partial K} \right)}{c_1'' - P_1' K_1 - \beta \sigma^2 \frac{\partial v_2}{\partial K} K_1} < 0 \end{aligned} \tag{14}$$

This proves (ii).

We now look at the limiting slope of $m_{next,1}^\circ(m, K)$. This equals

$$\begin{aligned} \lim_{m_1 \to 0} \frac{\partial m_{next,1}^\circ}{\partial m_1} &= \frac{\partial m_2^\circ}{\partial m_1}_{q_{d,1} fixed} + \frac{\partial m_2^\circ}{\partial q_{d,1}} \frac{dq_{d,1}}{dm_1} \\ &= 1 + 0 \\ &= 1 \end{aligned} \tag{15}$$

This proves (x).

Next consider the slope of the equilibrium fringe output. Differentiating the fringe FONC with respect to m_1 (and letting $q_{d,1}$ be a function of m_1)

$$\begin{aligned} & P_1' \left(K_1 q_{d,1} - K_1 q_{f,1} + m_1 K_1 \frac{dq_{d,1}}{dm_1} + (1-m_1) K_1 \frac{\partial q_{f,1}}{\partial m_1} \right) + \beta \sigma \frac{\partial v_{f,2}}{\partial m_2} \frac{\partial m_2}{\partial m_2^\circ} \frac{dm_2^\circ}{dm_1} \\ & + \beta \sigma \left[\frac{\partial v_{f,2}}{\partial m_2} \frac{\partial m_2}{\partial K_2} + \frac{\partial v_{f,2}}{\partial K_2} \right] \frac{dK_2}{dm_1} = c_{f,1}'' \frac{dq_{f,1}}{dm_1} \end{aligned} \tag{16}$$

Using the fact that statement (v) holds for $t = 2$, evaluating the above at the limit yields,

$$P_1' K_1 \frac{\partial q_{f,1}}{\partial m_1} + \beta \sigma \frac{\partial v_{f,2}}{\partial K_2} \frac{dK_2}{dm_1} = c_{f,1}'' \frac{\partial q_{f,1}}{\partial m_1} \tag{17}$$

Note that $K_2 = \sigma Q_1 = \sigma \left[m_1 q_{d,1} K_1 + (1-m_1) q_{f,1} K_1 \right]$, so in the limit,

$$\lim_{m_1 \to 0} \frac{dK_2}{dm_1} = \sigma K_1 \frac{\partial q_{f,1}(0, K_1)}{\partial m_1} \tag{18}$$

So we can rewrite (17) as

$$P_1' K_1 \frac{\partial q_{f,1}(0, K_1)}{\partial m_1} + \beta\sigma \frac{\partial v_{f,2}}{\partial K_2} \sigma K_1 \frac{\partial q_{f,1}(0, K_1)}{\partial m_1} = c_{f,1}'' \frac{\partial q_{f,1}(0, K_1)}{\partial m_1} \tag{19}$$

By looking at the perfectly competitive limit, it is straightforward to verify that in the limit where $m_2 = 0$ that $\frac{\partial v_{f,2}}{\partial K_2} < 0$. Condition (19) then implies that $\frac{\partial q_{f,1}(0,K_1)}{\partial m_1} = 0$ must hold, proving (iii).

Next note that $\frac{\partial q_{f,1}(0,K_1)}{\partial m_1} = 0$ implies that slope (18) $\frac{dK_2}{dm_1}$ is zero in the limit, which we use below.

We turn now to the second derivative of v_f. Differentiating (16) and evaluating at the limit yields

$$P' \left(2K_1 \frac{dq_{d,1}}{dm_1} + K_1 \frac{d^2 q_{f,1}}{dm_1^2} \right) + \beta\sigma \frac{d}{dm_1} \left[\frac{\partial v_{f,2}}{\partial m_2} \right] \frac{\partial m_2}{\partial m_2^\circ} \frac{dm_2^\circ}{dm_1} + \beta\sigma \frac{\partial v_{f,2}}{\partial K_2} \frac{d^2 K_2}{dm_1^2} - c_{f,1}'' \frac{dq_{f,1}^2}{dm_1^2} = 0 \tag{20}$$

Note

$$\begin{aligned} \frac{d}{dm_1} \left[\frac{\partial v_{f,2}}{\partial m_2} \right] &= \frac{\partial^2 v_{f,2}}{\partial m_2^2} \frac{dm_2}{dm_1} + \frac{\partial^2 v_{f,2}}{\partial m_2 \partial K_2} \frac{dK_2}{dm_1} \\ &= \frac{\partial^2 v_{f,2}}{\partial m_2^2} \frac{\partial m_2}{\partial m_2^\circ} \frac{dm_2^\circ}{dm_1} \\ &= \frac{\partial^2 v_{f,2}}{\partial m_2^2} \frac{\partial m_2}{\partial m_2^\circ} \end{aligned}$$

where

$$\begin{aligned} \frac{dm_2}{dm_1} &= \frac{\partial m_2}{\partial m_2^\circ} \frac{dm_2^\circ}{dm_1} + \frac{\partial m_2}{\partial K_2} \frac{dK_2}{dm_1} \\ \lim \frac{dm_2}{dm_1} &= \frac{\partial m_2}{\partial m_2^\circ} \frac{dm_2^\circ}{dm_1} \\ &= \frac{\partial m_2}{\partial m_2^\circ} \end{aligned}$$

Note the above uses the fact that $\lim \frac{dK_2}{dm_1} = 0$ and (15).

By differentiating $K_2 = \sigma Q_1 = \sigma \left[m_1 q_{d,1} K_1 + (1 - m_1) q_{f,1} K_1 \right]$ twice with respect to m_1 and evaluating at $m_1 = 0$, we obtain

$$\lim \frac{d^2 K_2}{dm_1^2} = \sigma \left[2K_1 \frac{dq_{d,1}}{dm_1} + K_1 \frac{d^2 q_{f,1}}{dm_1^2} \right]$$

Substitute this into (20) yields,

$$\begin{aligned}
& P'\left(2K_1\frac{dq_{d,1}}{dm_1}+K_1\frac{d^2q_{f,1}}{dm_1^2}\right)+\beta\sigma\frac{d}{dm_1}\left[\frac{\partial v_{f,2}}{\partial m_2}\right]\frac{\partial m_2}{\partial m_2^\circ}\frac{dm_2^\circ}{dm_1}+\beta\sigma\frac{\partial v_{f,2}}{\partial K_2}\frac{d^2K_2}{dm_1^2}-c_1''\frac{dq_{f,1}^2}{dm_1^2} \\
= \; & P'\left(2K_1\frac{dq_{d,1}}{dm_1}+K_1\frac{d^2q_{f,1}}{dm_1^2}\right)+\beta\sigma\frac{\partial^2 v_{f,2}}{\partial m_2^2}\left(\frac{\partial m_2}{\partial m_2^\circ}\right)^2+\beta\sigma\frac{\partial v_{f,2}}{\partial K_2}\sigma\frac{d^2K_2}{dm_1^2}-c_1''\frac{dq_{f,1}^2}{dm_1^2} \\
= \; & \left(P'+\beta\sigma^2\frac{\partial v_{f,2}}{\partial K_2}\right)\left(2K_1\frac{dq_{d,1}}{dm_1}+K_1\frac{d^2q_{f,1}}{dm_1^2}\right)+\beta\sigma\frac{\partial^2 v_{f,2}}{\partial m_2^2}\left(\frac{\partial m_2}{\partial m_2^\circ}\right)^2-c_1''\frac{dq_{f,1}^2}{dm_1^2} \\
= \; & 0
\end{aligned}$$

Solving, yields,

$$\frac{\partial^2 q_{f,1}(0,K_1)}{\partial m_1^2}=\frac{\left(P'+\beta\sigma^2\frac{\partial v_{f,2}}{\partial K_2}\right)2K_1\frac{dq_{d,1}}{dm_1}+\beta\sigma\frac{\partial^2 v_{f,2}}{\partial m_2^2}\left(\frac{\partial m_2}{\partial m_2^\circ}\right)^2}{c_1''-\left(P'+\beta\sigma^2\frac{\partial v_{f,2}}{\partial K_2}\right)K_1}$$

The numerator is strictly positive and bounded. The denominator is strictly positive and bounded. This proves (iv).

We next look at first-period total output, $Q_1=[m_1q_{d,1}K_1+(1-m_1)\,q_{f,1}K_1]$

$$\begin{aligned}
\frac{\partial Q_1}{\partial m_1} &= K_1(q_{d,1}-q_{f,1})+m_1K_1\frac{\partial q_{d,1}}{\partial m_1}+(1-m_1)\,K_1\frac{\partial q_{f,1}}{\partial m_1} \qquad (21)\\
\lim_{m_1\to 0}\frac{\partial Q_1}{\partial m_1} &= 0 \\
\lim_{m_1\to 0}\frac{\partial^2 Q_1}{\partial m_1^2} &= 2K_1\frac{\partial q_{d,1}}{\partial m_1}+K_1\frac{\partial^2 q_{f,1}}{\partial m_1^2}
\end{aligned}$$

5. The Slopes of the Value Functions

We begin with the fringe value. The level is

$$v_{f,1}=p_1q_{f,1}+\beta\sigma v_{f,2}q_{f,1}-c(q_{f,1})$$

Using the envelope theorem, the slope is

$$\frac{\partial v_{f,1}}{\partial m_1}=\frac{dp_1}{dm_1}q_{f,1}+\beta\sigma\frac{dv_{f,2}}{dm_1}q_{f,1}$$

Note that $\frac{dp_1}{dm_1}=0$ from above. The fact that (v) holds for $t=2$ and $\frac{dK_2}{dm_1}=0$ together imply that $\frac{dv_{f,2}}{dm_1}=0$. Hence $\frac{\partial v_{f,1}}{\partial m_1}=0$. Analogously, $\frac{\partial v_{d,1}}{\partial m_1}=0$. Thus statement (v) holds.

Differentiating again and taking the limit yields,

$$\frac{d^2 v_{f,1}(0,K)}{dm_1^2} = \frac{d^2 p_1}{dm_1^2} q_{f,1} + \beta\sigma \frac{dv_{f,2}^2}{dm_1^2} q_{f,1} \tag{22}$$

To derive the $\frac{dv_{f,2}^2}{dm_1^2}$ term, consider $\frac{dv_{f,2}}{dm_1}$.

$$\begin{aligned} \frac{dv_{f,2}}{dm_1} &= \frac{\partial v_{f,2}}{\partial m_2}\frac{dm_2}{dm_1} + \frac{\partial v_{f,2}}{\partial K_2}\frac{dK_2}{dm_1} \qquad (23) \\ \lim \frac{d^2 v_{f,2}}{dm_1^2} &= \left[\frac{\partial^2 v_{f,2}}{\partial m_2^2}\frac{dm_2}{dm_1} + \frac{\partial v_{f,2}}{\partial m_2 \partial K_2}\frac{dK_2}{dm_1}\right]\frac{dm_2}{dm_1} + \frac{\partial v_{f,2}}{\partial K_2}\frac{d^2 K_2}{dm_1^2} \\ &= \frac{\partial^2 v_{f,2}}{\partial m_2^2}\left[\frac{\partial m_2}{\partial m_2^\circ}\right]^2 + \frac{\partial v_{f,2}}{\partial K_2}\frac{d^2 K_2}{dm_1^2} \\ & \frac{\partial^2 v_{f,2}}{\partial m_2^2}\left[\frac{\partial m_2}{\partial m_2^\circ}\right]^2 + \sigma\frac{\partial v_{f,2}}{\partial K_2}\frac{d^2 Q_1}{dm_1^2} \end{aligned}$$

Note that the first term is strictly positive. The second term would also be positive if $\frac{d^2 Q_1}{dm_1^2} < 0$ which intuitively should hold but we do not have a proof. Instead we show that $\lim \frac{d^2 v_{f,2}}{dm_1^2} > 0$ with the following argument. Holding K_1 fixed, define the function $B(m_1)$ by $B(m_1) \equiv p_1(m_1) + \beta\sigma v_{2,f}(m_1)$, where p_1 and $v_{2,f}$ are the equilibrium values given m_1. Then

$$\begin{aligned} v_{1,f}(m_1) &= B(m_1) q_{d,1}(m_1) - c(q_{d,1}(m_1)) \\ \frac{dv_{1,f}(m_1)}{dm_1} &= \frac{dB(m_1)}{dm_1} q_{d,1}(m_1) \\ \lim \frac{d^2 v_{1,f}(m_1)}{dm_1^2} &= \lim \frac{d^2 B}{dm_1^2} q_{d,1}(m_1) \end{aligned}$$

Now from the fringe FONC we know that $B(m_1) = c'(q_{f,1}(m_1))$. Since $\lim \frac{dq_{f,1}}{dm_1} = 0$ and $\lim \frac{dq_{f,1}^2}{dm_1^2} > 0$, it is immediate that $\lim \frac{d^2 B}{dm_1^2} > 0$, which proves that $\lim \frac{d^2 v_{1,f}}{dm_1^2} > 0$, so statement (vi) holds.

Next look at the dominant firm value

$$v_{d,1} = p_1 q_{d,1} - c(q_{d,1}) + \beta \frac{w_2^\circ}{m_1 K_1}$$

But

$$\begin{aligned} w_2^\circ &= m_2 K_2 v_{d,2} - (m_2 - m_2^\circ) K_2 v_{f,2} \\ &= \sigma m_1 K_1 q_{d,1} v_{d,2} - \sigma m_1 K_1 q_{d,1} v_{d,2} + m_2 K_2 v_{d,2} - (m_2 - m_2^\circ) K_2 v_{f,2} \end{aligned}$$

So

$$\frac{w_2^\circ}{m_1 K_1} = \sigma q_{d,1} v_{d,2} - \sigma q_{d,1} v_{d,2} + \frac{m_2 K_2}{m_1 K_1} v_{d,2} - \frac{(m_2 - m_2^\circ) K_2}{m_1 K_1} v_{f,2}$$

Also note that

$$\sigma q_{d,1} m_1 K_1 = m_2^\circ K_2$$

by the definition of m_2°, so

$$\begin{aligned} v_{d,1} &= p_1 q_{d,1} - c(q_{d,1}) + \beta \sigma q_{d,1} v_{d,2} + \beta \left[\frac{(m_2 K_2 - \sigma q_{d,1} m_1 K_1)}{m_1 K_1} v_{d,2} - \frac{(m_2 - m_2^\circ) K_2}{m_1 K_1} v_{f,2} \right] \\ &= p_1 q_{d,1} - c(q_{d,1}) + \beta \sigma q_{d,1} v_{d,2} + \beta \left[\frac{(m_2 K_2 - m_2^\circ K_2)}{m_1 K_1} v_{d,2} - \frac{(m_2 - m_2^\circ) K_2}{m_1 K_1} v_{f,2} \right] \\ &= p_1 q_{d,1} - c(q_{d,1}) + \beta \sigma q_{d,1} v_{d,2} + \beta \frac{K_2}{K_1} \frac{(m_2 - m_2^\circ)}{m_1} (v_{d,2} - v_{f,2}) \end{aligned}$$

The slope is

$$\begin{aligned} \frac{dv_{d,1}}{dm_1} &= \frac{dp_1}{dm_1} q_{d,1} + \left(p_1 - c'_{d,1} + \beta \sigma v_{d,2}\right) \frac{dq_{d,1}}{dm_1} + \beta \sigma q_{d,1} \frac{dv_{d,2}}{dm_1} \\ &\quad + \beta \frac{d}{dm_1} \left[\frac{K_2}{K_1} \right] \frac{(m_2 - m_2^\circ)}{m_1} (v_{d,2} - v_{f,2}) \\ &\quad + \beta \frac{d}{dm_1} \left[\frac{(m_2 - m_2^\circ)}{m_1} \right] \frac{K_2}{K_1} (v_{d,2} - v_{f,2}) \\ &\quad \beta \frac{d}{dm_1} [v_{d,2} - v_{f,2}] \frac{K_2}{K_1} \frac{(m_2 - m_2^\circ)}{m_1} \end{aligned}$$

Taking the second derivative and evaluating at the limit yields

$$\begin{aligned} \lim \frac{dv_{d,1}^2}{dm_1^2} &= \frac{d^2 p_1}{dm_1^2} q_{d,1} - c''_{d,1} \left[\frac{dq_{d,1}}{dm_1} \right]^2 + \beta \sigma q_1 \frac{dv_{d,2}^2}{dm_1^2} \\ &\quad + \beta \left[\frac{d^2 v_{d,2}}{dm_1^2} - \frac{d^2 v_{f,2}}{dm_1^2} \right] \sigma q_1 \left[\frac{\partial m_2}{\partial m_2^\circ} - 1 \right] \end{aligned}$$

note all the complicated terms cancel except for the last because $(v_{d,2} - v_{f,2})$ is zero and so is $\frac{d}{dm_1}[v_{d,2} - v_{f,2}]$. Also note that $K_2/K_1 = \sigma q_1$ in the limit since $K_2 = \sigma q_1 K_1$. Analogous to above,

$$\lim \frac{d^2 v_{d,2}}{dm_1^2} = \frac{\partial^2 v_{d,2}}{\partial m_2^2} \left[\frac{\partial m_2}{\partial m_2^\circ} \right]^2 + \sigma \frac{\partial v_{d,2}}{\partial K_2} \frac{d^2 Q_1}{dm_1^2}$$

Now define Δ to be the difference in the limiting second derivatives,

$$\begin{aligned}
\Delta &= \lim \frac{d^2 v_{f,1}}{dm_1^2} - \lim \frac{d^2 v_{d,1}}{dm_1^2} \qquad (24) \\
&= \left[\frac{d^2 p_1}{dm_1^2} q_1 + \beta\sigma \frac{dv_{f,2}^2}{dm_1^2} q_1 \right] \\
&\quad - \left[\begin{array}{c} \frac{d^2 p_1}{dm_1^2} q_{d,1} - c_1'' \left[\frac{dq_{d,1}}{dm_1} \right]^2 + \beta\sigma q_1 \frac{dv_{d,2}^2}{dm_1^2} \\ + \beta \left[\frac{d^2 v_{d,2}}{dm_1^2} - \frac{d^2 v_{f,2}}{dm_1^2} \right] \sigma q_1 \left[\frac{\partial m_2}{\partial m_2^\circ} - 1 \right] \end{array} \right] \\
&= c_1'' \left[\frac{dq_{d,1}}{dm_1} \right]^2 + \beta \frac{\partial m_2}{\partial m_2^\circ} \left[\frac{dv_{f,2}^2}{dm_1^2} - \frac{dv_{d,2}^2}{dm_1^2} \right] \\
&= c_1'' \left[\frac{dq_{d,1}}{dm_1} \right]^2 + \beta \left[\frac{\partial m_2}{\partial m_2^\circ} \right]^3 \left[\frac{dv_{f,2}^2}{dm_2^2} - \frac{dv_{d,2}^2}{dm_2^2} \right] \\
&> 0
\end{aligned}$$

6. The Merger Function

The merger decision at time $t = 1$ is (to simplify notation in this section we leave out the $t = 1$ subscript).

$$\max_m m v_d - (m - m^\circ) v_f$$

Given $v_d(m) < v_f(m)$ for $m > 0$, the optimal merger is zero at $m^\circ = 0$, proving (vii).

For $m^\circ > 0$, the FONC for an interior optimum is

$$m \frac{\partial v_d}{\partial m} - [v_f - v_d] - (m - m^\circ) \frac{\partial v_f}{\partial m} = 0$$

The sufficient SOC

$$2 \frac{\partial v_d}{\partial m} + m \frac{\partial^2 v_d}{\partial m^2} - 2 \frac{\partial v_f}{\partial m} - (m - m^\circ) \frac{\partial^2 v_f}{\partial m^2} < 0$$

At the limit this is zero. To look near the limit, differentiate again and get

$$\begin{aligned}
& 3 \frac{\partial^2 v_d}{\partial m^2} + m \frac{\partial^3 v_d}{\partial m^3} - 3 \frac{\partial^2 v_f}{\partial m^2} - (m - m^\circ) \frac{\partial^3 v_f}{\partial m^3} \\
&= 3 \left[\frac{\partial^2 v_d}{\partial m^2} - \frac{\partial^2 v_f}{\partial m^2} \right], \text{ at limit} \\
&< 0
\end{aligned}$$

so the problem is strictly concave near the limit.

Now differentiate FONC with respect to m°,

$$\frac{\partial v_f}{\partial m}dm^\circ + \left[2\frac{\partial v_d}{\partial m} + m\frac{\partial^2 v_d}{\partial m^2} - 2\frac{\partial v_f}{\partial m} - (m - m^\circ)\frac{\partial^2 v_f}{\partial m^2}\right]dm = 0$$

Define

$$Z(m^\circ, m) = \left[2\frac{\partial v_d}{\partial m} + m\frac{\partial^2 v_d}{\partial m^2} - 2\frac{\partial v_f}{\partial m} - (m - m^\circ)\frac{\partial^2 v_f}{\partial m^2}\right]$$

So

$$\frac{\partial v_f}{\partial m}dm^\circ + Z(m^\circ, m)dm = 0$$

So

$$\frac{dm}{dm^\circ} = \frac{\frac{\partial v_f}{\partial m}}{-Z(m^\circ, m(m^\circ))}$$

Both the numerator and the denominator go to zero as m° goes to zero (and $m(m^\circ)$ goes to zero. We use l'Hôpital's rule to look at the ratio of the slopes. The slope of the denominator is

$$\frac{dZ}{dm^\circ} = \frac{\partial^2 v_f}{\partial m^2} + \left[3\frac{\partial^2 v_d}{\partial m^2} + m\frac{\partial^3 v_d}{\partial m^3} - 3\frac{\partial^2 v_f}{\partial m^2} - (m - m^\circ)\frac{\partial^3 v_d}{\partial m^3}\right]\frac{dm}{dm^\circ}$$

at the limit this equals

$$\lim\frac{dZ}{dm^\circ} = \frac{\partial^2 v_f}{\partial m^2} + \left[3\frac{\partial^2 v_d}{\partial m^2} - 3\frac{\partial^2 v_f}{\partial m^2}\right]\frac{dm}{dm^\circ}.$$

The slope of the numerator is

$$\frac{d\left[\frac{\partial v_f}{\partial m}\right]}{dm^\circ} = \frac{\partial^2 v_f}{\partial m^2}\frac{dm}{dm^\circ}$$

So

$$\begin{aligned}\lim\frac{dm}{dm^\circ} &= \frac{\lim\frac{\partial v_f}{\partial m}}{\lim -Z(m, m(m^\circ))}\\ &= \frac{\lim\frac{\partial^2 v_f}{\partial m^2}\frac{dm}{dm^\circ}}{-\lim\frac{\partial^2 v_f}{\partial m^2} - \lim\left[3\frac{\partial^2 v_d}{\partial m^2} - 3\frac{\partial^2 v_f}{\partial m^2}\right]\frac{dm}{dm^\circ}}\end{aligned}$$

So

$$1 = \frac{\frac{\partial^2 v_f}{\partial m^2}}{-\frac{\partial^2 v_f}{\partial m^2} - \left[3\frac{\partial^2 v_d}{\partial m^2} - 3\frac{\partial^2 v_f}{\partial m^2}\right]\frac{dm}{dm^\circ}}$$

or

$$-\frac{\partial^2 v_f}{\partial m^2} - \left[3\frac{\partial^2 v_d}{\partial m^2} - 3\frac{\partial^2 v_f}{\partial m^2}\right]\frac{dm}{dm^\circ} = \frac{\partial^2 v_f}{\partial m^2}$$

or

$$\left[-3\frac{\partial^2 v_d}{\partial m^2} + 3\frac{\partial^2 v_f}{\partial m^2}\right]\frac{dm}{dm^\circ} = 2\frac{\partial^2 v_f}{\partial m^2}$$

so

$$\lim \frac{dm}{dm^\circ} = \frac{2\frac{\partial^2 v_f}{\partial m^2}}{\left[-3\frac{\partial^2 v_d}{\partial m^2} + 3\frac{\partial^2 v_f}{\partial m^2}\right]}$$

This proves (viii) which completes the proof.■

2.2 Lemma A2

Lemma A2. Let $v(K)$ be the the value function for a competitive firm with one unit of capital in perfect competition when there is no dominant firm. Then the slope of the value function evaluated at the stationary competitive capital level K^*_{com} is

$$\frac{dv}{dK} = \frac{c''}{2\beta\sigma^2 K}\left[(1-\beta) + \lambda - \left[(1-\beta)^2 + \lambda^2 + 2(1+\beta)\lambda\right]^{\frac{1}{2}}\right]$$

for λ defined by

$$\lambda \equiv \frac{-P'K}{c''}.$$

Proof.

The policy and value functions must solve the first order condition and the Bellman equation,

$$\begin{aligned} p(q(K)K) + \beta\sigma v(\sigma q(K)K) - c'(q(K)) &= 0 \\ v(K) - p(q(K)K)q(k) + c(q(K)) - \beta\sigma v(\sigma q(K)K)q(K) &= 0 \end{aligned}$$

Differentiating and evaluating at $K = K^*$ yields

$$P'\left(\frac{dq}{dK}K + q\right) + \beta\sigma^2\frac{dv}{dK}\left(\frac{dq}{dK}K + q\right) - c''\frac{dq}{dK} = 0$$

and

$$\begin{aligned}
&\frac{dv}{dK}-\frac{dp}{dQ}\left(\frac{dq}{dK}K+q\right)q-\beta\sigma\sigma\frac{dv}{dK}\left(\frac{dq}{dK}K+q\right)q \\
&-\left(p-c'+\beta\sigma v\right)\frac{dq}{dK} \\
=\ &\frac{dv}{dK}-\frac{dp}{dQ}\left(\frac{dq}{dK}K+q\right)q-\beta\sigma\sigma\frac{dv}{dK}\left(\frac{dq}{dK}K+q\right)q=0
\end{aligned}$$

But using $\sigma q=1$, we have two conditions,

$$\begin{aligned}
P'\left(\frac{dq}{dK}K+q\right)+\beta\sigma^2\frac{dv}{dK}\left(\frac{dq}{dK}K+q\right)-c''\frac{dq}{dK} &= 0 \\
\frac{dv}{dK}-\frac{dp}{dQ}\left(\frac{dq}{dK}K+q\right)q-\beta\sigma\frac{dv}{dK}\left(\frac{dq}{dK}K+q\right) &= 0
\end{aligned}$$

Solve these two conditions for $\frac{dv}{dK}$ and $\frac{dq}{dK}$. Let

$$\begin{aligned}
x &= \frac{dq}{dK} \\
y &= \frac{dv}{dK}
\end{aligned}$$

Then

$$\begin{aligned}
P'\left(xK+q\right)+\beta\sigma^2 y\left(xK+q\right)-c''x &= 0 \\
y-P'\left(xK+q\right)q-\beta\sigma y\left(xK+q\right) &= 0
\end{aligned}$$

So

$$\begin{aligned}
y &= P'\left(xK+q\right)q+\beta\sigma y\left(xK+q\right) \\
&= \frac{1}{\sigma}\left(P'\left(xK+q\right)+\beta\sigma^2 y\left(xK+q\right)\right) \\
&= \frac{1}{\sigma}c''x
\end{aligned}$$

Substituting this into the first equation yields

$$\begin{aligned}
P'\left(xK+\frac{1}{\sigma}\right)+\beta\sigma^2\frac{1}{\sigma}c''x\left(xK+\frac{1}{\sigma}\right)-c''x &= 0 \\
P'Kx+P'\frac{1}{\sigma}+\beta\sigma c''Kx^2+\beta c''x-c''x &= 0
\end{aligned}$$

$$\begin{aligned}
\sigma P' K x + P' + \beta\sigma^2 c'' K x^2 + \beta\sigma c'' x - \sigma c'' x &= 0 \\
-\beta\sigma^2 c'' K x^2 + \left(\left(1-\beta\right)\sigma c'' - P'\sigma K\right) x - P' &= 0 \\
A x^2 + B x + C &= 0
\end{aligned}$$

where

$$\begin{aligned}
A &= -\beta\sigma^2 c'' K < 0 \\
B &= \left(\left(1-\beta\right)\sigma c'' - P'\sigma K\right) > 0 \\
C &= -P' > 0
\end{aligned}$$

Since $A < 0$, $B > 0$, and $C > 0$, there exists one negative root. The solution is

$$\begin{aligned}
x &= \frac{-B}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A} \\
y &= \frac{1}{\sigma} c'' x
\end{aligned}$$

This can be simplified.

$$\begin{aligned}
B^2 - 4AC &= \left(1-\beta\right)^2 \sigma^2 c''^2 + \left(P'\sigma K\right)^2 - 2(1-\beta)\sigma^2 c'' P' K - 4\beta\sigma^2 c'' K P' \\
&= \left(1-\beta\right)^2 \sigma^2 c''^2 + \left(P'\sigma K\right)^2 - \left(2 - 2\beta + 4\beta\right)\sigma^2 c'' P' K \\
&= \left(1-\beta\right)^2 \sigma^2 c''^2 + \left(P'\sigma K\right)^2 - 2(1+\beta)\sigma^2 c'' P' K
\end{aligned}$$

$$\begin{aligned}
x &= \frac{\left(1-\beta\right)\sigma c'' - P'\sigma K}{2\beta\sigma^2 c'' K} - \frac{\left[\left(1-\beta\right)^2 \sigma^2 c''^2 + \left(P'\sigma K\right)^2 - 2(1+\beta)\sigma^2 c'' P' K\right]^{\frac{1}{2}}}{2\beta\sigma^2 c'' K} \\
&= \frac{\left(1-\beta\right) c'' - P' K}{2\beta\sigma c'' K} - \frac{\left[\left(1-\beta\right)^2 c''^2 + \left(P' K\right)^2 - 2(1+\beta) c'' P' K\right]^{\frac{1}{2}}}{2\beta\sigma c'' K}
\end{aligned}$$

so

$$\begin{aligned}
\frac{dv}{dK} &= \frac{1}{\sigma} c'' x \\
&= \frac{1}{2\beta\sigma^2 K}\left[\left(1-\beta\right) c'' - P' K - \left[\left(1-\beta\right)^2 c''^2 + \left(P' K\right)^2 - 2(1+\beta) c'' P' K\right]^{\frac{1}{2}}\right]
\end{aligned}$$

$$= \frac{c''}{2\beta\sigma^2 K}\left[(1-\beta)+\left(\frac{-P'K}{c''}\right)-\left[(1-\beta)^2+\left(-\frac{P'K}{c''}\right)^2+2(1+\beta)\left(\frac{-P'K}{c''}\right)\right]^{\frac{1}{2}}\right]$$

$$= \frac{c''}{2\beta\sigma^2 K}\left[(1-\beta)+\lambda-\left[(1-\beta)^2+\lambda^2+2(1+\beta)\lambda\right]^{\frac{1}{2}}\right].$$

■

2.3 Lemma A3

The dynamics near the competitive steady state are determined by the slope of the transition function F, with respect to changes in m°, evaluated at perfect competition $m^\circ = 0$. For notation, we denote this slope as s.

It is convenient to state the following lemma about the transition here.

Lemma A3.

$$s \equiv \frac{\partial F}{\partial m^\circ}(0,K) = \frac{\partial \tilde{m}}{\partial m^\circ}(0,K).$$

Proof. By Proposition 1, we know that $\widetilde{m}(0) = 0$ (here we leave K implicit and evaluate the function only at m°). Part Lemma A1 above implies that $q_d(0) = q_f(0)$ must hold. From the proof of A1, it is immediate that $q_d(\widetilde{m}(m^\circ))$ and $q_f(\widetilde{m}(m^\circ))$ are differentiable at $m^\circ = 0$. Hence,

$$\frac{dF}{dm^\circ}(0) = \frac{d}{dm^\circ}\left[\frac{\widetilde{m}(m^\circ)\, q_d(\widetilde{m}(m^\circ))}{\widetilde{m}(m^\circ)\, q_d(\widetilde{m}(m^\circ)) + (1-\widetilde{m}(m^\circ))q_f(\widetilde{m}(m^\circ))}\right]\Bigg|_{m^\circ=0}$$

$$= \frac{q_d(0)}{q_f(0)}\frac{d\tilde{m}}{dm^\circ}(0) = \frac{d\tilde{m}}{dm^\circ}(0).$$

■

The reason for this result is that when the dominant firm market share is zero, the dominant firm and fringe investment rates are equal, since the dominant firm will forever have market share zero by Proposition 1. While an increase in share at zero results in a first-order difference between q_d and q_f, this effect is negligible in the limit since these differences are applied to a dominant firm share that is zero in the limit. Thus, next period's concentration m°_{next} will be the same as the post-merger concentration level $\widetilde{m}$, to a first order.

2.4 Lemma A4

Lemma A4. The transition slope s is the unique solution to

$$s = \frac{4}{3} - \left(\frac{4}{3} - \frac{1}{\psi + 1}\right)\beta s^3 \tag{25}$$

for ψ defined by

$$\psi \equiv \frac{1}{2}\left(\lambda + \left[(1-\beta)^2 + \lambda^2 + 2(1+\beta)\lambda\right]^{\frac{1}{2}} - (1-\beta)\right) > 0 \tag{26}$$

and λ defined by

$$\lambda \equiv \frac{-P'K}{c''},$$

evaluated at the stationary competitive levels Q^*_{com}, K^*_{com}, and $q^* = \frac{1}{1-\delta}$.

Proof.

Take T to infinity and evaluate K at the stationary level $K^*(\beta)$ for the infinite horizon case. The formula (24) for Δ then can be written

$$\begin{aligned}\Delta &= \frac{\partial v_f^2}{\partial m^2} - \frac{\partial v_d^2}{\partial m^2} \\ &= c''\left[\frac{\partial q_d}{\partial m}\right]^2 + \beta s^3 \Delta\end{aligned}$$

or

$$\Delta = \frac{c''\left[\frac{\partial q_d}{\partial m}\right]^2}{1 - \beta s^3} \tag{27}$$

Next consider $\frac{\partial v_f^2}{\partial m^2}$. Substituting in $\sigma q = 1$ (which holds at the stationary output level), the formula (22) for $\frac{\partial v_f^2}{\partial m^2}$ can be written

$$\begin{aligned}\frac{\partial v_f^2}{\partial m^2} &= \frac{d^2 p_1}{dm_1^2} q_{f,1} + \beta\frac{dv_{f,2}^2}{dm_1^2} \\ &= qP'\frac{d^2Q}{dm^2} + \beta\left[\frac{\partial v_f^2}{\partial m^2}s^2 + \sigma\frac{\partial v_f}{\partial K}\frac{d^2Q}{dm^2}\right]\end{aligned}$$

$$
\begin{aligned}
&= \beta \frac{\partial v_f^2}{\partial m^2} s^2 - \frac{1}{\sigma K}\left[-P'K - \sigma^2 \frac{\partial v_f}{\partial K}\right] \frac{d^2 Q}{dm^2} \\
&= \beta \frac{\partial v_f^2}{\partial m^2} s^2 - \frac{\xi}{\sigma K} \frac{d^2 Q}{dm_1^2}
\end{aligned}
$$

where ξ is defined by

$$
\xi \equiv -\left(P'K + \sigma^2 \frac{\partial v_f}{\partial K} K\right).
$$

Note that the second equality above uses (23).

From statement (iii) and (iv) of Lemma A1, we have (all evaluated at the limits)

$$
\begin{aligned}
\frac{\partial q_d}{\partial m} &= -\frac{\xi}{\sigma \left(c'' + \xi\right)} \\
\frac{\partial^2 q_f}{\partial m^2} &= -2 \frac{\xi}{\xi + c''} \frac{\partial q_d}{\partial m} + \frac{1}{\xi + c''} \beta \sigma s^2 \frac{\partial v_f^2}{\partial m^2}
\end{aligned}
$$

Using (21),

$$
\begin{aligned}
\frac{\partial^2 Q}{\partial m^2} &= 2K \frac{\partial q_d}{\partial m} + K \frac{\partial^2 q_f}{\partial m^2} \\
&= 2K \frac{\partial q_d}{\partial m} + K \left(-2 \frac{\xi}{\xi + c''} \frac{\partial q_d}{\partial m} + \frac{1}{\xi + c''} \beta \sigma s^2 \frac{\partial v_f^2}{\partial m^2}\right) \\
&= 2K \frac{\partial q_d}{\partial m} \left(\frac{c''}{\xi + c''}\right) + \frac{K}{\xi + c''} \beta \sigma s^2 \frac{\partial v_f^2}{\partial m^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v_f^2}{\partial m^2} &= \beta \frac{\partial v_f^2}{\partial m^2} s^2 - \frac{\xi}{\sigma K} \frac{d^2 Q}{dm_1^2} \\
&= \beta \frac{\partial v_f^2}{\partial m^2} s^2 - \frac{\xi}{\sigma K} \left(2K \frac{\partial q_d}{\partial m} \left(\frac{c''}{\xi + c''}\right) + \frac{K}{\xi + c''} \beta \sigma s^2 \frac{\partial v_f^2}{\partial m^2}\right) \\
&= \frac{\beta c''}{\xi + c''} \frac{\partial v_f^2}{\partial m^2} s^2 + 2c'' \left[\frac{\partial q_d}{\partial m}\right]^2
\end{aligned}
$$

Collecting the terms involving $\frac{\partial v_f^2}{\partial m^2}$ on the left-hand size, we have

$$
\begin{aligned}
\frac{\partial v_f^2}{\partial m^2} \left(1 - \frac{\beta c'' s^2}{\xi + c''}\right) &= 2c'' \left[\frac{\partial q_d}{\partial m}\right]^2 \\
\frac{\partial v_f^2}{\partial m^2} &= \frac{2c'' \left[\frac{\partial q_d}{\partial m}\right]^2}{\left(1 - \frac{\beta c'' s^2}{\xi + c''}\right)}
\end{aligned}
$$

Using statement (viii) from Lemma A1,

$$\begin{aligned} s &= \frac{2\frac{\partial^2 v_f}{\partial m^2}}{3\Delta} \\ &= \frac{\frac{4c''\left[\frac{\partial q_d}{\partial m}\right]^2}{\left(1-\frac{\beta c'' s^2}{\xi + c''}\right)}}{\frac{3c''\left[\frac{\partial q_d}{\partial m}\right]^2}{1-\beta s^3}} \\ &= \frac{4}{3}\frac{1-\beta s^3}{1-\frac{\beta c'' s^2}{\xi + c''}} \end{aligned}$$

We can rewrite this as

$$s - \frac{c''\beta s^3}{\xi + c''} = \frac{4}{3} - \frac{4}{3}\beta s^3$$

Or

$$s = \frac{4}{3} - \left(\frac{4}{3} - \frac{1}{\psi + 1}\right)\beta s^3$$

Where

$$\begin{aligned} \psi &\equiv \frac{\xi}{c''} \\ &= -\left(\frac{P'K}{c''} + \frac{\beta\sigma^2 K}{c''}\frac{\partial v_f}{\partial K}\right) \\ &= \left(\lambda - \frac{1}{2}\left[(1-\beta) + \lambda - \left[(1-\beta)^2 + \lambda^2 + 2(1+\beta)\lambda\right]^{\frac{1}{2}}\right]\right) \\ &= \frac{1}{2}\left(\lambda + \left[(1-\beta)^2 + \lambda^2 + 2(1+\beta)\lambda\right]^{\frac{1}{2}} - (1-\beta)\right). \end{aligned}$$

This uses Lemma A2 to substitute in the formula for $\frac{\partial v_f}{\partial K}$.

■

3. The Final Step: Proof of Proposition 5(ii)

Proposition 5. (ii) Suppose the horizon is infinite, $T = \infty$. There exists a $\beta' < 1$ such that if $\beta > \beta'$ and $K = K^*_{com}(\beta)$ and if the dominant firm's initial market share m° is positive but sufficiently small, then $\tilde{m}(m^\circ, K) < m^\circ$; i.e., the dominant firm sells capital.

Proof.

Based on the lemmas above, it is sufficient to show that for high enough β, $s > 1$. Taking the limit at β goes to 1 (and adjusting K according to $K = K^*_{com}(\beta)$), and using the formula (26) it is clear that

$$\lim_{\beta \to 1} \psi > 0.$$

From the equation (25) for s, it is then immediate that $s > 1$ for high enough β which completes the proof.

4. Comparative Statics Claims from Footnote 17

In footnote 17 we claim that for the constant elasticity case, the merger function increases in ε_D and decreases in ε_S near the limit where m° is close to zero. Here we prove this as well as some additional comparative statics claims.

Claim (i) In the constant elasticity case, s increases in ε_D, decreases in ε_S and is constant in δ. (ii) In the constant elasticity case, s decreases in β, assuming that $\varepsilon_D = 1$

Proof.

For notational simplicity it is convenient to let $\eta = \varepsilon_D$ and $\theta = \frac{1}{\varepsilon_S}$. We prove the comparative statics results in three steps.

4.1 Step 1.

We first show that

$$\begin{aligned} \lambda &= \frac{1}{\varepsilon_D} \frac{\varepsilon_S + (1-\beta)}{1 + \frac{1}{\varepsilon_S}} \\ &\equiv \frac{1}{\eta} \frac{\frac{1}{\theta} + (1-\beta)}{1+\theta}. \end{aligned} \tag{28}$$

From the definition of λ,

$$\begin{aligned} \lambda &\equiv \frac{-P'K}{c''} \\ &= \frac{-P'\sigma Q}{c''} \\ &= -\frac{\sigma Q}{\frac{dQ}{dP} c''} \\ &= \frac{1}{\eta} \frac{p^*_{com}\sigma}{c''} \end{aligned}$$

Observe that $\sigma q = 1$. So

$$\begin{aligned} c(q) &= q^{1+\theta} = \sigma^{-1-\theta} \\ c'(q) &= (1+\theta)\, q^{\theta} = (1+\theta)\, \sigma^{-\theta} \\ c''(q) &= \theta(1+\theta) q^{\theta-1} = \theta(1+\theta)\sigma^{-\theta+1} \end{aligned}$$

Recall from the text that

$$\begin{aligned} p^*_{com} &= (1-\beta)c'(q^*) + \beta\sigma c(q^*) \\ &= (1-\beta)(1+\theta)\sigma^{-\theta} + \beta\sigma\sigma^{-1-\theta} \\ &= ((1-\beta)(1+\theta)+\beta)\sigma^{-\theta} \end{aligned}$$

Thus

$$\begin{aligned} \frac{p^*_{com}\sigma}{c''} &= \frac{((1-\beta)(1+\theta)+\beta)\sigma^{-\theta+1}}{\theta(1+\theta)\sigma^{-\theta+1}} \\ &= \frac{((1-\beta)(1+\theta)+\beta)}{\theta(1+\theta)} \\ &= \frac{(1-\beta+\theta-\theta\beta+\beta)}{\theta(1+\theta)} \\ &= \frac{(1+\theta-\theta\beta)}{\theta(1+\theta)} \\ &= \frac{\frac{1}{\theta}+1-\beta}{1+\theta} \end{aligned}$$

so λ has the formula (28) as claimed.

4.2 Step 2.

Define the function

$$H(\varepsilon_D, \varepsilon_S, \beta, s) \equiv \frac{4}{3} - \left(\frac{4}{3} - \frac{1}{\psi(\varepsilon_D, \varepsilon_S, \beta)+1}\right)\beta s^3 - s. \tag{29}$$

Since $\psi > 0$, this function is strictly decreasing in s. Now $H(\psi, 0) > 0$ and $H(\psi, s) < 0$ for large enough s. Thus there exists a unique solution to the equation $H(s) = 0$. Let $s(\varepsilon_D, \varepsilon_S, \beta)$ be this unique solution. Note that since H is independent of δ, s is constant in δ.

To show that s increases in ε_D, since H decreases in s, it is necessary to show that H increases in ε_D. From (29), this is equivalent to showing that ψ decreases in ε_D. This follows since λ is decreasing in ε_D and ψ is increasing in λ.

By an analogous argument, to show that s decreases in ε_S, it is sufficient that λ increase in ε_S. This is immediate from inspection of (28).

Thus we have proved the comparative statics claims for ε_D, ε_S and δ.

4.3 Step 3

It remains to show that s is decreasing in β for $\varepsilon_D = 1$. From the discussion above, this is equivalent to showing that H decreases in β. We will show that for $\varepsilon_D = 1$, ψ is constant in β. The result immediately follows.

To show that ψ is constant in β, we write it as

$$\psi \equiv \frac{1}{2}\left(\lambda - (1-\beta) + \left[(1-\beta)^2 + \lambda^2 + 2(1+\beta)\lambda\right]^{\frac{1}{2}}\right) > 0$$

$$\lambda = \frac{\varepsilon + (1-\beta)}{1+\frac{1}{\varepsilon}}$$

where $\varepsilon = \varepsilon_S$ and in the formula for λ we use $\varepsilon_D = 1$. Observe that

$$\begin{aligned}\lambda - (1-\beta) &= \frac{\varepsilon + (1-\beta)}{1+\frac{1}{\varepsilon}} - \frac{\left(1+\frac{1}{\varepsilon}\right)(1-\beta)}{1+\frac{1}{\varepsilon}} \\ &= \frac{\varepsilon - \frac{1}{\varepsilon}(1-\beta)}{1+\frac{1}{\varepsilon}}\end{aligned}$$

Substituting into the formula for ψ and ignoring the multiplicative constant $\frac{1}{2}$, and multiplying through by $\left(1+\frac{1}{\varepsilon}\right)$, we need to show that following is constant in β:

$$\varepsilon - \frac{1}{\varepsilon}(1-\beta) + \left[\left(1+\frac{1}{\varepsilon}\right)^2 (1-\beta)^2 + \varepsilon + (1-\beta) + 2\left(1+\frac{1}{\varepsilon}\right)(1+\beta)\left(\varepsilon + (1-\beta)\right)\right]^{\frac{1}{2}}$$

We can write the inside of the bracketed term as

$$\begin{aligned}&\left(1+\frac{1}{\varepsilon}\right)^2 (1-\beta)^2 + (\varepsilon+1)^2 - 2(\varepsilon+1)\beta + \beta^2 + 2\left(1+\frac{1}{\varepsilon}\right)(1+\beta)\left(\varepsilon + (1-\beta)\right) \\ =\ &\left(1+\frac{1}{\varepsilon}\right)^2 - 2\left(1+\frac{1}{\varepsilon}\right)^2 \beta + \left(1+\frac{1}{\varepsilon}\right)^2 \beta^2\end{aligned}$$

$$
\begin{aligned}
& +(\varepsilon+1)^2-2(\varepsilon+1)\beta+\beta^2+2\left(1+\frac{1}{\varepsilon}\right)\left(1+\varepsilon-\beta+(1+\varepsilon)\beta-\beta^2\right) \\
= \; & \left(1+\frac{1}{\varepsilon}\right)^2-2\left(1+\frac{1}{\varepsilon}\right)^2\beta+\left(1+\frac{1}{\varepsilon}\right)^2\beta^2+(\varepsilon+1)^2-2(\varepsilon+1)\beta+\beta^2 \\
& +2\left(1+\frac{1}{\varepsilon}\right)(1+\varepsilon)+2\left(1+\frac{1}{\varepsilon}\right)\varepsilon\beta-2\left(1+\frac{1}{\varepsilon}\right)\beta^2 \\
= \; & \left[\left(1+\frac{1}{\varepsilon}\right)^2+1-2\left(1+\frac{1}{\varepsilon}\right)\right]\beta^2 \\
& -2\left(1+\frac{1}{\varepsilon}\right)^2\beta+\left(1+\frac{1}{\varepsilon}\right)^2+(\varepsilon+1)^2+2\left(1+\frac{1}{\varepsilon}\right)(1+\varepsilon) \\
= \; & \frac{1}{\varepsilon^2}\beta^2-2\left(1+\frac{1}{\varepsilon}\right)^2\beta+\left[\left(1+\frac{1}{\varepsilon}\right)+(1+\varepsilon)\right]^2 \\
= \; & \left(\left[\left(1+\frac{1}{\varepsilon}\right)+(1+\varepsilon)\right]-\frac{\beta}{\varepsilon}\right)^2
\end{aligned}
$$

Substituting this into the earlier bracketed term and taking the square root yields

$$
\begin{aligned}
& \varepsilon-\frac{1}{\varepsilon}(1-\beta)+\left(\left[\left(1+\frac{1}{\varepsilon}\right)+(1+\varepsilon)\right]-\frac{\beta}{\varepsilon}\right) \\
= \; & \varepsilon-\frac{1}{\varepsilon}+\left[\left(1+\frac{1}{\varepsilon}\right)+(1+\varepsilon)\right]
\end{aligned}
$$

which is constant in β as claimed.