Testing Option Pricing Models with Stochastic Volatility, Random Jump and Stochastic Interest Rate

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Abstract

In this paper, we propose a parsimonious GMM estimation and testing procedure for continuous-time option pricing models with stochastic volatility, random jump and stochastic interest rate. Statistical tests are performed on both the underlying asset return model and the risk-neutral option pricing model. Firstly, the underlying asset return models are estimated using GMM with valid statistical tests for model specification. Secondly, the preference related parameters in the risk-neutral distribution are estimated from observed option prices. Our findings confirm that the implied risk premiums for stochastic volatility, random jump and interest rate are overall positive and varying over time. However, the estimated risk-neutral processes are not unique, suggesting a segmented option market. In particular, the deep ITM call (or deep OTM put) options are clearly priced with higher risk premiums than the deep OTM call (or deep ITM put) options. Finally, while stochastic volatility tends to better price long-term options, random jump tends to price the short-term options better, and option pricing based on multiple risk-neutral distributions significantly outperforms that based on a single risk-neutral distribution.

Key Words: Stochastic Volatility, Poisson Jump, Stochastic Interest Rate, Option Pricing, Generalized Method of Moments (GMM).

JEL Classifications: G13, G14, C13, C52

1 Introduction

In this paper, we perform empirical tests of option pricing models with stochastic volatility, random jump and stochastic interest rate. The issue of testing certain option pricing model is two-fold as it involves both the underlying asset return process and the risk-neutral process. From a model specification point of view, one may be concerned whether the underlying process is a reasonable representation of the asset return dynamics. From an option pricing and risk hedging point of view, one may be concerned whether the risk-neutral process provides a reasonable tool for option pricing and risk hedging. In reality, few models share the simplicity with the Black-Scholes model where both the underlying process and the risk-neutral process are nicely tractable. That is, the underlying model can be easily tested and the option pricing and risk hedging performance can be easily evaluated based on the closed form option pricing formula. [1] Various extensions of the Black-Scholes model has been developed in the continuous-time framework. This is primarily due to the fact that the continuous-time models with affine structure can lead to closed form option pricing formulas, thanks to recently developed Fourier inversion technique by Heston (1993) and Scott (1994). Unfortunately estimation and statistical tests of these models are by no means straightforward. In financial applications, due to the unavailability of continuous sampling path, estimation has usually been performed by first discretizing the model and then applying various moment based estimation methods. In the econometrics and statistical literature, new estimation techniques are mostly developed based on simulation methods. The application of these approaches has had varying success due mainly to the need of both discretizing the model and simulating long sample paths. For this reason, in the option pricing literature empirical tests have mostly focused only on the risk-neutral process. [2]

In this paper, the option pricing models are empirically tested for both the underlying asset return process and the risk-neutral process. To test the underlying asset return process, we first derive *exact* moments of the model from the closed form characteristic function. The

[1] Ironically, this is also why the Black-Scholes model can be universally rejected based on its underlying model misspecification and systematic mispricing of options.

[2] The risk-neutral process can be estimated using information from the options market, for example, the nonparametric state price density (SPD) estimation by Aït-Sahalia and Lo (1998), the estimation of nonparametric American option early exercise boundaries by Broadie, Detemple, Ghysels, and Torres (2000), the artificial neutral network by Hutchinson, Lo and Poggio (1994), the implied binomial tree by Rubinstein (1994), and fitting risk-neutral density using Edgeworth expansion by Jarrow and Rudd (1982) and Longstaff (1995), etc.

estimation is then via the generalized method of moments (GMM), which involves neither discretization of the continuous-time process nor simulation of sample paths. In addition to computational efficiency, this procedure provides a valid statistical test of the underlying model specification. To test the risk-neutral process, the option pricing performance of alternative models is evaluated based on the out-of-sample comparison with observed market option prices. Only the preference related parameters in the risk-neutral process and the unobserved stochastic volatility are implied from options market.

Our approach is different from the *implied estimation* method, used in e.g. Bates (1996a, 2000) and Bakshi, Cao and Chen (1997), in which the risk neutral process is implied only from options market.[3] With GMM estimation, our approach provides a valid statistical test of the underlying asset return and interest rate process. Our approach is also different from the studies by e.g. Chernov and Ghysels (2000), Pan (2002) and Jones (2002) in which the underlying asset return process and risk-neutral process are jointly estimated. However, in these studies only a single option quote or equivalently implied volatility at each observation time is used in the estimation. In our approach, given the estimated underlying asset return and interest rate processes, the risk-neutral distribution is then estimated from different segments of the options market. Since the primitive information and the derivative information from different segments of the options market are combined, the estimated premiums of various risk factors are obviously more sensible measures of investors' preference. One of the major empirical findings in this paper is that the risk-neutral process estimated from different segments of the options market is not unique, suggesting a segmented option market. In particular, the deep ITM call (or deep OTM put) options are clearly priced with higher risk premiums than the deep OTM call (or deep ITM put) options. Our results are in line with Jones (2001) in which a non-linear factor analysis is performed on the S&P 500 index option returns. The implication of the above findings is that when the models are used in pricing options or hedging risks, multiple risk-neutral distributions instead of a single risk-neutral distribution should be used.

The structure of this paper is as follows. In section 2, the underlying model of asset return and interest rate is specified with a detailed discussion of the statistical properties, and the closed form option pricing formula is derived from the risk-neutral model. In section 3,

[3] For discussion of various issues, e.g. model identification and statistical tests, related to the implied estimation method, see Bates (1996a, 1996b).

an estimation procedure of the underlying asset return and interest rate models is proposed using GMM. Statistical tests on various model specifications are performed. In section 4, the preference related parameters and the unobserved stochastic volatility are implied from options market. The estimated risk-neutral process is used to evaluate the option pricing performance of alternative models. A robustness check on our results is also performed using the FTSE 100 index options. In section 5, we conclude.

2 The Option Pricing Model

2.1 The Underlying Process of Asset Return and Interest Rate

Empirical evidence overwhelmingly suggests that the original Black-Scholes (1973) option pricing model is inconsistent with the distribution of many financial asset returns and thus generates systematic biases of option prices. Various alternative models have been developed over the past two decades or so to relax the unrealistic assumptions. Firstly, the assumption of constant interest rate is relaxed to allow for stochastic interest rate.[4] Secondly, since the sampling paths of asset returns are believed to be discontinuous due to abnormal information shocks, the jump-diffusion (JD) models are introduced.[5] Thirdly, it is widely believed that the volatility of asset returns tends to be time-varying and occasionally clustered, which leads to various stochastic volatility (SV) models.[6] Combinations of the above extensions lead to the stochastic volatility and stochastic interest rate models,[7] as well as jump-diffusion with stochastic volatility and/or stochastic interest rate models.[8] Simulation and empirical studies have shown that the extension in each of the above directions can have important impact on option pricing and risk hedging.[9]

This paper focuses on the following data generating process (DGP) for the asset price S_t

[4]As in Merton (1973), Amin and Jarrow (1992), and Madan and Chang (1996).

[5]See Merton (1976) and Bates (1988).

[6]See e.g. Hull and White (1987), Johnson and Shanno (1987), Wiggins (1987), Scott (1987, 1991), Chesney and Scott (1989), Melino and Turnbull (1990), Stein and Stein (1991), Cao (1992), and Heston (1993).

[7]For example, Bailey and Stulz (1989), Amin and Ng (1993), and Scott (1997).

[8]See Bates (1996a, 2000), Scott (1997), Bakshi, Cao and Chen (1997), and Pan (2002).

[9]See e.g. Rabinovitch (1989), Merton (1976), Hull and White (1987), Bailey and Stulz (1989), and Jiang (2000) for simulation studies, and Ball and Torous (1985), Melino and Turnbull (1992), Bates (1996a,2000), Bakshi, Cao and Chen (1997), Jiang and van der Sluis (1999), Chernov and Ghysels (2000), and Pan (2002) for empirical studies.

as proposed in Bates (2000) and Pan (2002):

$$
\begin{aligned}
dS_t &= (r_t - d + \eta V_t + (\lambda\mu - \lambda^*\mu^*) + \alpha_0)S_t dt + V_t^{1/2} S_t dW_t \\
&\quad +(J_t - 1)S_t dq_t(\lambda) - \lambda\mu S_t dt \\
dV_t &= \kappa(\gamma - V_t)dt + \sigma_v V_t^{1/2} dW_t^v \\
dr_t &= \beta(\alpha - r_t)dt + \sigma_r r_t^{1/2} dW_t^r \\
dW_t dW_t^v &= \rho dt, \qquad t \in [0, T]
\end{aligned} \tag{1}
$$

where S_t, r_t and d denote the asset price (ex-dividend), the interest rate, and the dividend yield respectively, $dq_t(\lambda)$ is assumed to be statistically independent of $J_t, dW_t, dW_t^v, dW_t^r$, and dW_t, dW^v are assumed to be correlated with correlation ρdt, but uncorrelated with J_t, dW_t^r. Similar to Pan (2002), an explicit stochastic process could also be specified for the dividend yield. As we shall see from the data in Section 3.2, relative to the S&P 500 index return, the daily change in dividend yield is rather small in magnitude measured by both the mean and standard error. Therefore, for simplicity, we assume non-random dividend yield in this paper.

The above model assumes a jump-diffusion process with stochastic volatility for the asset price and a stochastic process for the interest rate (the SVJ-SI model, hereafter). The stochastic volatility follows the square-root process, as specified in e.g. Bailey and Stulz (1989) and Heston (1993), and can be correlated with the asset return process. A negative correlation ($\rho < 0$) would induce the stylized *leverage effect* for asset returns, see Black (1976). The jump component is a compound Poisson process and $q_t(\lambda)$ is assumed to be iid over time with Prob$(dq_t(\lambda) = 1) = \lambda dt$, Prob$(dq_t(\lambda) = 0) = 1 - \lambda dt$. The jump size J_t is assumed to be log-normally distributed with $\ln J_t \sim$ iid $\mathrm{N}(\mu_J, \sigma_J^2))$ where $\mu_J = \ln(1+\mu) - \frac{1}{2}\sigma_J^2$. The last term $-\lambda\mu dt$ compensates for the instantaneous change in expected asset return due to random jump. The drift term consists of the interest rate, the dividend yield, the risk premium ηV_t for stochastic volatility, the risk premium $(\lambda\mu - \lambda^*\mu^*)$ for random jump, and the risk premium α_0 associated with the price risk, etc. Since the last two risk premia can not be disentangled, in our estimation we denote $\mu_0 = (\lambda\mu - \lambda^*\mu^*) + \alpha_0$. The interest rate model follows the square-root process specified in Cox, Ingersoll and Ross (CIR, hereafter) (1985b). The SVJ-SI model has also been the subject of other studies, e.g. Bates (1996a, 2000), Scott (1997), Bakshi, Cao and Chen (1997), Bakshi and Madan (2000), and Pan (2002), and nests many other models as special cases. For instance, (i) the model with stochastic volatility and random jump (the SVJ model, hereafter) with parameter restrictions $\beta = \sigma_r = 0$, (ii) the model with random jump (the JD model, hereafter) with $\beta = \sigma_r = \kappa = \sigma_v = 0$, (iii) the model with stochastic volatility (the SV model, hereafter) with $\beta = \sigma_r = 0$ and $\lambda = 0$, and (iv) the model with constant volatility and constant interest rate, namely the Black-Scholes and Merton model (the BS model, hereafter) with $\beta = \sigma_r = \kappa = \sigma_v = 0$ and $\lambda = 0$. Exceptions are models with jumps in the underlying volatility, e.g. the regime-switching model of Naik (1993), models with random jump intensity proportional to the stochastic volatility, e.g. Andersen, Benzoni and Lund (2002) and Pan (2002), models with non-affine SV processes, e.g. Andersen, Benzoni and Lund (2002) and Jones (2002), and models with multivariate SV processes, e.g. Bates (2000).

2.2 Statistical Properties

The statistical properties of the square-root process for the stochastic volatility and interest rate are well-known in the literature. Due to the lack of explicit solution to the asset return process with SV and random jump as specified in (1), however, there have been few attempts to formally derive its statistical properties. To our knowledge, the only attempts are made by Das and Sundaram (1999) and Jiang and Knight (2002). In this paper, we rely on the Kolmogorov forward (or the Fokker-Planck) equation to solve for the characteristic functions of the asset return as well as the joint asset returns and derive both the conditional and unconditional moment conditions.

Lemma 1: Given the stochastic process defined in (1), the joint conditional characteristic function (CCF) of $\Delta \ln S_{t+\Delta} = \ln S_{t+\Delta} - \ln S_t$ and $r_{t+\Delta}$ on S_t, V_t, r_t can be derived as

$$\begin{aligned} & \psi(\Delta \ln S_{t+\Delta}, r_{t+\Delta}; \phi, \varphi | S_t, V_t, r_t) = E[e^{i\phi\Delta \ln S_{t+\Delta} + i\varphi r_{t+\Delta}} | S_t, V_t, r_t] \\ = \quad & \exp\{C(\phi, 0, \varphi, \Delta) + D(\phi, 0, \Delta)V_t + B(\phi, \varphi, \Delta)r_t + \Delta\lambda(e^{i\phi\mu_J - \frac{1}{2}\phi^2\sigma_J^2} - 1)\} \end{aligned} \tag{2}$$

and the joint unconditional CF of $\Delta \ln S_{t+\Delta} = \ln S_{t+\Delta} - \ln S_t$ and $r_{t+\Delta}$ can be derived as

$$\begin{aligned} & \psi(\Delta \ln S_{t+\Delta}, r_{t+\Delta}; \phi, \varphi) = E[e^{i\phi\Delta \ln S_{t+\Delta} + i\varphi r_{t+\Delta}}] \\ = \quad & \exp\{C(\phi, 0, \varphi, \Delta) - \frac{2\kappa\gamma}{\sigma_v^2} \ln\left(1 - \frac{\sigma_v^2 D(\phi, 0, \Delta)}{2\kappa}\right) \\ & -\frac{2\beta\alpha}{\sigma_r^2} \ln\left(1 - \frac{\sigma_r^2 B(\phi, \varphi, \Delta)}{2\beta}\right) + \Delta\lambda(e^{i\phi\mu_J - \frac{1}{2}\phi^2\sigma_J^2} - 1)\} \end{aligned} \tag{3}$$

where $C(\cdot), D(\cdot)$ and $B(\cdot)$ are given in the Appendix.

Proof: See Appendix.

Corollary to Lemma 1: The asset return $\Delta \ln S_t$ and interest rate r_t defined in (1) are stationary processes with the following first two conditional moments

$$
\begin{aligned}
&E[\Delta \ln S_{t+\Delta}|S_t, V_t, r_t] = (\mu_0 - d + \lambda\mu_J - \lambda\mu)\Delta + \alpha\Delta + \frac{1}{\beta}(r_t - \alpha)(1 - e^{-\beta\Delta}) + \psi_0 \\
&V[\Delta \ln S_{t+\Delta}|S_t, V_t, r_t] = \psi_1 + \frac{1}{\kappa}((e^{-\kappa\Delta} + \kappa\Delta - 1)\gamma + (1 - e^{-\kappa\Delta})V_t) + \lambda\Delta(\mu_J^2 + \sigma_J^2) \\
&E[r_{t+\Delta}|S_t, V_t, r_t] = r_t + (1 - e^{-\beta\Delta})(\alpha - r_t) \\
&V[r_{t+\Delta}|S_t, V_t, r_t] = \frac{\sigma_r^2}{2\beta}(1 - e^{-\beta\Delta})(2r_t e^{-\beta\Delta} + (1 - e^{-\beta\Delta})\alpha) \qquad (4)
\end{aligned}
$$

where $\mu_0 = (\lambda\mu - \lambda^*\mu^*) + \alpha_0, \psi_0 = \frac{(2\eta-1)e^{-\kappa\Delta}}{2\kappa}((e^{\kappa\Delta} - 1)(V_t - \gamma) + e^{\kappa\Delta}\kappa\gamma\Delta)$ and $\psi_1 = \frac{\sigma_r^2}{2\beta^3}((\alpha - 2r_t)e^{-2\beta\Delta} + 4(\alpha - \beta\Delta(r_t - \alpha))e^{-\beta\Delta} + 2(\alpha\Delta + r_t) - 5\alpha) + \frac{(2\eta-1)e^{-2\kappa\Delta}}{8\kappa^3}[2((2\eta - 1)(e^{2\kappa\Delta} + 2\Delta e^{\kappa\Delta} - 1)\sigma_v^2 + 4e^{\kappa\Delta}(e^{\kappa\Delta} + \Delta - 1)\kappa\rho\sigma_v)V_t + ((2\eta - 1)\sigma_v^2(1 + e^{2\kappa\Delta}(16\kappa^3\rho\Delta + 2\kappa\Delta - 5) + 4e^{\kappa\Delta}(\kappa\Delta + 1)) + 8e^{\kappa\Delta}\kappa\rho\sigma_v(\kappa\Delta + 2 - 2e^{\kappa\Delta}))\gamma]$ and the following first four unconditional moments for asset return

$$
\begin{aligned}
&E[\Delta \ln S_t] = (\mu_0 + \lambda\mu_J - \lambda\mu - d + \alpha + (\eta - \frac{1}{2})\gamma)\Delta \\
&V[\Delta \ln S_t] = \gamma\Delta + \Delta\lambda\mu_J^2 + \Delta\lambda\sigma_J^2 + \frac{\alpha\sigma_r^2\Delta}{\beta^2} - \frac{\alpha\sigma_r^2 - e^{-\beta\Delta}\alpha\sigma_r^2}{\beta^3} - \frac{\gamma\rho\sigma_v\Delta - 2\gamma\eta\rho\sigma_v\Delta}{\kappa} \\
&\quad + \frac{\gamma\rho\sigma_v - e^{-\kappa\Delta}\gamma\rho\sigma_v - 2\gamma\eta\rho\sigma_v + 2e^{-\kappa\Delta}\gamma\eta\rho\sigma_v + \gamma\sigma_v^2\Delta/4 - \gamma\eta\sigma_v^2\Delta + \gamma\eta^2\sigma_v^2\Delta}{\kappa^2} \\
&\quad + \frac{\gamma\eta\sigma_v^2 - \gamma\sigma_v^2/4 + e^{-\kappa\Delta}\gamma\sigma_v^2/4 - e^{-\kappa\Delta}\gamma\eta\sigma_v^2 - \gamma\eta^2\sigma_v^2 + e^{-\kappa\Delta}\gamma\eta^2\sigma_v^2}{\kappa^3} \\
&E[(\Delta \ln S_t - E[\Delta \ln S_t])^3] = \frac{1}{8\beta^5\kappa^5}\Big[e^{-(3\beta+\kappa)\Delta}(24e^{(2\beta+\kappa)\Delta}\alpha\kappa^5\sigma_r^4(2 + \beta\Delta) \\
&\quad + 3e^{3\beta\Delta}\beta^5\gamma\sigma_v(2\kappa\rho + (2\eta - 1)\sigma_v)(2(1 - 2\eta)^2\sigma_v^2 + (2\eta - 1)\kappa\sigma_v(8\rho + (2\eta - 1)\sigma_v\Delta) \\
&\quad + 4\kappa^2(1 + (2\eta - 1)\rho\sigma_v\Delta)) + 3e^{(3\beta+\kappa)\Delta}(8\alpha\kappa^5\sigma_r^4(-2 + \beta\Delta) \\
&\quad + \beta^5\gamma\sigma_v(2\kappa\rho + (2\eta - 1)\sigma_v)(2(2\eta - 1)^2\sigma_v^2 + 4\kappa^3\Delta - (2\eta - 1)\kappa\sigma_v(8\rho - (2\eta - 1)\sigma_v\Delta) \\
&\quad + 4\kappa^2(-1 + (2\eta - 1)\rho\sigma_v\Delta))) + 8e^{(3\beta+\kappa)\Delta}\beta^5\kappa^5\Delta\lambda\mu_J^3 + 24e^{(3\beta+\kappa)\Delta}\beta^5\kappa^5\Delta\lambda\mu_J\sigma_J^2)\Big] \\
&E[(\Delta \ln S_t - E[\Delta \ln S_t])^4] = 3V[\Delta \ln S_t]^2 + \frac{1}{32\beta^7\kappa^7}(e^{-2(\beta+\kappa)\Delta}\Big[32e^{2(\beta+\kappa)\Delta}\beta^7\kappa^7\Delta\lambda\mu_J^4 \\
&\quad + 192e^{2(\beta+\kappa)\Delta}\beta^7\kappa^7\Delta\lambda\mu_J^2\sigma_J^2 + 3(16e^{2\kappa\Delta}\alpha\kappa^7\sigma_r^6 + e^{2\beta\Delta}\beta^7\gamma(1 - 2\eta)^2\sigma_v^4(\sigma_v - 4\kappa\rho - 2\eta\sigma_v)^2 \\
&\quad + 64e^{(\beta+2\kappa)\Delta}\alpha\kappa^7\sigma_r^6(7 + 5\beta\Delta\beta^2\Delta^2) + e^{2(\beta+\kappa)\Delta}(16\alpha\kappa^7\sigma_r^6(-29 + 10\beta\Delta) \\
&\quad + \beta^7\gamma\sigma_v^2(-29(1 - 2\eta)^4\sigma_v^4 + 32\kappa^5(\Delta + 4\rho^2\Delta) + 2(2\eta - 1)^3\kappa\sigma_v^3(-116\rho + 5(2\eta - 1)\sigma_v\Delta) \\
&\quad - 16(1 - 2\eta)^2\kappa^2\sigma_v^2(6 + 35\rho^2 - 5(2\eta - 1)\rho\sigma_v\Delta) - 48(2\eta - 1)\kappa^3\sigma_v(8\rho + 8\rho^3 + (1 - 2\eta)\sigma_v\Delta
\end{aligned}
$$

$$
\begin{aligned}
&+4(1-2\eta)\rho^2\sigma_v\Delta)+32\kappa^4(-1-8\rho^2+6(2\eta-1)\rho\sigma_v\Delta+4(2\eta-1)\rho^3\sigma_v\Delta)))\\
&+4e^{(2\beta+\kappa)\Delta}\beta^7\gamma\sigma_v^2(7(1-2\eta)^4\sigma_v^4+(2\eta-1)^3\kappa\sigma_v^3(56\rho+5(2\eta-1)\sigma_v\Delta)\\
&+16\kappa^5\rho^2\Delta(2+(2\eta-1)\rho\sigma_v\Delta)+(1-2\eta)^2\kappa^2\sigma_v^2(24+136\rho^2+40(2\eta-1)\rho\sigma_v\Delta\\
&+(1-2\eta)^2\sigma_v^2\Delta^2)+4(2\eta-1)\kappa^3\sigma_v(24\rho^3+3(2\eta-1)\sigma_v\Delta\\
&+24(2\eta-1)\rho^2\sigma_v\Delta+2\rho(12+(1-2\eta)^2\sigma_v^2\Delta^2))+4\kappa^4(2+12(2\eta-1)\rho\sigma_v\Delta\\
&+16(2\eta-1)\rho^3\sigma_v\Delta+\rho^2(16+5(1-2\eta)^2\sigma_v^2\Delta^2)))+32e^{2(\beta+\kappa)\Delta}\beta^7\kappa^7\Delta\lambda\sigma_J^4))\Big] \qquad (5)
\end{aligned}
$$

Proof: See Appendix.

Remark: Most interesting are the third and fourth moments. In the third moment, the parameter ρ is associated with the diffusion part due to the presence of asymmetric volatility and the parameter μ_J is associated with the jump part due to the non-zero expected jump size. It can be verified that the asset returns can be asymmetric with either positive or negative skewness. From the fourth moment, we can see that the asset return distribution can have positive excess kurtosis and exhibit fat tails. In other words, the asset return distribution defined in the model can be skewed with fat tails, which is consistent with the stylized facts of asset returns.

Lemma 2: Given the stochastic process defined in (1), the joint unconditional CF of $\Delta\ln S_{t+\Delta}=\ln S_{t+\Delta}-\ln S_t$ and $\Delta\ln S_\Delta=\ln S_\Delta-\ln S_0$, $t\geq\Delta$, can be derived as

$$
\begin{aligned}
&\psi(\Delta\ln S_{t+\Delta},\Delta\ln S_\Delta;\phi,\varphi)=E[e^{i\phi\Delta\ln S_{t+\Delta}+i\varphi\Delta\ln S_\Delta}]\\
=\;&\exp\{C(\phi,0,0,\Delta)+C(0,-iD(\phi,0,\Delta),-iB(\phi,0,\Delta),t-\Delta)+C(\varphi,-iD^*,-iB^*,\Delta)\\
-\;&\frac{2\kappa\gamma}{\sigma_v^2}\ln\left(1-\frac{\sigma_v^2(iD^*+D(\varphi,-iD^*,\Delta))}{2\kappa}\right)-\frac{2\beta\alpha}{\sigma_r^2}\ln\left(1-\frac{\sigma_r^2(iB^*+B(\varphi,-iB^*,\Delta))}{2\beta}\right)\\
+\;&\Delta\lambda(e^{i\phi\mu_J-\frac{1}{2}\phi^2\sigma_J^2}+e^{i\varphi\mu_J-\frac{1}{2}\varphi^2\sigma_J^2}-2)\} \qquad (6)
\end{aligned}
$$

where $D^*=iD(\phi,0,\Delta)+D(0,-iD(\phi,0,\Delta),t-\Delta)$, $B^*=iB(\phi,0,\Delta)+B(0,-iB(\phi,0,\Delta),t-\Delta)$ and $C(\cdot)$, $D(\cdot)$ and $B(\cdot)$ are given in the Appendix.

Proof: See Appendix.

Corollary to Lemma 2: (a) The asset return $\Delta\ln S_t$ is correlated over time with

$$
Cov[\Delta\ln S_{t+\Delta},\Delta\ln S_\Delta]\;=\;\frac{1}{2\beta^3}e^{-\beta(t+\Delta)}(e^{\beta\Delta}-1)^2\alpha\sigma_r^2,\;\;t\geq\Delta \qquad (7)
$$

and (b) the squared asset return $(\Delta\ln S_t)^2$ is correlated over time with

$$
Cov[(\Delta\ln S_{t+\Delta})^2,(\Delta\ln S_\Delta)^2]
$$

$$
\begin{aligned}
&= \frac{\alpha\sigma_r^6}{4\beta^7}[e^{-2\beta t}(e^{-2\beta\Delta} + e^{2\beta\Delta} - 4(e^{-\beta\Delta} + e^{\beta\Delta}) + 6) + e^{-\beta t}(e^{\beta\Delta} - 8(1+\beta\Delta)(2 \\
&- (1+\beta\Delta)e^{-\beta\Delta})] + \frac{\gamma\sigma_v^2}{2\kappa^3}e^{-\kappa t}((e^{\Delta\kappa} + e^{-\kappa\Delta} - 2)(1+4\rho^2) + 4\kappa\Delta\rho^2(e^{-\kappa\Delta} - 1)) \quad (8)
\end{aligned}
$$

which is strictly positive but decreases as t increases.

Remark: It is noted that the squared return behaves quite differently than the volatility process.

2.3 The Closed-Form Option Pricing Formula

In a general equilibrium framework, such as CIR (1985a), Ahn and Thompson (1988), and Bates (1988, 1991), European options that pay off only at maturity are priced as their expected discounted payoffs under an equivalent "risk-neutral" representation. With the asset return process and spot interest rate process specified in (1), following CIR (1985b) and Bates (1988), we have the following lemma.

Lemma 3: The risk-neutral specification corresponding to the model defined in (1) under certain restrictions is given by[10]

$$
\begin{aligned}
dS_t/S_t &= (r_t - d)dt + V_t^{1/2}dW_t^* + (J_t^* - 1)dq_t^*(\lambda^*) - \lambda^*\mu^* dt \\
dV_t &= (\kappa(\gamma - V_t) + \Phi_v)dt + \sigma_v V_t^{1/2} dW_t^{v*} \\
dr_t &= (\beta(\alpha - r_t) + \Phi_r)dt + \sigma_r r_t^{1/2} dW_t^{r*} \\
dW_t^* dW_t^{v*} &= \rho dt, \quad t \in [0, T] \qquad (9)
\end{aligned}
$$

where $\Phi_v = Cov(dV_t, dU_w/U_w), \Phi_r = Cov(dr_t, dU_w/U_w), \lambda^* = \lambda E(1 + \Delta U_w/U_w), \mu^* = \mu + \frac{Cov(J_t, \Delta U_w/U_w)}{E[1+\Delta U_w/U_w]}$, and $q_t^*(\lambda^*)$ is a Poisson process with intensity λ^*, J_t^* is lognormally distributed with $\ln J_t^* \sim N(\mu_J^*, \sigma_J^2)$, U_w is the marginal utility of nominal wealth of the representative investor, $\Delta U_w/U_w$ is the random percentage jump conditional on a jump occurring and dU_w/U_w is the percentage shock in the absence of jumps.

Proof: See Appendix.

It is noted that in the "risk-neutral" specification, all risk factors are appropriately compensated. In this paper, we impose tractable functional forms on the risk premium

$$
\Phi_v = \xi V_t, \quad \Phi_r = \zeta r_t \qquad (10)
$$

[10] The basic restrictions include that firstly, the process of the optimally invested wealth follows a stochastic volatility jump diffusion with constant parameters. Secondly, the utility function is assumed to be time-separable and isoelastic, see detailed discussion in Bates (1996a) and CIR (1985b).

That is, the risk premium is proportional to the variance or risk of the process.[11] These risk premiums lead to the following risk-neutral process.

$$\begin{aligned} dS_t &= (r_t - d)S_t dt + V_t^{1/2} S_t dW_t^* + (J_t^* - 1)S_t dq_t^*(\lambda^*) - \lambda^* \mu^* S_t dt \\ dV_t &= (\kappa(\gamma - V_t) + \xi V_t)dt + \sigma_v V_t^{1/2} dW_t^{\sigma *} \\ dr_t &= (\beta(\alpha - r_t) + \zeta r_t)dt + \sigma_r r_t^{1/2} dW_t^{r*} \\ dW_t^* dW_t^{v*} &= \rho dt, \quad t \in [0, T] \end{aligned} \tag{11}$$

with $\ln J_t^* \sim N(\mu_J^*, \sigma_J^2)$ and $\mu_J^* = \ln(1 + \mu^*) - \frac{1}{2}\sigma_J^2$. The drift terms for the SV and SI processes can be rewritten as $(\gamma_v - \kappa^* V_t)$ and $(\gamma_r - \beta^* r_t)$ respectively with $\gamma_v = \kappa\gamma, \gamma_r = \beta\alpha, \kappa^* = \kappa - \xi, \beta^* = \beta - \zeta$. The last term $-\lambda^*\mu^* S_t dt$ in the return process compensates the instantaneous change in expected asset return due to random jump under the risk-neutral measure.

Corollary to Lemma 3: Given the risk-neutral process in (11), the price of a zero coupon bond with maturity $\tau = T - t$ is given by

$$B(t, \tau) = a(t, \tau)e^{-b(t,\tau)r_t} \tag{12}$$

where $a(t, \tau) = [\frac{2\Gamma e^{(\beta^* + \Gamma)\tau/2}}{(\beta^* + \Gamma)(e^{\Gamma\tau} - 1) + 2\Gamma}]^{2\gamma_r/\sigma_r^2}, b(t, \tau) = \frac{2(e^{\Gamma\tau} - 1)}{(\beta^* + \Gamma)(e^{\Gamma\tau} - 1) + 2\Gamma}, \Gamma = (\beta^{*2} + 2\sigma_r^2)^{1/2}$. And the price of an European call option at time t with maturity T and strike price K is given by

$$C(S_t, t) = S_t \Pi_1(S_t, t; K, T, r_t, V_t) - KB(t, \tau)\Pi_2(S_t, t; K, T, r_t, V_t) \tag{13}$$

where the risk-neutral probabilities, Π_1 and Π_2, are inverted from the respective characteristic functions as given by $\Pi_j(S_t, t; K, T, r_t, V_t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{exp\{-i\phi \ln K\} f_j(t, \tau, S_t, r_t, V_t; \phi)}{i\phi}]d\phi$ for $j = 1, 2$, with the characteristic functions f_j given in the appendix.

Proof: See Appendix.

Remark: The bond price is derived in CIR (1985b). Fourier inversion technique proposed by Heston (1993) and Scott (1994) can be applied to solve for closed form European option pricing formula, see e.g. Bates (1996a), Scott (1997), and Bakshi, Cao, and Chen (1997).

[11] Strict linearity of the volatility risk premium can be supported under log utility function when asset return volatility and market risk have a common component of a particular form, see Bates (1996a). The risk premium of stochastic volatility assumed here is the same as that in Bates (1996a) and the risk premium of interest rate is the same as that in CIR (1985b).

3 Estimation of Asset Return and Interest Rate Processes

In this paper, we identify and estimate the objective process and the risk-neutral process specified in the above section using information from both the underlying asset returns and the options market. The optimal procedure would be to estimate the whole parameter set using both the *primitive* information and the *derivative* information simultaneously.[12] Yet, in all available such attempts, e.g. Chernov and Ghysels (2000), Pan (2002), and Jones (2002), only a single observed option (namely a short-maturity at-the-money option) quote per observation time is used in estimation. Using only a single option quote or equivalently implied volatility, however, would exclude us from studying the properties of the overall options market. Since the preference related parameters are only determined by the information in the options market given the underlying asset return process, in this paper we propose a two-step estimation procedure. Namely in the first step the underlying model is estimated from asset return observations and in the second step the preference related parameters are estimated from options market. Apart from its relatively easy implementation, other advantages of the two-step estimation procedure include that the underlying model specification for asset return and interest rate can be empirically tested using valid statistics and the risk-neutral distribution can be estimated for different segments of the options market.

3.1 GMM Estimation of the Asset Return and Interest Rate Processes

Estimation of nonlinear latent variable models, such as the model in (1), is by no means a trivial task. The difficulty arises due to the fact that the latent variable, namely the stochastic volatility, is unobservable and thus the models cannot be estimated using standard maximum likelihood (ML) method. Over the past few years, remarkable progress has been made in the field of statistics and econometrics regarding the estimation of nonlinear latent variable models in general and SV models in particular. Various estimation methods for SV models have been proposed, but they are mostly simulation-based and very computationally intensive. For instance, Andersen, Benzoni, and Lund (2002) and Chernov and Ghysels (2000) use EMM to estimate the process of S&P 500 index returns, Jones (1998) and Eraker (2001)

[12] See the approach developed in Bates (1996a) and recent attempts of estimating the asset return process under both the objective and risk-neutral measures by Chernov and Ghysels (2000) using the efficient method of moment (EMM), by Pan (2002) using GMM and by Jones (2002) using the Markov chain Monte Carlo (MCMC) approach.

apply the Markov chain Monte Carlo (MCMC) to estimate the exchange rate process and individual stock return process respectively.

In this paper, we exploit the fact that the characteristic functions of the asset return and joint asset returns can be derived analytically, thus the *exact* moments of asset returns are available. The GMM estimation in this paper is based on exact moments of the continuous-time process.[13] Compared to the simulation based methods, the advantages of the GMM estimation are its relatively easy implementation in terms of computing time and the better finite sample properties. Let

$$\epsilon_t \quad = \quad \Delta \ln S_{t+\Delta} - E[\Delta \ln S_{t+\Delta}]$$

be the demeaned asset return process. The expectations of ϵ_t are calculated exactly as in Section 2. There are obviously infinitely many moments that may be used in GMM estimation. The primary guidance of the moment selection in this paper is the Monte Carlo evidence in Andersen and Sørensen (1996) on the GMM estimation of a discrete-time SV model. Firstly, in determining the number of moments used in the estimation, we keep in mind the following fundamental trade-off: inclusion of additional moments improves estimation performance for a given degree of precision in the estimation of the weighting matrix, but in finite samples this must be balanced against the deterioration in the estimate of the weighting matrix as the number of moments increases. Secondly, very high order moments should be avoided due to their erratic finite sample behavior caused by the presence of fat tails in the asset return distribution. Asymptotic normality of the GMM estimator requires finite variance of the moment conditions and good estimates of these quantities in finite samples. Thus our moment selection tends to focus on the lower order moments, which is consistent with Andersen and Sørensen (1996) and Jacquier, Polson and Rossi (1994). Thirdly, different from the discrete-time SV model, the absolute moments of the asset returns can not be derived for the continuous-time model. The Monte Carlo evidence in Andersen and Sørensen (1996), however, suggests that inclusion of these kinds of moments is in general unlikely to improve estimation performance and at best the gains are quite minor. Keeping in mind the

[13]GMM is also used by Andersen (1994), Andersen and Sørensen (1996) to estimate discrete-time SV models and Ho, Perraudin and Sørensen (1996) and Pan (2002) to estimate continuous-time SV models. Chacko and Viceira (1999), Singleton (2000) and Jiang and Knight (2002) propose using the empirical characteristic to estimate the general affine models.

above issues, we consider the following moments for the asset return process,

$$f_t(\theta) = \begin{bmatrix} \epsilon_t^k - E[\epsilon_t^k] \\ \epsilon_t \epsilon_{t+1} - E[\epsilon_t \epsilon_{t+1}] \\ \epsilon_t^2 \epsilon_{t+\tau}^2 - E[\epsilon_t^2 \epsilon_{t+\tau}^2] \end{bmatrix}$$

$$k = 1, 2, ...; \quad \tau = 1, 2, ...; \quad t = 1, 2, ..., T \tag{14}$$

The exact moment conditions are chosen with further considerations to the identification and estimation efficiency of particular model. First of all, the jump component is only reflected in the unconditional moments, thus the first group of moment conditions is important for the estimation of jump parameters. Since stochastic volatility and random jump both allow for skewness and excess kurtosis, it is important to include the fifth moment for the estimation of jump parameters, i.e. we set $k = 1, 2, 3, 4, 5$. Secondly, the autocorrelation of asset return is determined partly by the risk premium of stochastic volatility in the drift term and the autocorrelation of squared asset return is determined by the dynamics of the stochastic volatility process and its correlation with asset returns. Thus the second and third group of moment conditions is important for the identification of the volatility risk premium and volatility dynamics. Since the autocorrelation is varying over time, we use these moment conditions with different lags, namely $\tau = 1, 2, 3, 4, 5$.

Similarly, let

$$\epsilon_t \;\; = \;\; r_{t+\Delta} - E[r_{t+\Delta} \mid r_t], \quad t = 1, 2, ..., T$$

be the demeaned interest rate process. Again, the expectations of ϵ_t are calculated exactly as in Section 2. In this paper, the same moment conditions as in Chan, Karolyi, Longstaff and Sanders (1992) are used in the estimation of the interest rate process, namely the first two conditional moments in (4) with the lagged variable as instrumental variable, except that the moment conditions in our paper are exact as they are derived from the continuous-time model.

Denote the vector of the population moment conditions as listed in (14) by $f_t(\theta)$, and the vector of corresponding sample moment conditions with sample size T by $g_T(\theta)$, the GMM estimator is defined as

$$\hat{\theta}_T = \arg\min_{\theta \in \Theta} \{\mathcal{J}_T(\theta) = g_T'(\theta) \mathcal{W}_T(\theta) g_T(\theta)\} \tag{15}$$

where Θ denotes the permissible parameter space and $\mathcal{W}_T$ a positive definite weighting matrix which is chosen to yield the smallest asymptotic covariance matrix of the GMM estimator of θ as in Hansen (1982). Under regularity conditions, the estimator $\hat{\theta}_T$ is consistent and asymptotically normal, i.e.

$$T^{1/2}(\hat{\theta}_T - \theta) \sim AN(0, V_T) \tag{16}$$

and a consistent estimator of V_T is given by $\hat{V}_T = \frac{1}{T}(\hat{D}'(\theta)\hat{S}^{-1}(\theta)\hat{D}'(\theta))^{-1}$, where $\hat{D}(\theta)$ is the Jacobian matrix of $g_T(\theta)$ with respect to θ evaluated at the estimated parameters, and $\hat{S}(\theta)$ is a consistent estimator of $S(\theta) = E[f_t(\theta)f_t'(\theta)]$.

Under the null hypothesis that the model is true, the minimized value of $\mathcal{J}_T(\theta)$ in (15) is χ^2 distributed with degree of freedom equal to the number of orthogonality conditions net of the number of parameters to be estimated. This χ^2 statistic provides a goodness-of-fit test for the model, and a high value of this statistic suggests that the model is misspecified. Furthermore, to test restrictions imposed by various submodels on the general unrestricted model, the hypothesis tests developed by Newey and West (1987) can be used. They show that for a general null hypothesis of the form $H_0 : h(\theta) = 0$, where $h(\theta)$ is a vector of order q with q restrictions on the models, the test statistic

$$\mathcal{R} = T[\mathcal{J}_T(\widetilde{\theta}) - \mathcal{J}_T(\hat{\theta})] \tag{17}$$

is asymptotically χ^2-distributed with q degrees of freedom, where $\widetilde{\theta}$ is the restricted estimator of parameter θ. This test statistic is the normalized difference of the restricted ($\mathcal{J}_T(\widetilde{\theta})$) and unrestricted ($\mathcal{J}_T(\hat{\theta})$) objective function values for the efficient GMM estimator and is analogous to the likelihood ratio test.

3.2 The Data Set and Estimation Results

The data for S&P 500 index consists of daily observations over the period from 1980 to 1995. We set aside the last year of data (1995, i.e. the year in which options data are observed) in order to perform the out-of-sample test of the risk-neutral model, thus the model is estimated using the data over 1980 through 1994. The daily 3-month T-bill rates are used as proxy of the "instantaneous" rate.[14] The daily S&P 500 index dividend yield is extracted from

[14] As justified in Jiang (1998), the use of 3-month T-bill rates as the proxy of spot rate is a necessary comprise between literally taking an "instantaneous" rate, say overnight rates, and avoiding some of the associated spurious microstructure effects.

the S&P DRI data base. The summary statistics of daily S&P 500 index returns, 3-month T-bill rates and dividend yields are reported in Panel A of Table 1, from which we can see that the daily S&P 500 index returns are skewed and have positive excess kurtosis (> 3). Evidently, the daily changes in dividend yield are small in magnitude relative to the S&P 500 index returns as measured by both the mean and standard error, justifying the assumption of non-random dividend yield. As far as dynamic properties, the interest rate changes have much higher first-order autocorrelation than the index returns. For the squared S&P 500 index return series, the first order autocorrelation is low but not negligible, and higher order autocorrelations are in general diminishing.

Table 2 reports the parameter estimates, asymptotic standard errors, and GMM minimized criterion (χ^2) values for the SVJ model and for each of the three subnested models, namely the SV model, the JD model, and the BS model.[15] The SVJ-SI model is estimated jointly for the asset return process and the interest rate process using the moments derived for both the asset returns and interest rate, with χ^2=8.778 (d.f.=3) for the J-test and p-value=$3.24\ 10^{-2}$. However, the parameter estimates of the asset return process are essentially the same as those for the SVJ model and the parameter estimates of the interest rate model are only slightly different than those when the interest rate process is estimated separately. Thus the results for the SVJ-SI model are not reported. Nevertheless, when the risk-neutral parameters for the SVJ-SI model are implied from option prices in the next section, the parameter estimates for the SVJ-SI model are used. We report the results of the SI model in Table 2 based on separate estimation with the J-test for the specification of the interest rate model. The following observations are drawn from the estimation results.

Firstly, based on the p-values of the χ^2 tests, the only model that is not rejected at the 5% critical level is the SVJ model and the p-value ranks the SVJ, JD and SV models accordingly.[16] In other words, only the SVJ model has reasonable fit to the historical time series. Note that the data used in our estimation includes the 1987 crash with a very large negative daily return (-22.8%). As expected, the crash is driving the significance of the random jump

[15] The weighting matrix is estimated by the Barlett kernel proposed by Newey and West (1987) with a fixed lag length of 20. The JD model and the BS model are also re-estimated via the (constrained) ML method, which gives similar results.

[16] As discussed in Jiang and van der Sluis (1999), although the p-value is a monotone function of the actual evidence against the null hypothesis and we can rank the goodness-of-fit based on p-values, it is very dangerous to choose the best model of these specifications on basis of p-value, see Berger and Delampady (1987).

component. The interest rate model is not rejected at 1% critical level but there is no significant mean reversion in the interest rate process, which is consistent with the results in Chan, Karolyi, Longstaff and Sanders (1992).

Secondly, a further test proposed by Newey and West (1987) as outlined in the previous section suggests that all three subnested models of the SVJ model, i.e. the JD model, the SV model and the BS model, are strongly rejected. In other words, both random jump and stochastic volatility are essential to the modeling of S&P 500 index returns. Furthermore, the estimate of κ is significant in both the SV model and SVJ model, suggesting significant mean reversion in the volatility process. The estimate of η is also significant, suggesting a significant risk premium of stochastic volatility in the drift term.

Thirdly, since both asymmetric stochastic volatility and asymmetric random jump can contribute to the skewness and kurtosis of asset returns, it would be interesting to see how the parameter estimates are affected in various models. Comparing the jump parameter estimates in the JD model with those in the SVJ model, it can be seen that with stochastic volatility in the SVJ model, the random jump frequency becomes significantly lower and the jump size becomes significantly larger. Comparing the SV parameter estimates in the SV model with those in the SVJ model, it is clear that due to the jump component in the SVJ model, the mean reversion of the SV process becomes stronger and the level of asymmetry in terms of correlation between volatility and asset return is reduced to -0.358 from -0.603.

Finally, as we mentioned, the continuous-time SV process or SVJ process are also estimated in other studies using the S&P 500 index returns. Noticeable differences between our GMM parameter estimates and those obtained using simulation based methods are that both the value of the mean reversion parameter and that of the conditional variance in the SV process are much higher in our estimation. Our estimation results indicate a much stronger mean reversion and higher variance for the SV process. Using the exact expressions of various moments, it is noted that the model moments calculated using our parameter estimates from the GMM estimation closely match the empirical moments. The intuitive justification of our results is that, in the SV model framework, in order to incorporate the negative skewness and fat tails of the S&P 500 index return distribution, a negative correlation between asset return and volatility and a significant level of variation in volatility are required.

4 Estimation of Risk-Neutral Process and Option Pricing Performance of Alternative Models

In the second step of estimating the risk-neutral process, we estimate the preference related parameters from observed S&P 500 index option prices. Since option prices are observed over the year of 1995 and the underlying model is estimated based on S&P 500 index daily returns from 1980 to 1994, one may be concerned whether the parameter estimates in the underlying model are robust to the sampling period. Re-estimating the model based on returns from 1980 to 1995 shows that the parameter estimates are virtually the same, suggesting no significant structural change for the dynamics of S&P 500 index in 1995.

4.1 The Options Data Set

The S&P 500 index options data is obtained from the CBOE for the sample period from January 3, 1995 to December 29, 1995, which extends one year from the estimation period. Since we do not rely solely on option prices to obtain the parameter estimates through fitting the option pricing formula, such a sample size is adequate for our analysis. S&P 500 index options (SPX) are European-type and among the actively traded financial derivatives in the world. The original data set contains both call options and put options. However, all the deep in-the-money options for both puts and calls are very infrequently traded and their prices are thus notoriously unreliable. To circumvent this problem, we use the idea in Aït-Sahalia and Lo (1998), i.e. we replace the prices of all illiquid deep in-the-money call options with those of liquid put options at the relevant strike prices via put-call parity. The put options are by construction out-of-the-money options and thus liquid. After this procedure, all the information contained in liquid put prices has been extracted and resides in corresponding call prices. Therefore, put prices may now be discarded without any loss of reliable information.

The data set consists of intra-daily bid-ask quotes for the index options with various strike prices and expiration dates. To ease computational burden, for each business day in the sample only the last reported bid-ask quote during the trading session of each option contract is used in the empirical test. The index is simultaneously observed as the option's bid-ask quote, which avoids the issue of non-synchronous prices.[17] Following Ghysels, Harvey and

[17]A few filters are further applied to the data set. First of all, the data only include options with at least 5 days to expiration to reduce biases induced by liquidity-related issues. Secondly, option quotes which do not

Renault (1996), we define the degree of moneyness as

$$x = \ln(S_t / K e^{-\int_t^T (r_\tau - d_\tau) d\tau}) \tag{18}$$

In our partition, a call option is said to be *at-the-money* (ATM) if $-0.01 < x \le 0.02$, *out-of-the-money* (OTM) if $x \le -0.01$, and *in-the-money* (ITM) if $x > 0.02$. A finer partition according to moneyness and maturity results in 18 categories as in Panel A of Table 3. For each category, the average bid-ask midpoint price and its standard error, the average effective bid-ask spread (i.e. the ask price minus the bid-ask midpoint) and its standard deviation, as well as the number of observations are reported.

Figure 1 plots the implied Black-Scholes volatility against moneyness for options with different terms of maturity. The implied Black-Scholes volatilities are calculated from each option quote using the current yield of U.S. treasury instruments.[18] The Black-Scholes implied volatility exhibits obvious shape of "smirk" which is more pronounced for short-term options. These observations indicate that the short-term options are the most severely mispriced ones by the Black-Scholes model and present perhaps the greatest challenge to any alternative option pricing model.

4.2 Implied Risk Premiums of Stochastic Volatility, Random Jump, Interest Rate and Implied Stochastic Volatility

Using the observed option prices and bond prices, we back out the preference related parameters for each business day from January 3, 1995 to December 29, 1995. The parameter set includes λ^* and μ_J^* which are related to the random jump risk, κ^* related to the volatility risk and β^* related to the interest rate risk, as well as the unobserved stochastic volatility V_t. From an econometric modeling point of view, see e.g. Renault (1996), the extra free parameters offer a more flexible error structure in fitting into the option prices. As a result of our identification procedure, however, the implied parameters all bear economic meaning

satisfy arbitrage restrictions ($C(S_t, t) \ge \max(0, S_t - K, S_t - K e^{\int_t^T (r_\tau - d_\tau) d\tau})$) are excluded. Thirdly, options with prices below $3/8 are also excluded as for these options the market microstructure can have strong impact on the bid and ask.

[18] Namely, we use the 3-month T-bill rates for options with maturity less than 4 months, and 6-month T-bill rates for options with maturity longer than 4 months. All discount rates are converted to annualized compound rates.

and can offer insights for the understanding of the options market. Since the longest maturity of the options in our data set is roughly one year, we back out the interest rate risk related parameter β^* through fitting into the short-end yield curve, namely

$$\hat{\beta}_t^* = \arg\min_{\beta^*} \sum_{i=1}^{N} (B(t, \tau_i; \beta^*) - \hat{B}(t, \tau_i))^2 \tag{19}$$

where $N = 1, \tau_i =$3-, 6-month, and $\hat{B}(t, \tau_i)$ is the observed zero coupon bond price at time t calculated from yields with maturity τ_i. Similarly, the parameters and volatility, denoted by $\theta_t^* = (\lambda^*, \mu_J^*, \kappa^*, V_t)$, are backed out from the observed option prices at time t,

$$\hat{\theta}_t^* = \arg\min_{\theta_t^*} \sum_{i=1}^{N_t} (C_{t,i}(S_t, t; \tau_i, K_i; \theta_t^*) - \hat{C}_{t,i}(S_t, t; \tau_i, K_i))^2 \tag{20}$$

where $\hat{\theta}_t^* = (\hat{\lambda}_t^*, \hat{\mu}_{J,t}^*, \hat{\kappa}^*, \hat{V}_t)$, $\hat{C}_{t,i}(S_t, t; \tau_i, K_i)$ is the observed option price at time t with maturity τ_i and strike K_i, and N_t is the number of option prices observed at time t.

The objective function in (20) is defined as the sum of squared dollar errors, which may force the estimation to assign higher weights to relatively expensive options, e.g. long-term ITM options. An alternative is to minimize the sum of squared percentage pricing errors of all options, but that would put higher weight on relatively cheaper options, e.g. short-term OTM options. As Bakshi, Cao and Chen (1997) argue, using the objective function in (20) to imply parameters for alternative candidate models should in some sense give each model an “equal” chance, and it is also consistent with the existing practice of judging a model’s performance relative to that of the Black-Scholes model. Same objective function is also used in Bates (1996a), Dumas, Fleming and Whaley (1998), Longstaff (1995), Madan and Chang (1996), and Nandi (1996).

In Table 4, we report the mean and standard error of the preference related parameters implied from observed option and bond prices each day over the period of January 3, 1995 to December 29, 1995 for each model. If the risk-neutral parameters are implied from option and bond prices with the parameters of the underlying asset return and interest rate processes set equal to their estimates, the standard errors of the risk-neutral parameters would be inevitably under-estimated.[19] As a remedy, we perform the following two-step procedure. In the first step, we bootstrap the underlying parameter values (with 1,000 re-sampling) from

[19] As the referee correctly points out, this is because the uncertainty associated with the underlying parameters will also contribute to the uncertainty of the risk-neutral parameters.

their asymptotic normal distribution (with positivity restriction for certain parameters) based on the estimates and standard errors. In the second step, with each set of parameter values we imply the risk-neutral parameters from the option and bond prices. Finally, the mean and standard errors of the risk-neutral parameters are computed and reported in Table 4. We note that overall the standard errors obtained following the above bootstrapping procedure are slightly higher than those obtained using simply the estimates of the underlying parameters. Relatively, the mean-reverting parameter κ^* of the stochastic volatility process has the largest increase in standard error. For the stochastic volatility risk premium of the SV model the standard error increases from $9.271\ 10^{-7}$ to $1.064\ 10^{-6}$, and for the random jump risk premium of the JD model the standard error increases from $1.610\ 10^{-5}$ to $1.705\ 10^{-5}$. In both cases, all the implied risk-neutral parameters are highly significant. The main results are summarized as follows.

Firstly, for the Black-Scholes (BS) model, the implied volatility is varying over time and occasionally clustered. It provides further evidence that the model with constant volatility is misspecified.

Secondly, for the stochastic volatility (SV) model, in addition to the underlying volatility we also imply the mean reversion parameter κ^* in the risk-neutral process. As reported in Table 4, the risk premium of stochastic volatility, $\Phi_v = \xi V_t$, is overall positive and varying over time, which is consistent with the findings in other studies, e.g. Melino and Turnbull (1990), Lamoureux and Lastrapes (1993), and Pan (2002).

Thirdly, for the jump-diffusion (JD) model, we imply both the jump frequency λ^* and jump size μ_J^* from option prices. Similar to the SV model, the JD model has an extra free parameter than the Black-Scholes model and offers a more flexible error structure in fitting into option prices. As specified in the asset price process of equation (1), the difference between the expected downside jump risk from historical estimate and the expected downside jump risk implied from option prices, i.e. $\Phi_J = \hat{\lambda}\hat{\mu} - \hat{\lambda}^*\hat{\mu}^*$, measures the risk premium of random jump. As reported in Table 4, the random jump risk premium is also overall positive and varying over time.

Finally, the stochastic volatility with random jump (SVJ) model offers even more flexible error structure in fitting into option prices, as we have to back out both the stochastic volatility related parameters and the random jump related parameters. As reported in Table 4, similar to the SV model and the JD model, both the volatility risk premium and jump risk premium are overall positive and varying over time. For the stochastic volatility with random jump and stochastic interest rate (SVJ-SI) model, as reported in Table 4, the risk premium of interest rate implied from short-end yield curve is also positive. The rest of the parameters implied from option prices are almost identical to those in the SVJ models.

4.3 Segmentation of the Option Market

One of the implicit assumptions made in the above estimation procedure is that all option contracts are priced with the same risk premiums. However, an interesting hypothesis is that options market may be imperfect or segmented. To test this hypothesis, we divide the options contracts according to the degree of moneyness and length of maturity, and the preference related parameters are implied from each sub-group of the options data set by minimizing the objective function in (20). Such a procedure provides us with a much richer information set about investors' preference. While this practice is purely *ad hoc* for the Black-Scholes model, it is perfectly justifiable for the SV, JD, SVJ, and SVJ-SI models as the preference related parameters in these models may be dependent of the specific option contracts.

We divide the options data set into three sub-groups according to the degree of moneyness or the length of maturity with equal intervals of the respective indicator. As reported in Figure 3(a) for the SV model, the risk premium of stochastic volatility implied from short-term options and medium-term options do not have clear difference.[20] However, as reported in Figure 3(b) the risk premium implied from ITM call options (or OTM put options) is clearly higher than that implied from OTM call options (or ITM put options). From our discussion in Section 4.2, the uncertainty in the underlying parameters can also contribute to the uncertainty of the risk-neutral parameters. The evidence from Section 4.2 suggests that the difference in risk premium remains statistically highly significant even if the risk-neutral parameter uncertainty due to the underlying parameter uncertainty is also taken into account. Since the risk premium of stochastic volatility is equal to the product of volatility and market price of volatility risk, it would be interesting and important to further investigate whether the differences in risk premium, as shown in Figure 3(b), are due to the difference in the level of implied volatility or the difference in the market price of volatility risk. As plotted in Figure 3(c), the implied stochastic volatility in the SV model still exhibits the shape of

[20]It should be noted that in our options data set the longest maturity is only 242 business days, thus this result may not hold for long maturity option contracts, such as LEAPS.

"smirk", but its level of "smirkness" is much less pronounced than that of the implied Black-Scholes volatility. This suggests that the SV model still shows certain moneyness related biases, indicating mis-specification of the model to a certain degree or differential subjective belief on the level of volatility by investors in different segments of the options market. The combination of Figure 3(b) and 3(c) indicates that both the implied volatility and the market price of volatility risk implied from deep ITM call options are higher than those implied from deep OTM call options.

The evidence from the jump-diffusion (JD) model further confirms our findings in the SV model about investors' behaviour. Namely, the random jump risk premiums implied from short-term options and medium-term options do not have clear differences, but the jump risk premium implied from ITM call options is also clearly higher than that implied from OTM call options. It is important to point out that while the "smirk" shape of the implied Black-Scholes volatility may also suggest that the deep ITM call options (or deep OTM put options) are priced at significant premiums relative to the deep OTM call options, this argument is undermined by the assumption of complete market in the Black-Scholes world and the obvious model misspecification. While one may argue that the above conclusions can also be impaired by the misspecification of the underlying SV and JD models, the results based on the SVJ and SVJ-SI models would be more convincing. Through the matching of various moments in the GMM estimation procedure, the SVJ and SVJ-SI models reflect both the static properties, e.g. the negative skewness and fat tails, and the dynamic properties, e.g. the persistence of conditional volatility, of the underlying asset returns. The Hansen J-test for the SVJ model specification based on GMM estimation has a p-value of 0.0587 . For both the SVJ and SVJ-SI models, the estimation results show that the risk premiums of stochastic volatility and random jump implied from sub-grouped options data sets have the same qualitative relationship with respective to the degree of moneyness and maturity as in the SV and JD models.

The findings that deep OTM put options are priced with higher risk premiums than the deep OTM call options suggest a segmented options market. In other words, the risk factors are priced differently in different segments of the options market. A plausible explanation to this finding is that deep OTM put option are mostly used by fund managers to hedge their equity positions against market crash, while deep OTM call options are often dealt by traders for speculation. It's no surprise that the fund managers would be willing to pay higher premiums for options as hedging instruments. Our results are in line with Jones (2001) in which a non-linear factor analysis is performed on the S&P 500 index option returns.[21] The findings in Jones (2001) suggest that while allowing for more than one factor does reduce the degree of mispricing of many options, two or three factors are still insufficient to explain the abnormally negative returns on a wide range of put options. In particular, the short-term OTM put options have expected negative returns that are too negative to be consistent with a factor-based explanations.

4.4 Option Pricing Performance: Out-of-Sample Comparison based on a Single Risk-Neutral Distribution and Multiple Risk-Neutral Distributions

To gauge each model's option pricing performance, the model-generated option prices are compared to the observed market option prices. In order to perform out-of-sample comparison, the procedure of using current information to calculate option prices is outlined as follows. To price options at day $t+1$, the preference related parameters backed-out from option prices on day t are used, i.e. $\lambda^*_{t+1} = \hat{\lambda}^*_t, \mu^*_{J,t+1} = \hat{\mu}^*_{J,t}, \kappa^*_{t+1} = \hat{\kappa}^*_t$, and the volatility is predicted based on the information at time t, i.e. $V^*_{t+1} = E[V_{t+1} \mid V_t = \hat{V}_t]$, which is given in the following lemma.

Lemma 4: Given $V_t = \hat{V}_t$, the expected volatility in period $t+1$ is given by,

$$E[V_{t+1} \mid V_t = \hat{V}_t] = \gamma + e^{-\kappa}(\hat{V}_t - \gamma) \tag{21}$$

Proof: see Appendix.

For the Black-Scholes model, the implied volatility at time t is used as the volatility input at time $t+1$. Thus, all the models rely only on information available at current time, and the comparisons are based on out-of-sample performance. Option pricing biases are measured by the mean relative error (MRE) and the mean absolute relative error (MARE), given by

$$\begin{aligned} MRE &= \frac{1}{\sum_{t=1}^{T} N_t} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \frac{C^M_{t,i} - \hat{C}_{t,i}}{\hat{C}_{t,i}} \\ MARE &= \frac{1}{\sum_{t=1}^{T} N_t} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \frac{|C^M_{t,i} - \hat{C}_{t,i}|}{\hat{C}_{t,i}} \end{aligned} \tag{22}$$

[21]We wish to thank the referee for making us aware of this article.

where $\hat{C}_{t,i}$ and $C^M_{t,i}$ represent respectively the observed market option price and the model option price. The MRE statistic measures the average relative biases of the model option prices, while the MARE statistic measures the dispersion of relative biases of the model prices. The difference between MARE and MRE suggests the direction of the systematic bias of the model prices. Since the percentage errors are very sensitive to the magnitude of option prices which are determined by both moneyness and length of maturity, we also calculate MRE and MARE for each of the 18 moneyness-maturity categories in Table 5.

Table 5 reports the relative pricing errors (%) based on underlying volatility for alternative models. In general, all the models have similar patterns of mispricing, namely, overpricing of deep OTM options and underpricing of deep ITM options. The BS model has the largest pricing errors and the SVJ-SI model has the smallest pricing errors. For comparisons between alternative models, similar to Chernov and Ghysels (2000), we compute formal tests based on the following set of moment conditions,

$$\begin{aligned}
\frac{1}{\sum_{t=1}^{T} N_t}\sum_{t=1}^{T}\sum_{i=1}^{N_t}\frac{|C^{M1}_{t,i}-\hat{C}_{t,i}|}{\hat{C}_{t,i}} - e_{MARE} = 0 \\
\frac{1}{\sum_{t=1}^{T} N_t}\sum_{t=1}^{T}\sum_{i=1}^{N_t}\frac{|C^{M2}_{t,i}-\hat{C}_{t,i}|}{\hat{C}_{t,i}} - e_{MARE} = 0
\end{aligned} \tag{23}$$

where M1 and M2 represent model 1 and model 2 in the comparison. The overidentifying restriction test statistic based on the minimized GMM criterion is asymptotically distributed as $\chi^2_{(1)}$, which indicates whether the difference of MAREs between models are statistically significant. The test is robust to the correlated and conditionally heteroskedastic error structure. We compute the test statistics for each pair of models based on all options and options in each sub-category. Instead of reporting the results in a tedious way, we summarize the major results using the following terminologies, namely "largely outperform" for p-value less than 1%; "moderately outperform" for p-value between 5% and 1%; "slightly outperform" for p-value between 10% and 5%; and "no clear difference" for p-value greater than 10%.

First of all, the SV, JD, SVJ and SVJ-SI models all "largely outperform" the BS model. Secondly, there is "no clear difference" between the SVJ model and SVJ-SI model, suggesting the stochastics of interest rate only has minimal impact on option pricing, which is consistent with Bakshi, Cao and Chen (1997). Thirdly, the SVJ and SVJ-SI models "slightly outperform" both the SV model and the JD model. Finally, also similar to the findings in Bakshi, Cao and Chen (1997), while the JD model "moderately outperform" the SV model for short-term options, the SV model "slightly outperform" the JD model for long-term options. The last two findings are of particular interest as they suggest that the SV model and JD model can be used as complementary tools in building option pricing models. Further analysis based on the implied Black-Scholes volatilities suggests that the SV model fails to capture the skewness and kurtosis of the asset return distributions over very short time horizon. The intuition is that in the SV model since the volatility is highly persistent, change in volatility and asymmetry between asset return and volatility can only be realized through time evolution. On the other hand, the random Poisson jump can induce skewness and kurtosis over short time horizon, but fails to capture the persistence of asset return volatility.

The implicit assumption in the above option pricing procedure is that all option contracts are priced with the same risk premiums. As we have noticed, while the medium-term and short-term call options imply similar risk premiums, the deep ITM call options are priced with higher risk premiums than the deep OTM call options. A remedy for this drawback is to price options based on multiple risk-neutral distributions by resorting to a much richer information set. In this section, we divide the options data set into three sub-groups according to the degree of moneyness or the length of maturity using the partition in Table 3. We first imply the model parameters from each subset of observed option prices, then use them to price options that belong to the same sub-group in the following day. In other words, multiple risk-neutral distributions are used in pricing options.

As expected, our results show that when the option contracts are divided according to maturity there is no obvious improvement in option pricing performance, while when the option contracts are divided according to degree of moneyness there is a substantial improvement in option pricing performance. This is because the risk premiums implied from options, as our empirical results suggest, are more sensitive to the moneyness of the options. Table 6 reports the relative pricing errors (%) of alternative models using parameters implied from moneyness-based subsets of options. Compared to Table 5, the results in Table 6 show that for all models, the percentage pricing errors are dramatically reduced, in particular for OTM options. This suggests that a richer information set of investors' preference implied from different segments of the options market can provide much more accurate option prices. Since the analysis is based on out-of-sample performance, it also suggests that investors' preferences are relatively persistent and the dynamic structure of the risk-neutral SVJ and SVJ-SI models are rather robust over time. It is noted that among different models, the Black-Scholes model still has the largest option pricing errors and the SVJ and SVJ-SI models have the smallest option pricing errors. Based on model-to-model comparison, similar conclusions are reached as in last section. Only for short-term deep OTM options, the pricing errors remain high for the SVJ and SVJ-SI models. However, a 15% relative pricing error for these options translates to an absolute error of only $\$0.075$ on the average, which is even smaller than the average effective bid-ask spread (i.e. half of the bid-ask spread). It is obvious that to further reduce the option pricing errors, one can divide the option contracts into finer sub-groups and base on the implied multiple risk-neutral distributions to price options.

4.5 Robustness Check: Evidence from the FTSE 100 Index Options

As a robustness check, we perform similar analysis using the prices of the FTSE 100 index options. For brevity, we summarize the main results here in comparison with those based on the S&P 500 index. We first estimate the underlying asset return and interest rate processes using daily FTSE 100 index returns and the daily U.K. 3-month t-bill rates. The data covers the period from April 1984 to December 1998. Summary statistics of the FTSE 100 index returns and U.K. interest rates are given in the Panel B of Table 1. Overall, the FTSE 100 index exhibits similar properties as the S&P 500 index, but with less negative skewness and lower kurtosis. The variation of the FTSE 100 index is also more persistent over time.

The asset return models with alternative specifications, namely the BS, SV, JD, SVJ, and SVJ-SI, are estimated based on the daily returns of the FTSE 100 index. Similar to the results for the S&P 500 index, all three nested return models (BS, SV and JD) are strongly rejected for the FTSE 100 index. In other words, both stochastic volatility and random jump are also essential components of the FTSE 100 index returns. Similarly, the CIR process for the UK interest rate is not rejected at the 1% critical level. There are also some noticeable differences between the estimation results for the S&P 500 index and FTSE 100 index. The SV process for the FTSE 100 index is less mean-reverting and has lower magnitude of negative correlation with asset returns. The jump component has higher jump frequency but smaller jump size as measured by both the mean and standard deviation of the jump distribution.

In the second step, the risk-neutral parameters are estimated from option prices with given underlying asset return and interest rate processes. The option prices cover the year of 1999, one year extending the sampling period of asset returns used in the estimation of the underlying models. Only call option prices are used in our analysis in order to minimize the early exercise premium of the American-style options. Various filters similar to the S&P 500 index options are also applied. The data set contains 10,900 observations, with summary statistics reported in Panel B of Table 3. The plots of Black-Scholes implied volatility from the FTSE 100 index options have similar "smirk" pattern as that from S&P 500 index options. The "smirk" is also more pronounced for the short-term options. The risk-neutral parameters and the unobserved stochastic volatility are implied from the bond prices and option prices following the same procedure as in Section 4.2. For interest rate process, the U.K. 6-month t-bill yield is used to imply the market price of interest rate risk. The risk premiums for stochastic interest rate, stochastic volatility, and random jump are overall positive and varying over time.

To investigate potential segmentation of option market, we divide the option contracts into three subsets according to either the maturity or the degree of moneyness as in Panel B of Table 3. The risk premiums of stochastic volatility and random jump are then implied from each subset. Such a procedure provides us with a much richer information set about investors' risk preference. Figure 3 plots the risk premium of stochastic volatility implied from different subsets of FTSE 100 index options. As shown in Panel (a), the risk premium of stochastic volatility implied from short-term options and medium-term options do not have clear difference. However, as shown in Panel (b) the risk premium implied from ITM call options (or OTM put options) is clearly higher than that implied from OTM call options (or ITM put options). Further as shown in Panel (c), the implied stochastic volatility in the SV model still exhibits the shape of "smirk", but its level of "smirkness" is much less pronounced than that of the implied Black-Scholes volatility. The combination of Panels (b) and (c) indicates that both the implied volatility and the market price of volatility risk implied from deep ITM call options are higher than those implied from deep OTM call options. Similarly, the random jump risk premium under the jump-diffusion (JD) model exhibits the same pattern across the option maturity and moneyness. The evidence from the SVJ model, however, indicates that when both the stochastic volatility and random jump are present in the model, the difference of random jump risk premium between deep ITM and OTM options becomes less significant. This result indicates that the segmentation of FTSE 100 index option market is mainly due to the difference of the market price of volatility risk across the options market.

Out-of-sample option pricing performance based on alternative models is further investigated based on the single risk-neutral distribution implied from the whole option market and the multiple risk-neutral distributions implied from different segments of option market. As expected, our results show that when the option contracts are divided according to maturity there is no obvious improvement in option pricing performance, while when the option contracts are divided according to degree of moneyness there is a substantial improvement in option pricing performance. Similar to the findings for S&P 500 index options, this is because the risk premiums implied from options are more sensitive to the moneyness of the options. The results based on FTSE 100 index options thus further confirm that the option market is segmented and option pricing based on multiple risk-neutral distributions significantly outperforms that based on a single risk-neutral distribution.

5 Conclusion

In this paper we propose a two-step procedure for the estimation and testing of continuous-time option pricing models with stochastic volatility, random jump and stochastic interest rate. Namely, in the first step the underlying process is estimated from asset return observations via GMM based on the *exact* moment conditions of the continuous-time model with a valid statistical test of model specification, and in the second step the preference related parameters in the risk-neutral process are estimated from different segments of the options market. One of the major empirical findings in this paper is that the risk-neutral process estimated from different segments of the options market is not unique, suggesting an imperfect options market. In particular, our empirical results suggest that while the short-term and medium-term option contracts are priced with similar risk premiums, the ITM call options (or OTM put options) are clearly priced with higher risk premiums than the OTM call options (or ITM put options). More importantly, option pricing based on multiple risk-neutral distributions significantly outperforms that based on a single risk-neutral distribution. These findings are closely in line with Jones (2001) which is based on a nonlinear factor analysis of the S&P 500 index option returns. The results in Jones (2001) suggest that the short-term OTM put options behave differently than other options with expected negative returns too negative to be consistent with a factor-based explanations.

The approach proposed in this paper can be easily extended to the evaluation of the hedging performance of alternative option pricing models, which is currently under investigation by the author, and to the study of other underlying security and contingent claims. For future study, the efficiency of the GMM estimation procedure proposed in this paper can be improved based on Monte Carlo studies. The risk-neutral model considered in this paper can be extended to a semi-parametric specification in a more general setting.

Appendix

1. *Proof of Lemma 1:* Given the process of S_t as defined in (1), apply Itô's lemma, we have the jump diffusion process for $\ln S_t$:

$$d\ln S_t = (\mu_0 - \lambda\mu + r_t - d + (\eta - \frac{1}{2})V_t)dt + V_t^{1/2}dW_t + \ln J_t dq_t(\lambda)$$

Let $\psi(\Delta \ln S_{t+\Delta}, V_{t+\Delta}, r_{t+\Delta}; \phi_1, \phi_2, \phi_3 | S_t, V_t, r_t) = E[e^{i\phi_1 \Delta \ln S_{t+\Delta} + i\phi_2 V_{t+\Delta} + i\phi_3 r_{t+\Delta}} | S_t, V_t, r_t]$ be the joint conditional characteristic function (CCF) of $\Delta \ln S_{t+\Delta} = \ln S_{t+\Delta} - \ln S_t$, $V_{t+\Delta}$ and $r_{t+\Delta}$ given S_t, V_t, r_t, it can be verified through iterative expectations that $\psi(\cdot)$ is a martingale. Applying Itô's lemma and using its martingale property, we can derive the Kolmogorov forward (or Fokker-Planck) equation for $\psi(\cdot)$. The CCF for the jump component can be easily derived as $\Delta\lambda(e^{i\phi\mu_J - \frac{1}{2}\phi^2\sigma_J^2} - 1)$. Given the affine structure of the model, the solution of the CCF for the SV part is of the following structure (see e.g. Duffie, Pan and Singleton, (2000)):

$$\begin{aligned} & \psi(\Delta \ln S_{t+\Delta}, V_{t+\Delta}, r_{t+\Delta}; \phi_1, \phi_2, \phi_3 | S_t, V_t, r_t) \\ = \; & \exp\{C(\phi_1, \phi_2, \phi_3, \Delta) + (i\phi_2 + D(\phi_1, \phi_2, \Delta))V_t + (i\phi_3 + B(\phi_1, \phi_3, \Delta))r_t\} \end{aligned}$$

and solving the Ricatti equations derived from the Kolmogorov forward equation, we have

$$\begin{aligned} C(\phi_1, \phi_2, \phi_3, \Delta) &= i\phi_1\Delta(\mu_0 - \lambda\mu - d) \\ &\quad + i\phi_2\Delta\kappa\gamma + \frac{\kappa\gamma}{\sigma_v^2}[(\kappa - i\phi_1\rho\sigma_v - i\phi_2\sigma_v^2 - h)\Delta - 2\ln(\frac{1 - ge^{-h\Delta}}{1-g})] \\ &\quad + i\phi_3\Delta\beta\alpha + \frac{\beta\alpha}{\sigma_r^2}[(\beta - i\phi_3\sigma_r^2 - k)\Delta - 2\ln(\frac{1 - le^{-k\Delta}}{1-l})] \\ D(\phi_1, \phi_2, \Delta) &= \frac{\kappa - i\phi_1\rho\sigma_v - i\phi_2\sigma_v^2 - h}{\sigma_v^2} \cdot \frac{1 - e^{-h\Delta}}{1 - ge^{-h\Delta}} \\ B(\phi_1, \phi_3, \Delta) &= i\phi_3 + \frac{\beta - i\phi_3\sigma_r^2 - k}{\sigma_r^2} \cdot \frac{1 - e^{-k\Delta}}{1 - le^{-k\Delta}} \end{aligned}$$

where $h(\phi_1) = \sqrt{(\kappa - i\phi_1\rho\sigma_v)^2 + \sigma_v^2(\phi_1^2 - i\phi_1(2\eta - 1))}$, $g(\phi_1) = (\kappa - i\phi_1\rho\sigma_v - i\phi_2\sigma_v^2 - h)/(\kappa - i\phi_1\rho\sigma_v - i\phi_2\sigma_v^2 + h)$, and $k(\phi_1) = \sqrt{\beta^2 - 2i\phi_1\sigma_r^2}$, $l(\phi_1, \phi_3) = (\beta - i\phi_3\sigma_r^2 - k)/(\beta - i\phi_3\sigma_r^2 + k)$.

Now using the fact that V_t follows a gamma distribution with $f(V_t) = \frac{\omega^q}{\Gamma(q)}V_t^{q-1}e^{-\omega V_t}$, where $\omega = 2\kappa/\sigma_v^2, q = 2\kappa\gamma/\sigma_v^2$ and r_t follows a gamma distribution with $f(r_t) = \frac{\omega^q}{\Gamma(q)}r_t^{q-1}e^{-\omega r_t}$, where $\omega = 2\beta/\sigma_r^2, q = 2\beta\alpha/\sigma_r^2$, we have the joint unconditional characteristic function (CF) of $\Delta \ln S_{t+\Delta} = \ln S_{t+\Delta} - \ln S_t$ and $r_{t+\Delta}$ given by

$$\begin{aligned} & \psi(\Delta \ln S_{t+\Delta}, r_{t+\Delta}; \phi_1, \phi_3) = E[e^{i\phi_1 \Delta \ln S_{t+\Delta} + i\phi_3 r_{t+\Delta}}] \\ = \; & e^{C(\phi_1, 0, \phi_3, \Delta)} \cdot \left(1 - \frac{\sigma_v^2 D(\phi_1, 0, \Delta)}{2\kappa}\right)^{-2\kappa\gamma/\sigma_v^2} \cdot \left(1 - \frac{\sigma_r^2(i\phi_3 + B(\phi_1, \phi_3, \Delta))}{2\beta}\right)^{-2\beta\alpha/\sigma_r^2} \end{aligned}$$

2. *Proof of Corollary to Lemma 1:* Taking derivatives of the cumulant function as derived in Lemma 1 and then let $\phi_1 = \phi_2 = \phi_3 = 0$ to obtain various cumulants, the moments can be derived from the relationship between the cumulants and moments.

3. *Proof of Lemma 2:* Now we derive the joint characteristic function of $\Delta \ln S_{t+\Delta} = \ln S_{t+\Delta} - \ln S_t$ and $\Delta \ln S_\Delta = \ln S_\Delta - \ln S_0$, i.e.,

$$\begin{aligned}
& \psi(\Delta \ln S_{t+\Delta}, \Delta \ln S_\Delta; \phi, \varphi) = E[e^{i\phi\Delta \ln S_{t+\Delta} + i\varphi\Delta \ln S_\Delta}] \\
= \;& E[E[e^{i\phi\Delta \ln S_{t+\Delta} + i\varphi\Delta \ln S_\Delta} \mid S_t, V_t, r_t]] \\
= \;& E[E[e^{i\varphi\Delta \ln S_\Delta + C(\phi,0,0,\Delta) + D(\phi,0,\Delta)V_t + B(\phi,0,\Delta)r_t} \mid S_\Delta, V_\Delta, r_\Delta]] \\
= \;& e^{C(\phi,0,0,\Delta) + C(0,-iD(\phi,0,\Delta),-iB(\phi,0,\Delta),t-\Delta)} \\
& E[e^{i\varphi\Delta \ln S_\Delta + [iD(\phi,0,\Delta) + D(0,-iD(\phi,0,\Delta),t-\Delta)]V_\Delta + [iB(\phi,0,\Delta) + B(0,-iB(\phi,0,\Delta),t-\Delta)]r_\Delta}]
\end{aligned}$$

Let $D^* = iD(\phi, 0, \Delta) + D(0, -iD(\phi, 0, \Delta), t-\Delta)$, $B^* = iB(\phi, 0, \Delta) + B(0, -iB(\phi, 0, \Delta), t-\Delta)$, we have

$$\begin{aligned}
& \psi(\Delta \ln S_{t+\Delta}, \Delta \ln S_\Delta; \phi, \varphi) \\
= \;& e^{C(\phi,0,0,\Delta) + C(0,-iD(\phi,0,\Delta),-iB(\phi,0,\Delta),t-\Delta)} E[e^{i\varphi\Delta \ln S_\Delta + D^* V_\Delta + B^* r_\Delta}] \\
= \;& e^{C(\phi,0,0,\Delta) + C(0,-iD(\phi,0,\Delta),-iB(\phi,0,\Delta),t-\Delta) + C(\varphi,-iD^*,-iB^*,\Delta)} \\
& E[e^{[iD^* + D(\varphi,-iD^*,\Delta)]V_0 + [iB^* + B(\varphi,-iB^*,\Delta)]r_0} | V_0, r_0] \\
= \;& e^{C(\phi,0,0,\Delta) + C(0,-iD(\phi,0,\Delta),-iB(\phi,0,\Delta),t-\Delta) + C(\varphi,-iD^*,-iB^*,\Delta)} \\
& (1 - \frac{\sigma_v^2(iD^* + D(\varphi, -iD^*, \Delta))}{2\kappa})^{-2\kappa\gamma/\sigma_v^2} (1 - \frac{\sigma_r^2(iB^* + B(\varphi, -iB^*, \Delta))}{2\beta})^{-2\beta\alpha/\sigma_r^2}
\end{aligned}$$

4. *Proof of Corollary to Lemma 2:* See the proof of the Corollary to Lemma 1.

5. *Proof of Lemma 3:* The lemma follows directly from the results in CIR (1985b) and Bates (1988, 1991).

6. *Proof of Corollary to Lemma 3:* The derivation of the option pricing formula for jump-diffusion process with stochastic volatility and stochastic interest rates follows the techniques in Heston (1993), Scott (1997), Bates (1996a), and Bakshi, Cao and Chen (1997). Let $s_t = \ln S_t$, then insert the conjectured solution into the PDE which results in the PDEs for the risk-neutral probabilities Π_j for $j = 1, 2$. The resulting PDEs are the Fokker-Planck forward equations for probability functions. This implies that Π_1 and Π_2 are valid probability functions with values bounded between 0 and 1 and the PDEs must be solved separately subject to the terminal condition $\Pi_j(S_T, T) = 1_{S_T \geq K}, j = 1, 2$. The corresponding characteristic functions f_1 and f_2 for Π_1 and Π_2 also satisfy similar PDEs given by

$$\begin{aligned}
\frac{1}{2} V_t \frac{\partial^2 f_1}{\partial s_t^2} + (r_t - d - \lambda^* \mu^* + \frac{1}{2} V_t) \frac{\partial f_1}{\partial s_t} + \rho \sigma_v V_t \frac{\partial^2 f_1}{\partial s_t \partial V_t} + \frac{1}{2} \sigma_v^2 V_t \frac{\partial^2 f_1}{\partial V_t^2} \\
+ (\gamma_v - (\kappa^* - \rho\nu) V_t) \frac{\partial f_1}{\partial V_t} + \frac{1}{2} \sigma_r^2 r_t \frac{\partial^2 f_1}{\partial r_t^2} + (\gamma_r - \beta^* r_t) \frac{\partial f_1}{\partial r_t} + \frac{\partial f_1}{\partial t} \\
-\lambda^* \mu^* f_1 + \lambda^* E[(1 + \ln J_t^*) f_1(t, \tau; s_t + \ln J_t^*, r_t, V_t) - f_1(t, \tau; s_t, r_t, V_t)] = 0
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{1}{2}V_t\frac{\partial^2 f_2}{\partial s_t^2}+(r_t-d-\lambda^*\mu^*-\frac{1}{2}V_t)\frac{\partial f_2}{\partial s_t}+\rho\sigma_v V_t\frac{\partial^2 f_2}{\partial s_t\partial V_t}+\frac{1}{2}\sigma_v^2 V_t\frac{\partial^2 f_2}{\partial V_t^2}+(\gamma_v-\kappa^* V_t)\frac{\partial f_2}{\partial V_t}\\
&\qquad+\frac{1}{2}\sigma_r^2 r_t\frac{\partial^2 f_2}{\partial r_t^2}+(\gamma_r-(\beta^*-\frac{\sigma_r^2}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r_t})r_t)\frac{\partial f_2}{\partial r_t}+\frac{\partial f_2}{\partial t}-\lambda^*\mu^* f_2\\
&\qquad+\lambda^* E[f_2(t,\tau;s_t+\ln J_t^*,r_t,V_t)-f_2(t,\tau;s_t,r_t,V_t)]=0
\end{aligned}
$$

with boundary conditions $f_j(T,0;\phi)=\exp\{i\phi s_T\}, j=1,2$.

Conjecture that the solutions of f_1 and f_2 are respectively given by $f_1=\exp\{u(\tau)+x_r(\tau)r_t+x_v(\tau)V_t+i\phi s_t\}$ and $f_2=\exp\{v(\tau)+y_r(\tau)r_t+y_v(\tau)V_t+i\phi s_t-\ln B(t,\tau)\}$ with $u(0)=x_r(0)=x_v(0)=0$ and $v(0)=y_r(0)=y_v(0)=0$. Substitute in the conjectured solutions and solve the resulting systems of differential equations and note that $B(T,0)=1$, we have the following solutions

$$
\begin{aligned}
f_1(t,\tau) &= \exp\{-\frac{\gamma_r}{\sigma_r^2}[2\ln(1-\frac{(1-e^{-\xi_r\tau})(\xi_r-\beta^*)}{2\xi_r})+(\xi_r-\beta^*)\tau]\\
&- \frac{\gamma_v}{\sigma_v^2}[2\ln(1-\frac{(1-e^{-\xi_v\tau})(\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v)}{2\xi_v}]\\
&- \frac{\gamma_v}{\sigma_v^2}[\xi_v-\kappa^*+(1+i\phi)\rho\sigma_v]\tau+i\phi s_t+\frac{2i\phi(1-e^{-\xi_r\tau})}{2\xi_r-(1-e^{-\xi_r\tau})(\xi_r-\beta^*)}r_t\\
&+ \lambda^*\tau(1+\mu^*)[(1+\mu^*)^{i\phi}e^{i\phi(1+i\phi)\sigma_v^2/2}-1]-i\phi(\lambda^*\mu^*+d)\tau\\
&+ \frac{i\phi(i\phi+1)(1-e^{-\xi_v\tau})}{2\xi_v-(1-e^{-\xi_v\tau})(\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v)}V_t\}
\end{aligned}
$$

and

$$
\begin{aligned}
f_2(t,\tau) &= \exp\{-\frac{\gamma_r}{\sigma_r^2}[2\ln(1-\frac{(1-e^{-\xi_r^*\tau})(\xi_r^*-\beta^*)}{2\xi_r^*})+(\xi_r^*-\beta^*)\tau]\\
&- \frac{\gamma_v}{\sigma_v^2}[2\ln(1-\frac{(1-e^{-\xi_v^*\tau})(\xi_v^*-\kappa_v+i\phi\rho\sigma_v)}{2\xi_v^*}+(\xi_v^*-\kappa^*+i\phi\rho\sigma_v)\tau]\\
&+ i\phi s_t-\ln B(t,\tau)+\frac{2(i\phi-1)(1-e^{-\xi_r^*\tau})}{2\xi_r^*-(1-e^{-\xi_r\tau})(\xi_r^*-\beta^*)}r_t\\
&+ \lambda^*\tau[(1+\mu^*)^{i\phi}e^{i\phi(i\phi-1)\sigma_v^2/2}-1]-i\phi(\lambda^*\mu^*+d)\tau\\
&+ \frac{i\phi(i\phi-1)(1-e^{-\xi_v^*\tau})}{2\xi_v^*-(1-e^{-\xi_v^*\tau})(\xi_v^*-\kappa_v+i\phi\rho\sigma_v)}V_t\}
\end{aligned}
$$

where $\xi_r=\sqrt{\beta^{*2}-2\sigma_r^2 i\phi}$, $\xi_v=\sqrt{(\kappa^*-(1+i\phi)\rho\sigma_v)^2-i\phi(1+i\phi)\sigma_v^2}$, $\xi_r^*=\sqrt{\beta^{*2}-2\sigma_r^2(i\phi-1)}$ and $\xi_v^*=\sqrt{(\kappa^*-i\phi\rho\sigma_v)^2-i\phi(i\phi-1)\sigma_v^2}$.

7. *Proof of Lemma 4:* The solution of the squared process can be written as

$$
V_t=\gamma+e^{-\kappa(t-t_0)}(V_{t_0}-\gamma)+\sigma_v e^{-\kappa t}\int_{t_0}^t e^{\kappa\tau}V_\tau^{1/2}dW_\tau
$$

The last term has zero expectation conditional on V_{t_0}, thus $E[V_t|V_{t_0}]=\gamma+e^{-\kappa(t-t_0)}(V_{t_0}-\gamma)$.

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Table 1: Summary Statistics

A. Daily S&P 500 Index Returns, Dividend Yield, and U.S. Interest Rate

(1) Static Properties

	N	Mean	St. Dev.	Skewness	Kurtosis	Max	Min
$100 \times \Delta s_t$	3790	$3.821\ 10^{-2}$	$9.999\ 10^{-1}$	-3.314	79.87	8.71	-22.8
$100 \times \Delta r_t$	3790	$-1.691\ 10^{-3}$	$1.400\ 10^{-1}$	0.238	18.99	1.34	-1.27
$100 \times \Delta d_t$	3790	$8.198\ 10^{-6}$	$4.291\ 10^{-3}$	0.118	4.496	0.141	-0.157

(2)Dynamic Properties (autocorrelations)

	N	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(4)$	$\rho(5)$	$\rho(10)$	$\rho(20)$
$100 \times \Delta s_t$	3790	0.050	-0.035	-0.032	-0.044	0.047	-0.023	0.006
$(100 \times \Delta s_t)^2$	3790	0.107	0.149	0.076	0.019	0.137	0.008	0.005
$100 \times \Delta r_t$	3790	0.120	0.034	-0.010	0.043	0.036	0.035	0.031
$100 \times \Delta d_t$	3790	-0.048	-0.067	-0.030	0.008	0.011	0.010	0.020

B. Daily FTSE 100 Index Returns and U.K. Interest Rate

(1) Static Properties

	N	Mean	St. Dev.	Skewness	Kurtosis	Max	Min
$100 \times \Delta s_t$	3727	$4.479\ 10^{-2}$	$9.491\ 10^{-1}$	-0.881	13.52	7.60	-13.03
$100 \times \Delta r_t$	3727	$-1.167\ 10^{-3}$	$1.093\ 10^{-1}$	-2.296	37.70	0.71	-1.87

(2)Dynamic Properties (autocorrelations)

	N	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(4)$	$\rho(5)$	$\rho(10)$	$\rho(20)$
$100 \times \Delta s_t$	3727	0.072	0.019	-0.033	0.053	-0.017	0.034	0.030
$(100 \times \Delta s_t)^2$	3727	0.434	0.356	0.161	0.225	0.068	0.117	0.016
$100 \times \Delta r_t$	3727	0.042	-0.008	0.021	-0.018	0.020	0.053	-0.028

Note: Sample period for S&P 500 index is from January 1980 to December 1994, and sample period for FTSE 100 index is from April 1984 to December 1998. $\Delta s_t = \Delta \ln S_t$ are daily returns of S&P 500 index and FTSE 100 index, Δr_t are daily changes of U.S. and U.K. three month t-bill rates, and Δd_t are daily changes of S&P 500 index dividend yields.

Table 2: GMM Estimation Results of Alternative Models

S&P 500 Index Return Process

	Model &	SVJ	JD	SV	BS
Return	μ_0	$3.291\ 10^{-4}$	$3.762\ 10^{-4}$	$3.068\ 10^{-4}$	$3.725\ 10^{-4}$
Parameter		$(1.353\ 10^{-4})$	$(1.305\ 10^{-4})$	$(1.337\ 10^{-4})$	$(1.560\ 10^{-4})$
	η	0.805		0.911	
		(0.183)		(0.197)	
SV	$\sqrt{\gamma}$	$8.709\ 10^{-3}$	$8.303\ 10^{-3}$	$9.918\ 10^{-3}$	$9.970\ 10^{-3}$
Parameter		$(6.167\ 10^{-4})$	$(3.401\ 10^{-4})$	$(2.401\ 10^{-4})$	$(1.954\ 10^{-4})$
	κ	1.630		0.878	
		$(4.211\ 10^{-2})$		$(3.710\ 10^{-2})$	
	σ_v	$5.177\ 10^{-2}$		$6.645\ 10^{-2}$	
		$(2.710\ 10^{-3})$		$(2.903\ 10^{-3})$	
	ρ	-0.358		-0.603	
		$(1.820\ 10^{-2})$		$(1.435\ 10^{-2})$	
Jump	μ_J	$-4.689\ 10^{-2}$	$-2.380\ 10^{-2}$		
Parameter		$(1.907\ 10^{-2})$	$(1.299\ 10^{-2})$		
	λ	$2.230\ 10^{-3}$	$4.911\ 10^{-3}$		
		$(1.343\ 10^{-3})$	$(1.322\ 10^{-3})$		
	σ_J	$8.017\ 10^{-2}$	$7.566\ 10^{-2}$		
		$(1.219\ 10^{-2})$	$(5.557\ 10^{-3})$		
J-test (χ^2)		5.670 (d.f.= 2)	15.17 (d.f.= 6)	14.90 (d.f.= 5)	34.91 (d.f.= 9)
p-value		$5.87\ 10^{-2}$	$1.90\ 10^{-2}$	$1.08\ 10^{-2}$	$6.18\ 10^{-5}$
Nested test (χ^2)			13.06 (d.f.= 4)	12.55 (d.f.= 3)	30.78 (d.f.= 7)
p-value			$1.10\ 10^{-2}$	$5.72\ 10^{-3}$	$6.82\ 10^{-5}$

Stochastic Interest Rate Process (SI)

Parameter	α	β	σ_r	J-test (χ^2)	p-value	d.f.
	$5.767\ 10^{-2}$	$1.766\ 10^{-3}$	$5.598\ 10^{-3}$	6.37	$1.16\ 10^{-2}$	1
	$(7.944\ 10^{-3})$	$(1.225\ 10^{-3})$	$(1.105\ 10^{-4})$			

Note: The moment conditions in the GMM estimation procedure for both the S&P 500 index return process and interest rate process are given in Section 3.1. The numbers in brackets are standard errors. The blank cell indicates that the parameter is pre-set to zero. The SVJ-SI model is estimated jointly for the asset return process and the interest rate process, with χ^2=8.778 (d.f.=3) for the J-test and p-value=$3.24\ 10^{-2}$. However, the parameter estimates of the asset return process are essentially the same as those for the SVJ model and the parameter estimates of the interest rate model are only slightly different than those when the interest rate process is estimated separately. Thus the results for the SVJ-SI model are not reported. We report the results of the SI model based on separate estimation with the J-test for the specification of interest rate process.

Table 3: Sample Properties of Option Prices

Panel A: S&P 500 Index Options

	Moneyness	Days-to-Expiration (T-t [5, 242])			
	$[-0.16,\ 0.32]$	$\leq$30	$30--80$	≥ 80	subtotal
OTM	$x \leq -0.04$	0.492 (0.135) 0.082 (0.040) {256}	0.757 (0.335) 0.090 (0.042) {997}	2.401 (1.523) 0.113 (0.045) {621}	{1874}
	$-0.04 < x \leq -0.01$	1.080 (0.635) 0.094 (0.052) {692}	2.351 (1.248) 0.110 (0.055) {1151}	6.085 (2.377) 0.273 (0.110) {281}	{2124}
ATM	$-0.01 < x \leq 0.00$	2.763 (1.066) 0.123 (0.050) {269}	5.398 (1.096) 0.265 (0.094) {396}	10.29 (2.150) 0.376 (0.108) {75}	{740}
	$0.00 < x \leq 0.02$	6.773 (1.955) 0.295 (0.102) {519}	9.150 (2.160) 0.314 (0.109) {748}	12.96 (1.935) 0.494 (0.110) {180}	{1447}
ITM	$0.02 < x \leq 0.10$	25.07 (9.245) 0.760 (0.160) {1805}	25.38 (8.550) 0.764 (0.194) {2485}	27.97 (7.881) 0.775 (0.196) {791}	{5081}
	$x > 0.10$	52.56 (9.553) 1.000 (0.000) {277}	65.54 (21.48) 1.000 (0.000) {863}	72.16 (22.52) 1.000 (0.000) {1194}	{2334}
subtotal		{3794}	{6656}	{3150}	{13600}(total)

Panel B: FTSE 100 Index Options

	Moneyness	Days-to-Expiration (T-t [5, 249])			
	$[-0.18,\ 0.35]$	$\leq$30	$30--80$	≥ 80	subtotal
OTM	$x \leq -0.01$	1.777 (0.805) 0.301 (0.150) {761}	4.951 (1.828) 0.872 (0.387) {1801}	10.67 (2.141) 1.005 (0.430) {670}	{3232}
ATM	$-0.01 < x \leq 0.02$	22.90 (5.988) 0.770 (0.165) {752}	47.37 (10.79) 1.015 (0.022) {1087}	81.60 (11.72) 1.125 (0.014) {635}	{2474}
ITM	$x > 0.02$	123.8 (23.34) 1.502 (0.001) {1140}	171.5 (10.07) 1.712 (0.003) {2765}	235.2 (35.77) 1.771 (0.006) {1289}	{6194}
subtotal		{2653}	{5653}	{2594}	{10900}(total)

Note: In each cell from top to bottom are: the average bid-ask midpoint call option prices with standard error in parentheses, the average effective bid-ask spread (ask price minus the bid-ask midpoint) with standard error in parentheses, which are calculated from the original bid-ask quotes, and the number of option price observations (in curly brackets) for each moneyness-maturity category. The option price sample covers the year 1995 for S&P 500 index and 1999 for FTSE 100 index.

Table 4: Implied Parameters from S&P 500 Index Options and U.S. Bond Markets

Implied Volatility and Risk Premiums of Stochastic Volatility and Random Jump

	Parameter	SVJ	JD	SV	BS
SV	$\sqrt{V_t}$	$8.813\ 10^{-3}$		$1.039\ 10^{-2}$	$1.044\ 10^{-2}$
		$(1.605\ 10^{-4})$		$(3.316\ 10^{-4})$	$(8.787\ 10^{-4})$
	κ^*	1.450		0.731	
		$(5.920\ 10^{-3})$		$(1.223\ 10^{-2})$	
	Φ_v	$5.006\ 10^{-6}$		$1.547\ 10^{-5}$	
		$(2.996\ 10^{-7})$		$(1.064\ 10^{-6})$	
Jump	$\lambda^*\mu_J^*$	$-1.470\ 10^{-4}$	$-2.113\ 10^{-4}$		
		$(7.011\ 10^{-6})$	$(1.877\ 10^{-5})$		
	Φ_J	$3.812\ 10^{-5}$	$9.300\ 10^{-5}$		
		$(6.710\ 10^{-6})$	$(1.705\ 10^{-5})$		

Implied Market Price of Interest Rate Risk

Parameter	β^*	Φ_r
	$1.214\ 10^{-3}$ $(1.975\ 10^{-4})$	$3.001\ 10^{-5}$ $(1.086\ 10^{-5})$

Note: The preference related parameters for each risk-neutral model are implied, with the estimated historical models as given, from the observed S&P 500 index option prices during the period of January 3, 1995 through December 29, 1995. The market price of interest rate risk is implied from the bond market using observed T-bill rates over the same sample period. The numbers in brackets are standard errors of the estimates. $\Phi_v = \hat{\xi}V_t$ is the risk premium of stochastic volatility. $\lambda^*\mu_J^*$ is the expected downsize risk due to random jump, where λ^* and μ_J^* are respectively the implied jump frequency and expected jump size with $\mu_J^* = \ln(1+\mu^*) - \frac{1}{2}\sigma_J^2$. $\Phi_J = \hat{\lambda}\hat{\mu} - \hat{\lambda}^*\hat{\mu}^*$ is the measure of the risk premium of random jump. $\Phi_r = \hat{\zeta}r_t$ is the risk premium of stochastic interest rate.

Table 5: Option Pricing Errors (%) with Parameters Implied from All Options

	Moneyness $x = \ln(S_t/KB(t,T))$ $[-0.16,\ 0.32]$	Days-to-Expiration T-t [5, 242] ≤30	 $30 - -80$	 ≥ 80	 Overall
OTM	$x \leq -0.04$	57.68 74.75 36.32 53.61 28.36 28.24 34.93 40.00 34.33 38.70	80.15 81.54 21.55 30.08 43.60 44.34 33.22 34.52 32.12 33.71	50.79 52.21 20.23 20.50 37.07 37.08 15.71 33.53 15.72 33.56	67.35 70.89 26.32 33.74 39.35 41.10 26.10 37.30 25.08 36.19
	$-0.04 < x \leq -0.01$	52.63 53.35 29.85 34.49 28.70 29.40 23.70 32.25 23.57 32.03	24.80 25.33 -7.87 16.70 16.01 16.32 7.04 13.05 7.02 13.01	-1.83 7.25 -2.79 2.80 3.18 4.18 2.74 4.38 2.72 4.36	30.34 32.06 7.73 21.01 18.45 18.98 11.90 19.79 11.86 19.67
ATM	$-0.01 < x \leq 0.00$	12.26 13.93 14.87 17.47 7.57 8.84 -2.20 2.20 -2.20 2.21	2.51 5.54 -7.15 9.74 3.61 4.64 -3.39 3.55 -3.36 3.54	-7.65 7.82 -2.07 2.07 0.80 1.77 0.30 1.34 0.31 1.34	5.02 8.82 -0.57 11.69 4.77 5.86 -2.45 3.02 -2.45 3.01
	$0.00 < x \leq 0.02$	-0.62 4.99 12.11 12.38 0.76 3.18 -0.87 0.88 -0.86 0.87	-3.94 4.83 -0.29 5.15 0.09 2.35 -0.37 0.41 -0.37 0.40	-8.83 8.86 -1.35 1.68 -1.15 1.61 -0.75 1.56 -0.84 1.64	-3.36 5.39 2.51 5.77 0.18 2.56 -0.62 1.08 -0.71 1.19
ITM	$0.02 < x \leq 0.10$	-4.01 4.02 3.08 3.17 -1.36 1.42 -0.72 1.22 -0.70 1.20	-6.91 6.91 2.82 3.07 -2.04 2.94 -1.33 1.62 -1.32 1.61	-10.27 10.27 -0.25 0.41 -2.42 2.42 -1.27 1.28 -1.32 1.33	-6.40 6.40 1.08 2.28 -1.85 1.90 -1.03 1.47 -1.04 1.48
	$x > 0.10$	-1.71 1.71 0.53 0.62 -0.43 0.43 -0.73 0.81 -0.73 0.82	-2.87 2.87 1.82 1.82 -0.60 0.65 -0.22 0.27 -0.23 0.28	-5.13 5.13 0.24 0.25 -0.84 0.95 -0.43 0.69 -0.44 0.71	-3.89 3.89 0.69 0.99 -0.70 0.78 -0.52 0.55 -0.53 0.57
Overall		11.71 17.85 11.85 14.03 6.83 9.38 7.04 10.27 6.91 10.11	13.28 20.71 2.85 10.12 8.79 11.01 7.73 9.08 7.70 9.04	4.76 16.28 3.33 3.66 4.69 8.87 2.57 5.67 2.53 5.69	10.87 18.89 5.87 10.99 6.73 10.05 5.74 7.33 5.68 7.29

Note: In each cell, from top to bottom are the mean relative errors and mean absolute relative errors for the BS, SV, JD, SVJ, and SVJ-SI models respectively.

Table 6: Option Pricing Errors (%) with Parameters Implied from Moneyness-based Subgroup Options

	Moneyness $x = \ln(S_t/KB(t,T))$ $[-0.16,\ 0.32]$	Days-to-Expiration T-t [5, 242]							
		≤30		$30 - -80$		≥ 80		Overall	
OTM	$x \leq -0.04$	23.94	31.02	19.65	29.80	1.90	25.92	17.25	18.68
		15.08	22.25	7.86	10.98	1.00	1.14	6.60	13.64
		11.77	11.71	5.93	6.21	1.40	1.41	5.91	6.62
		14.50	16.60	2.14	2.62	0.80	1.05	3.56	5.09
		14.25	16.06	1.74	2.32	0.80	1.06	3.15	4.64
	$-0.04 < x \leq -0.01$	12.79	23.10	-2.68	13.97	-19.52	19.57	-1.73	17.68
		6.90	7.92	-1.30	2.18	-0.55	1.58	0.96	3.57
		5.40	6.71	-0.81	1.01	-1.56	2.28	1.18	3.46
		5.26	5.96	-0.88	1.20	-1.40	1.82	0.86	2.91
		5.21	5.87	-0.87	1.18	-1.34	1.77	0.84	2.85
ATM	$-0.01 < x \leq 0.00$	10.94	12.43	0.89	4.98	-9.61	9.72	3.26	8.17
		3.27	5.59	1.41	1.73	-2.57	2.58	0.52	3.81
		1.74	1.88	1.24	1.48	-0.99	2.20	0.42	2.04
		1.96	1.96	1.05	1.19	-0.37	1.67	0.57	1.80
		1.96	1.97	1.02	1.18	-0.39	1.67	0.55	1.79
	$0.00 < x \leq 0.02$	-1.53	4.83	-5.08	5.54	-10.22	10.25	-4.45	5.87
		1.72	1.98	-0.33	1.91	-1.56	1.95	-0.73	1.95
		0.73	1.07	-0.10	1.70	-1.33	1.86	-0.50	1.78
		0.84	0.85	-0.42	0.47	-0.87	1.80	-0.48	1.18
		0.83	0.84	-0.42	0.46	-0.97	1.90	-0.57	1.26
ITM	$0.02 < x \leq 0.10$	2.12	2.58	2.68	3.33	-4.82	4.82	0.90	3.29
		1.98	2.03	1.36	1.48	-0.12	0.19	0.56	1.17
		0.87	0.91	0.98	1.42	-1.14	1.14	0.65	0.98
		0.46	0.78	0.64	0.78	-0.60	0.60	0.53	0.76
		0.45	0.77	0.64	0.78	-0.62	0.62	0.53	0.76
	$x > 0.10$	-1.64	1.64	-2.49	2.49	-4.02	4.02	-3.17	3.17
		-0.51	0.59	-1.58	1.58	-0.19	0.20	-0.56	0.81
		-0.41	0.41	-0.52	0.56	-0.66	0.74	-0.57	0.64
		-0.70	0.78	-0.19	0.23	-0.34	0.54	-0.42	0.45
		-0.70	0.79	-0.20	0.24	-0.34	0.56	-0.43	0.46
Overall		5.53	8.92	0.90	9.50	-4.88	10.44	2.50	9.56
		1.92	3.01	0.61	2.64	-1.13	1.34	0.98	2.55
		0.83	1.68	0.39	2.05	-2.01	2.09	0.40	1.99
		0.52	1.13	0.35	1.17	-0.65	0.84	0.29	1.10
		0.45	1.05	0.35	1.15	-0.62	0.85	0.28	1.09

Note: See Table 5.

Figure 1: Implied Black-Scholes Volatility from Observed S&P 500 Index Option Prices

Plot of the implied Black-Scholes volatility from each option quote against the degree of moneyness for different ranges of maturities.

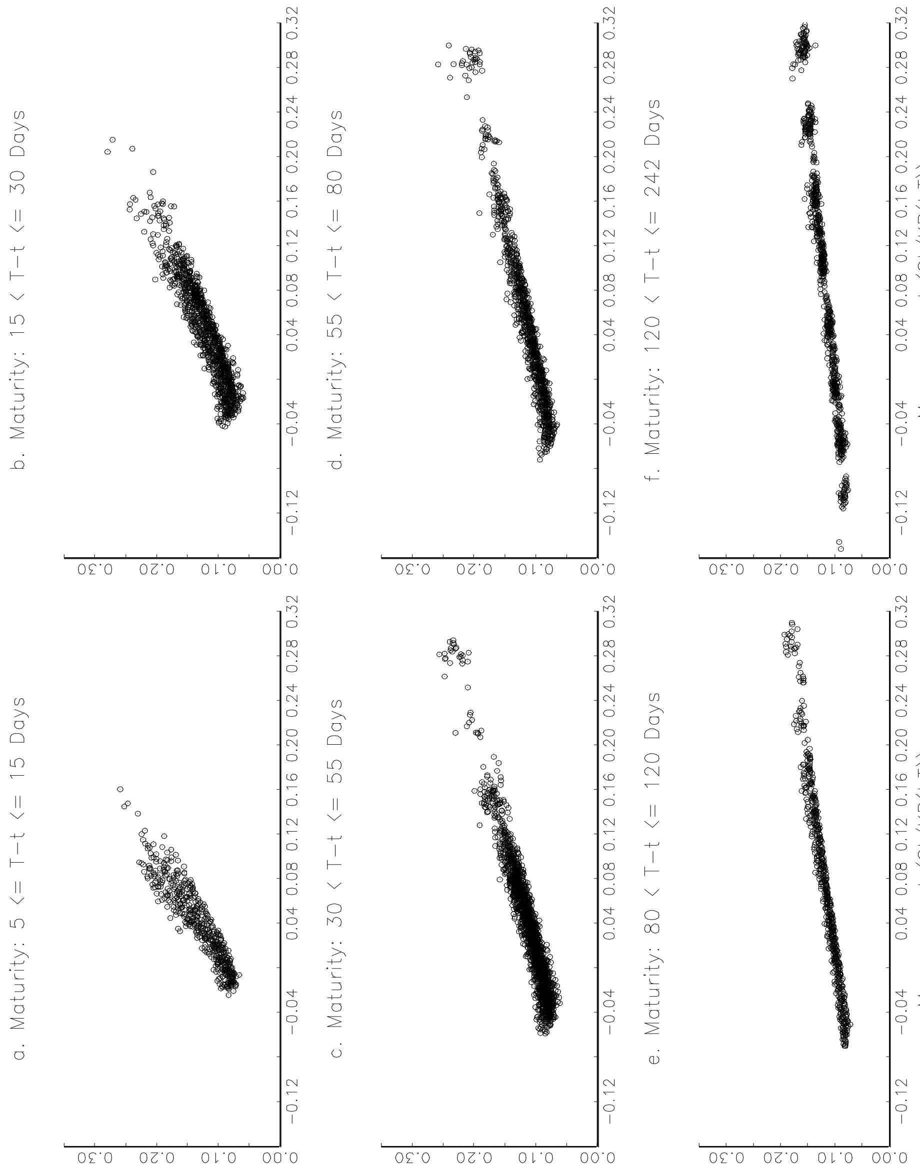

Figure 2: The Implied Risk Premium of Stochastic Volatility from S&P 500 Index Options

(a) Plot of the average risk premium of stochastic volatility against maturity, (b) Plot of the average risk premium of stochastic volatility against moneyness, and (c) Plots of the implied volatility for both the SV model and the Black-Scholes model.

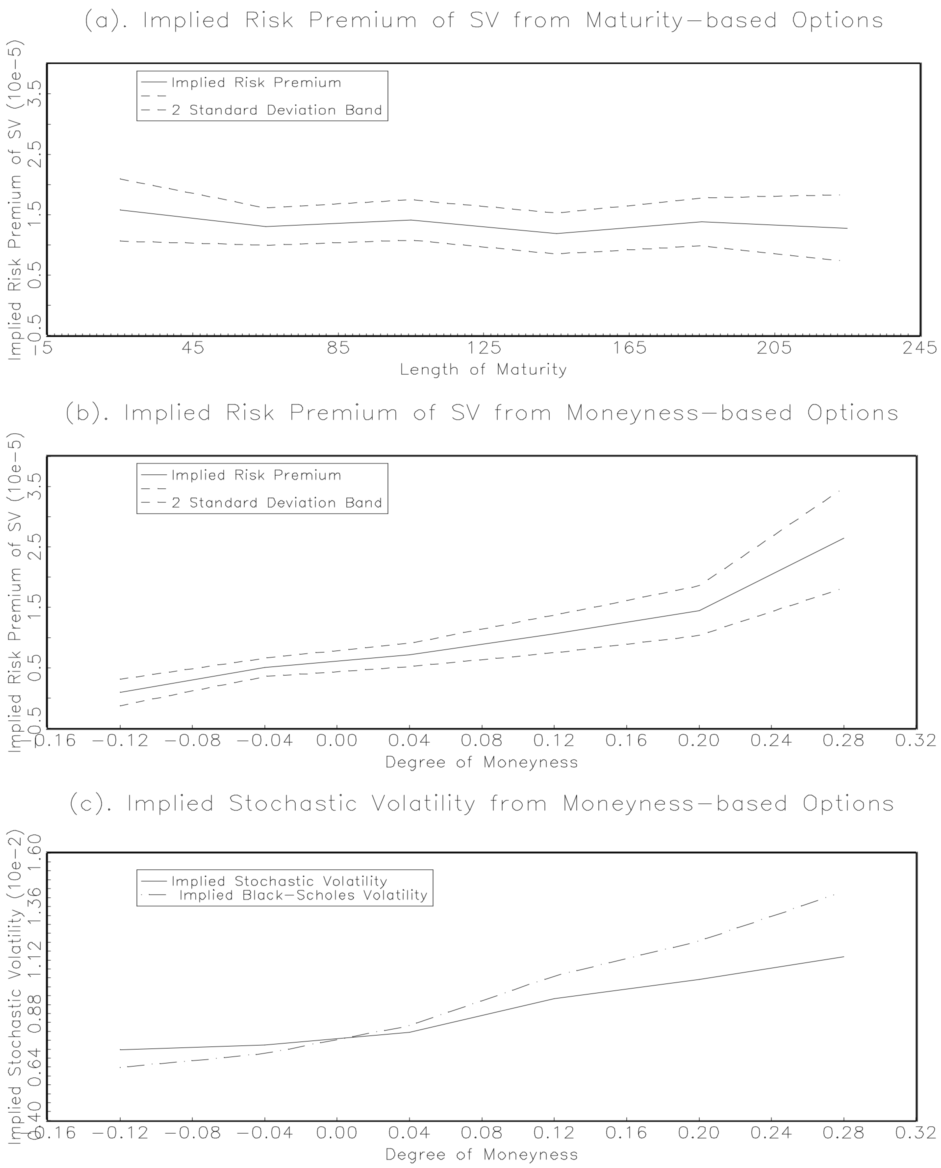

Figure 3: The Implied Risk Premium of Stochastic Volatility from FTSE 100 Index Options

(a) Plot of the average risk premium of stochastic volatility against maturity, (b) Plot of the average risk premium of stochastic volatility against moneyness, and (c) Plots of the implied volatility for both the SV model and the Black-Scholes model.