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Linear-quadratic term structure models – Toward the understanding of jumps in interest rates

George Jiang [a], Shu Yan [b,*]

[a] University of Arizona, Finance Department, Eller College of Management, Tucson, AZ 85721, United States
[b] University of South Carolina Finance Department, Moore School of Business, Columbia, SC 29208, United States

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ABSTRACT

We study linear-quadratic term structure models with random jumps in the short rate process where the jump arrival rate follows a stochastic process. Empirical results based on the US data show that incorporating stochastic jump intensity significantly improves model fit to the dynamics of both interest rate and volatility term structure. Our results also show that jump intensity is negatively correlated with interest rate changes and the average size is larger on the downside than upside. Examining the relation between jump intensity and macroeconomic shocks, we find that at monthly frequency, jumps are neither triggered by nor predictive of changes in macroeconomic variables. At daily frequency, however, we document interesting patterns for jumps associated with information shocks.

1. Introduction

Most existing term structure models assume that interest rates move continuously, following diffusion type processes. Recent studies, however, provide strong empirical evidence that interest rates contain "surprise elements" or unexpected discontinuous changes of large magnitude. There is now a growing body of literature that explicitly incorporates jumps in modeling the term structure dynamics of interest rates. Das (2002), for example, extends the Vasicek (1977) model to a jump-diffusion model and shows that incorporating jumps captures many empirical features of the Fed Funds rate that can not be explained by the continuous diffusion models. Johannes (2004) develops a statistical test of jumps and finds significant evidence for the presence of jumps in the 3-month Treasury bill rate. Piazzesi (2001, 2005) models the Fed's target rate as a jump process and shows that introducing jumps helps to fit the entire yield curve. Similarly, Farnsworth and Bass (2003) model the Fed fund rate process with jumps and find their model explaining well the shifts in the yield curve. Andersen et al. (2004) provide evidence that jumps are important to ensure the quality of model fit, especially in accommodating the extreme outlier behavior in the short-term interest rate process. Using data from the interest rate derivative market, Jarrow et al. (2007) find that incorporating jumps in the LIBOR rates is essential in explaining the volatility smiles of interest rate caps.

In the jump-diffusion term structure literature, jumps are generally modeled as a Poisson process with restrictive form of jump intensity, or jump arrival rate. Most papers assume a constant jump intensity. There is, however, strong evidence that the jump intensity is time-varying and stochastic. This is because large changes in the interest rates are often triggered by unexpected informational shocks to the market, and information flow is generally believed to be time-varying and clustered. To relax the assumption of constant jump intensity, Johannes (2004) allows the jump intensity to be a function of the level of the short rate. Motivated by the important effects of macroeconomic news announcements and policy related events on interest rates, Piazzesi (2001, 2005) assumes the jump intensity to be constant most of time but a linear function of certain variables such as the short rate, interest rate volatility, and Fed rate target around Federal Open Market Committee (FOMC) meetings and macroeconomic news announcements. In Farnsworth and Bass (2003), the jump intensity is specified as a function of the spread between the Fed fund rate and target.

In this paper, we propose a class of jump-diffusion term structure models where the jump intensity of the short rate follows its own stochastic process. This specification of jump arrival rate can be viewed as an extension of Farnsworth and Bass (2003), Johannes (2004), and Piazzesi (2001, 2005). In terms of model structure, our models belong to the linear-quadratic term structure models (LQTSMs) of Piazzesi (2001) and Cheng and Scaillet (2007),

* Corresponding author. Tel.: +1 803 777 4925; fax: +1 803 777 6876.
E-mail addresses: gjiang@email.arizona.edu (G. Jiang), syan@moore.sc.edu (S. Yan).

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which extend the existing affine term structure models (ATSMs) and quadratic term structure models (QTSMs).[1] Thus, the model nests both ATSM and QTSM as special cases. In particular, in the absence of jumps the model becomes the hybrid affine-quadratic diffusion model examined by Ahn et al. (2003). This nesting feature is very useful in assessing the additional explanatory power of incorporating jumps in the term structure dynamics. As shown in Piazzesi (2001) and Cheng and Scaillet (2007), the model remains tractable despite the additional complexity. Following the approach of Duffie and Kan (1996) and Duffie et al. (2000), we obtain a near closed-form bond pricing formula.

The more general specification of the jump intensity allows us to empirically examine the behavior of jumps in the short rate. It is known that in contrast to diffusive (or continuous) changes in the short rate, jumps have significantly different implications on asset allocation, derivatives pricing, and risk management. In addition, with empirical estimates of jump intensity we can further investigate what drives jumps in interest rates. Many existing studies have examined the impact of macroeconomic news announcements on interest rate changes.[2] One main difference of our paper from these studies is that we focus on jumps in interest rates. In particular, we investigate to what extent jumps in interest rates are related to pre-scheduled macroeconomic news announcements and to what extent they are caused by unscheduled events or unanticipated informational shocks.

In our empirical analysis, we focus on three models (SV, SVJ, and SVJT) specified under the linear-quadratic framework. The most general SVJT model incorporates both stochastic volatility and random jump of stochastic intensity in the short rate process. To examine model fit to the entire yield curve, we use the US term structure data for the period of 1984 through 2002, the so-called "post-disinflation" period. The yields along the yield curve have maturities ranging from 3 months to 30 years. We employ the implied-state GMM approach proposed in Pan (2002) for model estimation. Daily changes of bond yields with different maturities are used as moment conditions with lagged bond yields as instruments. To imply the state variables, namely the stochastic volatility and the stochastic jump intensity, we use the information from the T-bill futures market. The key identifying restriction is the dynamic relation between the observed state variables and the unobserved latent state variables.[3]

Our empirical results show that the stochastic jump intensity significantly improves the model fit to the term structure dynamics. Incorporating only stochastic volatility or modeling jumps with constant intensity in the short rate process proves to be too restrictive for the term structure dynamics. In particular, our analytical results on moment conditions show that jumps with stochastic intensity improve model fitting through the short term kurtosis. The flexible model specification improves model fitting not only to the dynamics of interest rate term structure but also to that of the volatility term structure. Furthermore, estimation results suggest that jump intensity tends to be negatively correlated with interest rate changes, with the average jump intensity implying 9 to 10 jumps per year. Overall, downside jumps tend to have a larger size (roughly 17 basis points) than upside jumps (roughly 15 basis points).

In the spirit of studies such as Ang and Piazzesi (2003) and Duffee (2006) that examine the relations between macroeconomic variables and term premia, we use the time series estimates of stochastic volatility and random jumps to investigate the relations between interest rate term structure dynamics and various macroeconomic variables. Our results suggest that at monthly frequency, while there is evidence that stochastic volatility has certain predictive power of future inflation, jumps are neither triggered by nor predictive of changes in macroeconomic variables. Once focusing on daily frequency, however, we find interesting patterns for jumps associated with various informational shocks in the financial market. Consistent with Johannes (2004), we find that while many jumps occur during the scheduled macroeconomic news release dates or FOMC meeting dates, there are a considerable number of jumps associated with unanticipated news or geopolitical development. Interestingly, interest rate jumps as a result of macroeconomic shocks are dominantly more negative than positive. These findings have important implications for understanding and managing jump risk in the bond market.

The rest of the paper is structured as follows. Section 2 presents the linear-quadratic term structure models with bond pricing formula. Section 3 contains three specific models that are closely related to those in the existing literature. These models are then examined in our empirical analysis in Section 4. Further analysis of jumps in interest rates and how they are related to macroeconomic shocks is conducted in Section 5. Section 6 concludes.

2. Linear-quadratic term structure models

In this section, we first briefly review the ATSMs and QTSMs with references to Dai and Singleton (2000) and Ahn et al. (2002) for details of model classification and identification. In the spirit of Piazzesi (2001) and Cheng and Scaillet (2007), we then show how to integrate ATSMs and QTSMs into LQTSMs. For the general specification analysis of LQTSMs, we refer to Cheng and Scaillet (2007).

2.1. General model framework

We fix a probability space $(\Omega, \mathscr{F}, P)$ with information filtration $\mathscr{F} = \{\mathscr{F}_t : t \geqslant 0\}$. As in Duffie et al. (2000), we let $X_1(t)$, the n_1-dimensional vector of affine-type state variables, follow the jump-diffusion process:

$$\mathrm{d}X_1 = \left(\mu_{X_1} - \lambda\mu_{Q_{X_1}}\right)\mathrm{d}t + \Sigma_{X_1}\mathrm{d}Z_1 + Q_{X_1}\mathrm{d}N, \tag{1}$$

where $Z_1(t)$ is an n_1-dimensional standard Brownian motion, $N(t)$ is a Poisson process with arrival intensity λ, and Q_{X_1}, the n_1-dimensional random jump size vector, follows an i.i.d. distribution with mean $\mu_{Q_{X_1}}$. N and Q_{X_1} are independent of each other and are independent of Z_1. The affine structure is defined by the condition that λ and entries of μ_{X_1} and covariance matrix $\Sigma_{X_1}\Sigma_{X_1}^{\top}$ are affine functions of X_1. In the ATSMs, the instantaneous interest rate, $r(t)$, is an affine function of X_1. The ATSMs can incorporate jumps with time-varying jump intensity. In the absence of jumps, Dai and Singleton (2000) examine the specifications and relative goodness of fit of the ATSMs.

[1] The literature on the ATSMs dates back to Vasicek (1977), and Cox et al. (1985b). Extensions include Longstaff and Schwartz (1992), Sun (1992), Pearson and Sun (1994), Balduzzi et al. (1996), Chen and Scott (1993), Chen (1996), Duffie and Kan (1996), Dai and Singleton (2000, 2002), and Chacko and Das (2002) among others. See Dai and Singleton (2003) for a review of the literature and the references. Some early examples of the QTSMs are Longstaff (1989), Beaglehole and Tenney (1992), and Constantinides (1992). More recent studies include Ahn et al. (2002), Gourieroux and Sufana (2003), and Leippold and Wu (2002, 2003). As shown in Cheng and Scaillet (2007), quadratic models can be reduced to affine structure by introducing additional state variables. However, for parsimonious presentation, the model specification in quadratic structure is often convenient and useful.

[2] Studies examining the effects of macroeconomic news announcements on price movements in the bond market include Balduzzi et al. (2001), Bollerslev et al. (2000), Fleming and Remolona (1999), Green (2004), and Kuttner (2001) among others. These papers document evidence that news surprises lead to significant movements in bond yields.

[3] Alternatively, the latent state variables can be implied using information contained in swap, caps, and swaption data as in Jagannathan et al. (2003). The intraday futures data are used in our study since they are available for the entire sample period.

Among non-affine term structure models, the QTSMs proposed by Ahn et al. (2002) and Leippold and Wu (2002) have recently attracted a lot of attention. Specifically, we let $X_2(t)$, the n_2-dimensional vector of quadratic-type state variables, follow the Gaussian diffusion process:

$$dX_2 = \mu_{X_2} dt + \Sigma_{X_2} dZ_2, \quad (2)$$

where $Z_2(t)$ is an n_2-dimensional standard Brownian motion, μ_{X_2} is an affine function of X_2, and Σ_{X_2} is a constant matrix. The quadratic structure of the QTSMs is defined by the condition that the instantaneous interest rate, $r(t)$, is a quadratic function of X_2. As shown by Ahn et al. (2002), the above specified QTSMs are maximally flexible and include some existing non-linear models such as those of Longstaff (1989), Beaglehole and Tenney (1992), and Constantinides (1992) as special cases. One advantage of QTSMs is that the positivity of $r(t)$ is guaranteed under certain restrictions on the model parameters. Similar to Dai and Singleton (2000) for the ATSM, Ahn et al. (2002) provide an analysis of empirical identification for the QTSMs.

The new term structure model examined in this paper belongs to the general LQTSMs of Piazzesi (2001) and Cheng and Scaillet (2007), and is a hybrid of ATSM and QTSM. Combining X_1 and X_2, we assume that $X(t) \equiv \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, the $(n_1 + n_2)$-dimensional vector of state variables, follows the jump-diffusion process:

$$dX = \left(\mu_X - \lambda\mu_{Q_X}\right)dt + \Sigma_X dZ + Q_X dN, \quad (3)$$

where $Z(t) = \begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix}$ is an $(n_1 + n_2)$-dimensional standard Brownian motion, and $N(t)$ is a Poisson process with arrival intensity λ. Only the first n_1 components of the random jump size vector Q_X are non-degenerate, i.e., $Q_X = \begin{pmatrix} Q_{X_1} \\ 0 \end{pmatrix}$, where Q_{X_1} is similarly defined as that in the ATSM. N and Q_X are independent of each other and are independent of Z. The drift and diffusive covariance matrix are of the form: $\mu_X = \begin{pmatrix} \mu_{X_1} \\ \mu_{X_2} \end{pmatrix}$, and $\Sigma_X\Sigma_X^\top = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^\top & \Omega_{22} \end{pmatrix}$. The linear-quadratic structure is defined by the condition that λ and entries of μ_{X_1} and Ω_{11} are linear in X_1 and quadratic in X_2, entries of μ_{X_2} and Ω_{12} are affine in X_2, and Ω_{22} is a constant matrix. Moreover, the instantaneous interest rate, $r(t)$, is the sum of a linear function of X_1 and a quadratic function of X_2.

Note that model (3) is not simply stacking equations (1) and (2) together. For example, X_2 appears (quadratically) in the drift and diffusion terms of X_1. Moreover, X_2 shows up (quadratically) in the jump intensity. This allows more flexibility in describing the dynamics of λ. The spot interest rate is driven by both affine-type (X_1) and quadratic-type (X_2) factors. This model nests both ATSM and QTSM as special cases. If X_2 is absent, the model simplifies to the ATSM. On the other hand, if X_1 is absent, the model reduces to the QTSM. In the absence of jumps, the model becomes the hybrid affine-quadratic diffusion model examined by Ahn et al. (2003). Cheng and Scaillet (2007) provide the structural constraints, as well as the admissibility conditions for the LQTSM class.

2.2. Bond pricing

To price bonds, we need the dynamics of the state variables under the risk-neutral probability measure. We show in the Appendix A that this can be obtained by solving a representative investor's utility optimization problem in equilibrium and the result is summarized in the following proposition.

Proposition 1. Given the jump-diffusion process (3), there is an equivalent risk-neutral probability measure $(\Omega, \mathscr{F}, P^*)$ under which the state variables follow the jump-diffusion process:

$$dX = \left(\mu_X^* - \lambda^*\mu_{Q_X}^*\right)dt + \Sigma_X dZ^* + Q_X^* dN^*, \quad (4)$$

where all the random variables and parameters with a star are similarly defined as those under the real probability measure $(\Omega, \mathscr{F}, P)$.

The proposition illustrates that there is a particular transformation from the objective probability measure to the risk-neutral probability measure. The risk-neutral specification of the model enables us to use the general transform analysis of Cheng and Scaillet (2007). For our purpose, we can price bonds in near closed-form as shown in the following proposition.

Proposition 2. Let $P(X_1, X_2, \tau)$ be the price of a zero-coupon bond with maturity τ, as a function of the state variables. And let θ denote the parameter vector of the model under the risk-neutral probability measure. Then the bond price is given by

$$P(X_1, X_2, \tau; \theta) = e^{A(\tau)+B(\tau)^\top X_1 + C(\tau)^\top X_2 + X_2^\top D(\tau) X_2}, \quad (5)$$

where $(A, B, C, D) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2 \times n_2}$, and D is symmetric. (A, B, C, D) are solutions to the ordinary differential equations (ODEs) given in the Appendix A.

For the purposes of model estimation, it is also useful to derive the conditional characteristic function (CCF) of the state variables. The CCF offers a very convenient way to compute the conditional moments of state variables and establish relations between the moments of observed variables and unobserved variables. We provide the main result of the CCF in the following proposition.

Proposition 3. The CCF of $X(t)$ under the risk-neutral probability measure is given by

$$f(s, X(t), \tau; \theta) = e^{i\left[A^*(s,\tau)+B^*(s,\tau)^\top X_1 + C^*(s,\tau)^\top X_2 + X_2^\top D^*(s,\tau) X_2\right]}, \quad (6)$$

where $i = \sqrt{-1}$, $s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ with $(s_1, s_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $(A^*, B^*, C^*, D^*) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2 \times n_2}$, and D^* is symmetric. (A^*, B^*, C^*, D^*) are solutions to the ODEs given in the Appendix A.

3. Examples

In this section, we present three representative models for illustration of model properties. The benchmark models are the stochastic volatility (SV) model and stochastic volatility model with constant jump intensity (SVJ) within the ATSMs. The third model is specified in the LQTSMs framework incorporating both stochastic volatility and random jumps with stochastic intensity (SVJT). All models are specified under the risk-neutral probability measure with bond price and CCF formulas given in the Appendix A.

3.1. Affine SV and SVJ models

Both SV and SVJ models are two-factor affine models where the state variables are the instantaneous interest rate r and its volatility v. Since the SV model is a restricted case of the SVJ model, we first present the SVJ model under which r and v have the following dynamics:

$$dr = (\mu_r^* + \kappa_{rr}^* r + \kappa_{rv}^* v - \lambda^*\mu_{Q_r}^*)dt + \sqrt{v}dZ_r^* + Q_r^* dN^*,$$
$$dv = \left(\mu_v^* + \kappa_{vr}^* r + \kappa_{vv}^* v\right)dt + \sigma_v\sqrt{v}dZ_v^*,$$

where Z_r^* and Z_v^* are Brownian motions with correlation $dZ_r^* dZ_v^* = \rho dt$.[4] N^* is a Poisson process with constant arrival intensity, λ^*. We assume that the jump size, Q_r^*, has a Bernoulli distribu-

[4] To see this model belonging to the ATSMs defined by Eq. (1), notice that $X_1 = \begin{pmatrix} r \\ v \end{pmatrix}$ and r is just the first component of X_1. Note that in all three examples considered in the paper, the instantaneous interest rate, r, can take negative values. This is an undesirable feature but seems a necessary compromise in order to maintain tractability. Our empirical analysis, however, indicates that the probability of r being negative is very small based on the parameter estimates of these models.

tion with probability p to take a positive value $\mu_+^* > 0$ and probability $1-p$ to take a negative value $\mu_-^* < 0$. The expected average jump size is $\mu_{Q_r}^* = p\mu_+^* + (1-p)\mu_-^*$. This specification of the jump distribution has a couple of advantages. First, it allows asymmetric jumps in the instantaneous interest rate. Second, both the conditional positive and negative jump sizes μ_+^* and μ_-^* are assumed to be constant, instead of following a distribution with infinite support. Given the finite jump size and infrequent occurrence of jumps, this specification alleviates the concern of negative interest rates. In the most general form, we allow the interest rate and volatility to enter the drift terms of both variables.

In the absence of jumps ($\lambda^* = 0$ and $Q_r^* = 0$), the SVJ model reduces to the diffusive SV model. The SV model can be regarded as a restricted version of the three-factor maximally flexible model $A_1(3)$ of Dai and Singleton (2000) by imposing certain restrictions, especially by dropping the central tendency variable. We choose this model as the benchmark for two reasons. First, it is nested within the SVJ model, allowing us to examine the effect of adding jumps on model fit to the yield curve. Second, we also have examined various three-factor models with either stochastic mean or bivariate stochastic volatilities. Compared to the extension that incorporates jumps, the three-factor models with either stochastic mean or bivariate stochastic volatilities all have poorer model fit.

With restrictions $\kappa_{vr}^* = 0$, and $\mu_v^* > 0$ and $\kappa_{vv}^* < 0$, the conditional variance of the short rate in the SV model follows a mean-reverting squared-root process. It can be shown that the variance is autoregressive with $E[v_t|v_{t-1}] = e^{\kappa_{vv}^*}v_{t-1} + (e^{\kappa_{vv}^*} - 1)(\mu_v^*/\kappa_{vv}^*)$ and $\mathrm{Var}[v_t|v_{t-1}] = \left(\frac{\sigma_v^2}{\kappa_{vv}^*}\right)(e^{2\kappa_{vv}^*} - e^{\kappa_{vv}^*}v_{t-1}) + \mu_v^*\left(\frac{\sigma_v^2}{2\kappa_{vv}^{*2}}\right)(1 - e^{\kappa_{vv}^*})^2$. Namely, both the variance of interest rate and the variance of variance are determined by the lagged sate variable v_{t-1} in a linear fashion. The model specification is also restrictive on the distribution of daily short rate changes, as reflected in the skewness and kurtosis, where moments of daily short rate changes can be derived from the CCF in the Appendix A. The diffusive nature of the short rate process and the persistence of variance imply very smooth changes in interest rate with relatively low level of kurtosis.

Similar to the SV model, the SVJ model can be regarded as a restricted version of the three-factor maximally flexible model $A_1(3)$ of Dai and Singleton (2000) augmented by a jump component in the spot rate process. In the terminology of Cheng and Scaillet (2007), the $A_1(3)$ class of Dai and Singleton is in the $LQ_1^0(3)$ class of LQTSMs. Also similar to the SV model, the conditional variance of the short rate in the SVJ model follows a mean-reverting squared-root process when we impose the restrictions $\kappa_{vr}^* = 0$, and $\mu_v^* > 0$ and $\kappa_{vv}^* < 0$. However, the moments of short rate changes, as derived from the CCF in the Appendix A, allow for skewness and high level of kurtosis due to the presence of random jumps. In particular, the third and fourth moments of the compensated jump component are, respectively $\lambda^*(p\mu_+^{*3} + (1-p)\mu_-^{*3})$ and $\lambda^*(p\mu_+^{*4} + (1-p)\mu_-^{*4})$. The jump component is nevertheless time-homogeneous with λ^* being constant and thus excludes day-to-day variability in the probability of jump occurrence.

3.2. Linear-quadratic SVJT model with stochastic jump intensity

The third example is a jump-diffusion model with time-varying jump intensity. That is, λ^* now follows a stochastic process correlated with the instantaneous interest rate and volatility processes:

$$\begin{aligned} dr &= \left(\mu_r^* + \kappa_{rr}^* r + \kappa_{rv}^*\sqrt{v} + \kappa_{r\lambda}^*\sqrt{\lambda^*} + \zeta_{vv}^* v + 2\zeta_{v\lambda}^*\sqrt{v\lambda^*} + \zeta_{\lambda\lambda}^*\lambda^* - \lambda^*\mu_{Q_r}^*\right)dt \\ &\quad + \sqrt{v}dZ_r^* + Q_r^* dN^*, \\ d\sqrt{v} &= \left(\mu_v^* + \kappa_{vv}^*\sqrt{v} + \kappa_{v\lambda}^*\sqrt{\lambda^*}\right)dt + \sigma_v dZ_v^*, \\ d\sqrt{\lambda^*} &= \left(\mu_\lambda^* + \kappa_{\lambda v}^*\sqrt{v} + \kappa_{\lambda\lambda}^*\sqrt{\lambda^*}\right)dt + \sigma_\lambda dZ_\lambda^*, \end{aligned}$$

where dZ_r^*, dZ_v^*, and dZ_λ^* are Brownian motions with correlation matrix: $\Sigma = \begin{pmatrix} 1 & \rho_{rv} & \rho_{r\lambda} \\ \rho_{rv} & 1 & \rho_{v\lambda} \\ \rho_{r\lambda} & \rho_{v\lambda} & 1 \end{pmatrix}$. The jump size, Q_r^*, again follows a Bernoulli distribution as in the SVJ model, allowing asymmetry between positive and negative jumps.

The model belongs to the LQTSMs defined by equations (3). In fact, according to the classification of Cheng and Scaillet (2007), this model is in the $LQ_0^2(3)$ family. That is, there are three state variables: one affine state variable in $X_1 = r$ and two quadratic state variables in $X_2 = \begin{pmatrix} \sqrt{v} \\ \sqrt{\lambda^*} \end{pmatrix}$. The stochastic variance and jump intensity, v and λ^*, are squares of the state variables in X_2.[5] Cheng and Scaillet (2007) classify all admissible three-factor LQTSMs. Although not having the maximal flexible structure, we choose this model specification mainly for the reason of parsimony. Note, however, that this model does not nest the SVJ model because volatility is quadratic in this model but affine in the SVJ model.

Compared to the SV and SVJ models, the SVJT model offers a more flexible structure for both the variance of interest rate and variance of variance. The instantaneous variance of the short rate is given by $v_t + \lambda_t^* p(1-p)(\mu_+^* - \mu_-^*)^2$ with the first term measuring the diffusive variance and the second term capturing the variance of the jump component. Under the restrictions $\kappa_{v\lambda}^* = \kappa_{\lambda v}^* = 0$ and $\mu_v^* > 0, \mu_\lambda^* > 0, \kappa_{vv}^* < 0, \kappa_{\lambda\lambda}^* < 0$, both $\sqrt{v_t}$ and $\sqrt{\lambda_t^*}$ follow mean-reverting Gaussian processes. For instance, the following properties for v_t and λ_t^* can be derived:

$$\begin{aligned} \mathrm{Var}[v_t|v_{t-1}] &= 4\left(\frac{\mu_v^*\sigma_v^2}{2\kappa_{vv}^{*2}}\right)(e^{-2\kappa_{vv}^*} - e^{-4\kappa_{vv}^*})v_{t-1} \\ &\quad + 4\left(\frac{\mu_v^{*2}\sigma_v^2}{\kappa_{vv}^{*3}}\right)(1 - e^{-2\kappa_{vv}^*})(e^{-\kappa_{vv}^*} - e^{-2\kappa_{vv}^*})\sqrt{v_{t-1}} \\ &\quad + 2\left(\frac{\mu_v^*\sigma_v^2}{2\kappa_{vv}^{*2}}\right)(1 - e^{-2\kappa_{vv}^*})\left(2\left(\frac{\mu_v^*}{\kappa_{vv}^*}\right)^2(1 - e^{-\kappa_{vv}^*})^2 \right. \\ &\quad \left. + \left(\frac{\mu_v^*\sigma_v^2}{2\kappa_{vv}^{*2}}\right)(1 - e^{-2\kappa_{vv}^*})\right) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}[\lambda_t^*|\lambda_{t-1}^*] &= 4\left(\frac{\mu_\lambda^*\sigma_\lambda^2}{2\kappa_{\lambda\lambda}^{*2}}\right)(e^{-2\kappa_{\lambda\lambda}^*} - e^{-4\kappa_{\lambda\lambda}^*})\lambda_{t-1}^* \\ &\quad + 4\left(\frac{\mu_\lambda^{*2}\sigma_\lambda^2}{\kappa_{\lambda\lambda}^{*3}}\right)(1 - e^{-2\kappa_{\lambda\lambda}^*})(e^{-\kappa_{\lambda\lambda}^*} - e^{-2\kappa_{\lambda\lambda}^*})\sqrt{\lambda_{t-1}^*} \\ &\quad + 2\left(\frac{\mu_\lambda^*\sigma_\lambda^2}{2\kappa_{\lambda\lambda}^{*2}}\right)(1 - e^{-2\kappa_{\lambda\lambda}^*})\left(2\left(\frac{\mu_\lambda^*}{\kappa_{\lambda\lambda}^*}\right)^2(1 - e^{-\kappa_{\lambda\lambda}^*})^2 \right. \\ &\quad \left. + \left(\frac{\mu_\lambda^*\sigma_\lambda^2}{2\kappa_{\lambda\lambda}^{*2}}\right)(1 - e^{-2\kappa_{\lambda\lambda}^*})\right) \end{aligned}$$

It can also be seen from the above moment conditions that the bi-variate processes of v_t and λ_t^* offer a flexible structure for the conditional variance of the short rate as well as the variance of the conditional variance. As shown in our empirical estimation, the improvement of model specification of SVJT over the SV and SVJ models comes from its flexibility of fitting into both the term structure of interest rate variance as well as the variance of variance. Similar to the SVJ model, the jump component allows for

[5] Note that in our specification, $\sqrt{v}$ and $\sqrt{\lambda^*}$ follow Gaussian processes and can be negative but v and λ^* are always non-negative because they are squares of their corresponding square-roots. Technically speaking, this is different from directly restricting v and λ^* to be non-negative processes. The main difference is the boundary properties at $v = 0$ and $\lambda^* = 0$. For the exact effect of such restrictions, please refer to the results in Ball and Roma (1994). Also see Cheng and Scaillet (2007) for a discussion of the sign-changing behavior for general linear-quadratic jump-diffusion models.

skewness and higher level of kurtosis in short rate changes. The third and fourth moments of the compensated jump component are respectively $\lambda_t^*(p\mu_+^{*3} + (1-p)\mu_-^{*3})$ and $\lambda_t^*(p\mu_+^{*4} + (1-p)\mu_-^{*4})$. The key difference with the SVJ model is that λ_t^* is now time-varying over time. This allows for day-to-day variation of jump behavior in the short rate process.

4. Empirical analysis

4.1. Estimation of the linear-quadratic models

It is well known that the statistical inference of continuous-time jump-diffusion models with latent state variables presents a great challenge, at least for the following two reasons. First, the transition density of the continuous-time jump-diffusion process is, with the exception of a few special cases, generally unknown or unavailable in closed-form so that the conventional likelihood-based inference is inapplicable. Second, any inference procedure has to deal with the unobserved variables, which typically involves integrating out the latent variables in the likelihood function. In this paper, we employ the implied-state generalized method of moments (IS-GMM) approach, as proposed in Pan (2002), for the estimation of the LQTSMs. The IS-GMM approach effectively deals with both of the above issues. First, instead of relying on the transition density function, the IS-GMM approach uses moment conditions for model estimation. Second, by implying the latent state variables from observed information, the IS-GMM approach avoids intensive simulations in the estimation process.

Specifically, let

$$\epsilon_{t+1,i} = y_{t+1}(\tau_i) - E_t[y_{t+1}(\tau_i)]$$

be the unexpected changes of bond yield with maturity $\tau_i (i = 1, 2, \ldots, M)$. The expectation of bond yield on day $t+1$ is taken conditional on information up to day t and under specific model specification. In this paper, we focus on the following moment conditions involving the unexpected daily changes of yields, with lagged unexpected daily changes of yields as instrumental variables, i.e.,

$$u_t(\theta) = \begin{bmatrix} \epsilon_{t+1,1} \\ \cdot \\ \epsilon_{t+1,M} \\ \epsilon_{t,1}\cdot\epsilon_{t+1,1} \\ \cdot \\ \epsilon_{t,M}\cdot\epsilon_{t+1,M} \end{bmatrix}, \quad t = 0, 1, \ldots, T-1, \tag{7}$$

where $u_t(\theta)$ is the vector of moment conditions with θ being the set of model parameters. The above moment conditions emphasize the fit of model generated yield curve to the observed yield curve.

Note that the above moment conditions depend not only on the model parameters (θ), but also on the latent state variables because the expected bond yields $E_t[y_{t+1}(\tau_i)]$ are functions of the latent state variables, in addition to the short rate on day t. Under the IS-GMM approach, the latent state variables are implied from additional information in the market. For instance, Pan (2002) uses the near term ATM option prices to imply the stochastic volatility. In this paper, we identify the latent state variables using information in the futures market of Treasury bills. The key identifying restriction is the dynamic relation derived directly under the model specification between the observed state variables and the unobserved latent state variables.

Formally, under the linear-quadratic model framework with the CCF given in Proposition 4, the following relation between the cumulants of the spot interest rate and the state variables can be derived:

Proposition 4. Given the CCF of the state variables $f(s, r(t+\tau), X_1(t+\tau), X_2(t+\tau)|r(t) = r, X_1(t) = X_1, X_2(t) = X_2) = e^{i[A^*(s,\tau)+B_r^*(s,\tau)r+B_1^*(s,\tau)^\top X_1+C^*(s,\tau)^\top X_2+X_2^\top D^*(s,\tau)X_2]}$ with the initial conditions $A^*(s,0)=0, B_r^*(s,0)=s_r, B_1^*(s,0)=s_1, C^*(s,0)=s_2,\ D^*(s,0)=0,$ where $s=(s_r,s_1,s_2)$, we have the following relation:

$$K^{(l)}(r(t+\tau)|\mathscr{F}_t) = \frac{\partial^l A^*}{\partial s_r^l}|_{s=0} + \frac{\partial^l B_r^*}{\partial s_r^l}|_{s=0} r + \frac{\partial^l B_1^{*\top}}{\partial s_r^l}|_{s=0} X_1 + \frac{\partial^l C^{*\top}}{\partial s_r^l}|_{s=0} X_2 + X_2^\top \frac{\partial^l D^*}{\partial s_r^l}|_{s=0} X_2, \tag{8}$$

where $l = 1, 2, \ldots, L$, and $K^{(l)}(r(t+\tau)|\mathscr{F}_t)$ denotes the lth order cumulant of $r(t+\tau)$ conditional on $\mathscr{F}_t$ under the P^*-measure.

The above proposition is a direct application of Proposition 4 with interest rate $r(t)$ specified explicitly as an affine-type state variable in our general linear-quadratic jump-diffusion model. The rest of the variables in (X_1, X_2) are unobserved or latent state variables of dimension $n_1 - 1$ and n_2. The general principle is that with given parameter values, the latent state variables can be identified from a set of moment conditions of $r(t+\tau)$, i.e. $\{K^{(l)}(r(t+\tau)|\mathscr{F}_t)\}_{l=1}^L$. The only requirements on A^*, B_r^*, B_1^*, C^*, and D^* are that there exist solutions to a system of linear-quadratic equations given by (8). In our empirical analysis, the SV and SVJ models involve only the stochastic volatility which is identified from the realized variance computed from intraday changes of the T-bill yields in the futures market, while the SVJT model involves stochastic volatility and stochastic jump intensity which are identified from the conditional variance and kurtosis (the fourth moment) calculated from intraday changes of yields in the futures market. It is known that, compared to stochastic volatility, jumps have a distinct impact on higher order moments of interest rate changes (such as skewness and kurtosis). Based on simulations of the sampling paths of the SVJT model, we find that, relative to skewness, kurtosis is a more robust moment. The use of kurtosis thus helps to disentangle jumps from stochastic volatility in the T-bill futures yields.

Denote the vector of the population moment conditions in (7) by $u_t(\theta)$, and the vector of corresponding sample moment conditions with sample size T by $g_T(\theta)$, the GMM estimator is given by

$$\hat{\theta}_T = \arg\min_{\theta\in\Theta}\{\mathscr{J}_T(\theta) = g_T'(\theta)\mathscr{W}_T(\theta)g_T(\theta)\} \tag{9}$$

where Θ denotes the permissible parameter space and $\mathscr{W}_T$ a positive definite weighting matrix which is chosen to yield the smallest asymptotic covariance matrix of the GMM estimator of θ as in Hansen (1982). Following Hansen (1982) and Pan (2002), under certain regularity conditions the estimator $\hat{\theta}_T$ is consistent and asymptotically normal, i.e. $T^{1/2}(\hat{\theta}_T - \theta) \overset{a}{\sim} \mathscr{N}(0, V_T)$, and a consistent estimator of V_T is given by $\widehat{V}_T = \frac{1}{T}(\widehat{D}'(\theta)\widehat{S}^{-1}(\theta)\widehat{D}'(\theta))^{-1}$, where $\widehat{D}(\theta)$ is the Jacobian matrix of $g_T(\theta)$ with respect to θ evaluated at the estimated parameters, and $\widehat{S}(\theta)$ is a consistent estimator of $S(\theta) = E[u_t(\theta)u_t'(\theta)]$. Further, because the GMM criterion function $\mathscr{J}_T(\theta)$ measures the distance between $g_T(\theta)$ and zero, it serves as a goodness-of-fit test for the model. As shown by Hansen (1982), under the null hypothesis of correct model specification we have $\mathscr{J}_T(\theta) \overset{a}{\sim} \mathscr{X}_{q-p}^2$ where q and p are, respectively the dimensions of u_t and θ.

4.2. The data

The data used in our empirical analysis consist of daily US T-bill, T-note and T-bond yields with maturities 3-, 6-month, 1-, 2-, 3-, 5-, 7-, 10-, and 30-year from January 3, 1984 to February 15, 2002. This period belongs to the so-called "post-disinflation" period (see, e.g., Duffee, 2006). Note that 2002 was the year the long term bond with 30-year maturity was discontinued. We use the 3-month T-bill yields as proxy of the short rate. Fig. 1 plots the time series of the daily 3-month Treasury yields and

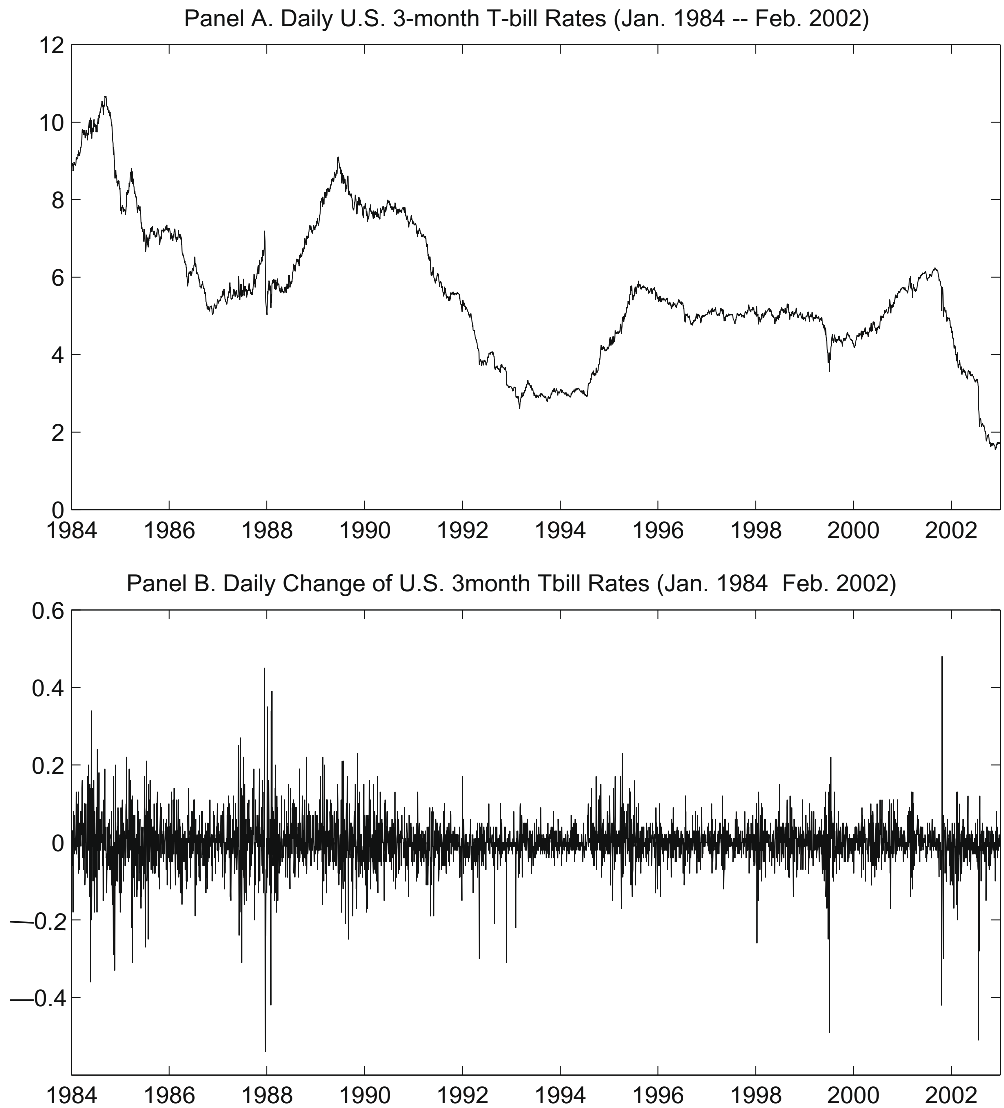

Fig. 1. Daily and daily changes of US 3-month T-bill yields. This figure plots the time series of the daily 3-month Treasury bill yields and the daily changes in panels A and B respectively. The sample period is from January 3, 1984 to February 15, 2002.

the daily changes in panels A and B, respectively. The plot reflects some large changes of the 3-month T-bill yields from day to day. Descriptive statistics of the data are reported in Table 1. Both skewness and kurtosis statistics indicate that interest rates are non-normally distributed. As in Aït-Sahalia (1996), we also report the first five orders of autocorrelations of the monthly interest rates as well as monthly interest rate changes in Table 1. The autocorrelation coefficients are in general very high, reflecting high persistence of interest rates.

Table 2 reports the principal components of daily interest rate changes and weekly volatility changes where weekly volatility is calculated from daily changes in interest rates. The principal component analysis reports similar results as in Litterman and Scheinkman (1991) and other existing studies. Namely, there are mainly three factors driving the dynamics of yield curve, the level, the slope and the curvature. They explain almost 96% variation of the yield curve. The principal component analysis of weekly volatility changes reveals a similar pattern for the dynamics of volatility term structure. That is, there are also mainly three factors driving the dynamics of volatility term structure, albeit these factors explain a lesser portion of the variation of the entire term structure.

Information in the futures market contains the intraday 3-month T-bill quotes with a maturity cycle of March, June, September, and December. The futures data we use are the 15-min 3-month T-bill futures quotes. Due to illiquidity and market microstructure concerns, we choose the futures contract with the shortest maturity among the March, June, September, and December cycle but with more than one week remaining to maturity. The choice of sampling frequency follows Andersen et al. (2003) in order to further avoid market microstructure effect associated with high frequency data. Sampled at 15-min frequency, the changes of the 3-month T-bill future yields used in our analysis have, respectively, a first order autocorrelation of −5.79% and second order autocorrelation of 2.58%. Higher orders of autocorrelation are in general diminishing and negligible. To ensure the robustness of our analysis, we also follow Hansen and Lunde (2006) and Zhou (1996) to correct for the first order autocorrelation in the variance estimation, there are no material changes for the results.

4.3. Estimation results and model fitting

The models specified in Section 2 under the risk-neutral probability measure are estimated following the IS-GMM procedure de-

Table 1
Summary statistics of interest rates

	Mean*	Standard deviation	Skew	Kurt	Min	Max	Autocorrelations of monthly series				
							ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
Panel A: Daily interest rates											
R3M	5.597	1.780	0.406	0.029	1.55	10.67	0.98	0.96	0.94	0.92	0.89
R6M	5.708	1.789	0.429	0.182	1.59	10.77	0.98	0.96	0.94	0.91	0.89
R1Y	6.190	1.980	0.585	0.429	1.93	12.34	0.98	0.96	0.93	0.91	0.88
R2Y	6.635	1.989	0.772	0.678	2.32	13.17	0.98	0.95	0.93	0.90	0.87
R3Y	6.834	1.990	0.877	0.775	2.70	13.49	0.98	0.95	0.92	0.89	0.86
R5Y	7.120	1.956	1.031	0.991	3.47	13.84	0.97	0.95	0.92	0.89	0.85
R7Y	7.341	1.938	1.060	0.970	3.95	13.95	0.97	0.95	0.92	0.89	0.85
R10Y	7.435	1.932	1.026	0.872	4.16	13.99	0.97	0.95	0.92	0.89	0.85
R30Y	7.672	1.817	1.044	0.931	4.70	13.94	0.97	0.95	0.92	0.89	0.86
Panel B: Daily interest rate changes											
ΔR3M	−1.609	0.059	−0.569	11.53	−0.54	0.48	0.12	0.04	0.00	0.07	0.05
ΔR6M	−1.626	0.057	−0.967	13.39	−0.78	0.31	0.08	0.04	0.04	0.04	0.06
ΔR1Y	−1.746	0.063	−0.751	10.57	−0.83	0.36	0.08	0.03	0.02	0.05	0.04
ΔR2Y	−1.743	0.068	−0.530	8.11	−0.84	0.36	0.09	0.03	0.02	0.03	0.04
ΔR3Y	−1.677	0.069	−0.347	6.34	−0.79	0.40	0.08	0.04	0.03	0.01	0.03
ΔR5Y	−1.615	0.069	−0.247	5.38	−0.77	0.41	0.06	−0.08	0.03	−0.00	0.03
ΔR7Y	−1.574	0.069	−0.179	5.27	−0.77	0.42	0.04	−0.02	0.03	−0.00	0.02
ΔR10Y	−1.545	0.067	−0.188	5.46	−0.75	0.39	0.02	−0.02	0.03	−0.01	0.03
ΔR30Y	−1.448	0.059	−0.433	7.58	−0.76	0.32	−0.04	−0.02	0.03	−0.00	0.02

Panels A reports the summary statistics of the daily interest rates with maturities of 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, 10-year and 30-year from January 3, 1984 to February 15, 2002. Panel B reports the summary statistics of the daily interest rate changes. The mean for the daily changes of interest rates has a magnitude of 10^{-4}. The autocorrelations are calculated from the monthly interest rates data.

scribed in Section 3.1. Dai and Singleton (2000), in their study of the ATSMs, point out that highly parameterized models may be over-identified and overfitting the data. Following their suggestion, we restrict $\kappa^*_{vr} = 0$ for the SV and SVJ models. For the SVJT model, we impose more restrictions for parsimony, namely $\kappa^*_{rv} = \kappa^*_{r\lambda} = \kappa^*_{v\lambda} = \kappa^*_{\lambda v} = \zeta^*_{v\lambda} = \zeta^*_{\lambda\lambda} = 0$. As mentioned earlier, the moment conditions include the daily unexpected changes of yields with nine different maturities, including the 3-month T-bill yields. The lagged unexpected daily changes of yields are used as the instrumental variable. This results in a total of 18 moment conditions.[6]

The estimation results are reported in Table 3.[7] All parameter estimates are based on daily annualized bond yields. The results of all three models suggest that the interest rate process is highly persistent with very slow mean reversion. On the other hand, the stochastic variance process for the SV model has a high mean reverting parameter and a very high volatility of volatility. The Hansen *J*-test, with a *p*-value of 0.02%, suggests that the SV model fits poorly to the yield curve.

The SVJ model also fits the yield curve poorly, considering the *p*-value of 0.19% for the Hansen J-test. However, it does suggest a significant jump component in the interest rate process. The jump frequency is statistically significant implying, on average, 9–10 jumps per year. This high jump frequency is most likely driven by the large changes of the short rate in the 80's. The results also suggest that whenever a jump occurs, there is a slightly higher chance (but statistically insignificant) for a negative jump than a positive jump ($1 - p = 51.01\%$). In addition, there is on average a larger size for negative jumps ($\mu^*_- = -0.1694\%$ or roughly 17 bps) than for positive jumps ($\mu^*_+ = 0.1536\%$ or roughly 15 bps). The difference is statistically different at 10% critical level. Finally, there is a significant but low level of negative correlation between stochastic volatility and interest rate changes.

The SVJT model represents a significant improvement in model fit to the yield curve. The Hansen J-test has a *p*-value of 2.64%, much higher than those of the SV and SVJ models. It should be noted that other than the additional jump intensity process in the SVJT model, both the variance and jump intensity processes are specified for their square roots, which also likely helps model fit. It is clear that the volatility (square root of variance) process for the SVJT model is well behaved with relatively low conditional volatility over time. The jump intensity process, however, is very volatile with strong mean reversion. Similar to the SVJ model, we identify on average a larger size for negative jumps ($\mu^*_- = -0.1689\%$) than for positive jumps ($\mu^*_+ = 0.1518\%$). Both stochastic volatility and jump intensity processes are negatively correlated with the changes of the short rate. There is a non-negligible correlation between the jump intensity and interest rate changes at −4.331%. In addition, a downside jump tends to be of a larger magnitude than an upside jump.

As pointed out by Ahn et al. (2003), a challenge for any interest rate model is its ability to fit the dynamics of both interest rate term structure and volatility term structure. While the above model specification test suggests that the SVJT model fits the dynamics of interest rate term structure relatively well, it is unknown how it fits the dynamics of volatility term structure. To examine model fit to the volatility term structure, we use the observed daily short rate and the estimates of latent state variables to simulate the yield curve. For instance, for the SVJT model we simulate bond yields with various maturities based on the daily observations of spot rate and the estimates of stochastic volatility and jump intensity. This way, we obtain a panel of model implied yields with the same structure as the empirical data. Then, similar to Table 2 we perform principal component analysis on the weekly volatility changes. For comparison purpose, similar results are also obtained for the SVJ model. The results are not reported for brevity. Compared to SVJ model, the second and third factors of the SVJT model are more pronounced. Specifically, the second and third factors explain, respectively, 14.6% and 4.2% variation of the volatility term structure. In particular, based on loadings of the third factor the SVJT model

[6] Both the moment conditions for bond yields and the identifying restrictions for latent variables involve numerical solutions of some ODEs. In this paper, we employ a procedure using the implicit backward-difference method (see, e.g., Aiken, 1985). The effectiveness and accuracy of the numerical procedure is verified for the CIR model using the closed-form bond pricing formula, and the affine SV model using theMonte Carlo simulations, see further details in Appendix A.5.

[7] To gauge the extent of negative interest rates, we simulate the sampling pathes of the SVJT model based on the parameter estimates. Using a Euler scheme with 100 intervals per day and starting value $r_0 = 5.6\%$ (the sample mean), we find that out of 10,000 sampling paths over a 1-year period, there are 61 of them reaching negative values.

Table 2
Principal components of daily interest rate changes and weekly volatility changes

Factor	Factor loadings									Variation (%)
Panel A: Principal components of daily interest rate changes										
1	2.41	3.02	3.91	4.40	4.56	4.57	4.48	4.25	3.55	0.822
2	−2.82	−1.99	−1.15	−0.27	0.16	0.65	1.10	1.20	1.33	0.106
3	1.27	−0.04	−0.81	−0.92	−0.66	−0.14	0.39	0.61	1.00	0.030
4	0.72	−1.10	−0.38	0.37	0.48	0.29	0.02	−0.08	−0.52	0.015
5	−0.11	0.65	−0.95	0.04	0.26	0.31	0.11	0.02	−0.39	0.010
6	0.01	0.02	−0.33	0.66	0.06	−0.37	−0.31	−0.20	0.55	0.007
7	−0.00	−0.00	−0.02	0.45	−0.59	−0.14	0.31	0.27	−0.31	0.005
8	0.00	−0.01	−0.00	0.09	−0.38	0.63	−0.24	−0.22	0.14	0.004
9	0.00	−0.00	−0.00	−0.01	−0.02	−0.02	0.46	−0.48	0.07	0.003
Panel B: Principal components of weekly volatility changes										
1	0.40	0.69	0.98	1.09	1.12	1.13	1.04	0.96	0.75	0.732
2	−0.83	−0.57	−0.36	−0.07	0.05	0.19	0.31	0.32	0.34	0.141
3	0.43	−0.05	−0.27	−0.24	−0.19	−0.01	0.15	0.22	0.33	0.049
4	0.22	−0.38	−0.09	0.15	0.17	0.08	−0.01	−0.04	−0.18	0.027
5	−0.02	0.21	−0.33	0.06	0.09	0.10	0.03	−0.02	−0.12	0.017
6	0.00	−0.01	−0.08	0.23	0.01	−0.12	−0.10	−0.09	0.20	0.012
7	−0.00	−0.00	−0.01	0.15	−0.19	−0.07	0.11	0.10	−0.11	0.009
8	0.00	−0.01	0.01	0.04	−0.14	0.21	−0.09	−0.04	0.03	0.007
9	0.01	−0.01	0.01	−0.01	−0.02	0.01	0.15	−0.17	0.02	0.005

This table reports the principal components of daily interest rate changes and weekly volatility changes. The rows represent 9 principal components while the first 9 columns represent the loadings of principal components on the 9 factors (changes in yields or volatility). The last column represents the percentage of total variation of the yield or volatility term structure explained by each of the individual principal components.

Table 3
Estimation results of alternative term structure models

	SV	SVJ	SVJT
Panel A: GMM estimation results			
μ_r^*	3.258×10^{-5} (4.157×10^{-5})	3.412×10^{-5} (4.258×10^{-5})	3.344×10^{-5} (4.346×10^{-5})
κ_{rr}^*	-8.459×10^{-4} (8.401×10^{-4})	-5.591×10^{-4} (8.688×10^{-4})	-4.993×10^{-4} (8.897×10^{-4})
κ_{rv}^* or ζ_{vv}^*	-4.770×10^{-3} (2.164×10^{-3})	-3.143×10^{-3} (3.882×10^{-3})	-2.816×10^{-2} (5.067×10^{-2})
$\zeta_{\lambda\lambda}^*$			-6.163×10^{-5} (1.005×10^{-4})
μ_v^*	4.233×10^{-4} (4.578×10^{-5})	2.492×10^{-4} (5.239×10^{-5})	1.761×10^{-3} (3.769×10^{-4})
κ_{vv}^*	-1.201×10^{-1} (6.037×10^{-2})	-8.422×10^{-2} (4.225×10^{-2})	-3.306×10^{-2} (2.513×10^{-2})
σ_v	5.645×10^{-2} (3.706×10^{-3})	3.199×10^{-2} (1.434×10^{-3})	6.445×10^{-3} (2.090×10^{-4})
ρ_{rv}	-5.923×10^{-3} (2.107×10^{-3})	-3.645×10^{-4} (1.308×10^{-4})	-3.133×10^{-3} (1.432×10^{-3})
μ_λ^*			1.631×10^{-2} (3.421×10^{-3})
$\kappa_{\lambda\lambda}^*$			-2.077×10^{-1} (9.325×10^{-2})
σ_λ			1.389×10^{-1} (9.271×10^{-3})
$\rho_{r\lambda}$			-4.331×10^{-2} (2.198×10^{-2})
λ^*		3.807×10^{-2} (9.052×10^{-3})	
p		4.899×10^{-1} (1.761×10^{-1})	4.932×10^{-1} (1.619×10^{-1})
μ_+^*		1.536×10^{-3} (6.159×10^{-4})	1.518×10^{-3} (5.431×10^{-4})
μ_-^*		-1.694×10^{-3} (8.314×10^{-4})	-1.689×10^{-3} (6.655×10^{-4})
Panel B: GMM test of overidentifying restrictions			
χ^2	34.99	22.76	9.23
d.o.f.	11	7	3
p-value	0.02%	0.19%	2.64%

This table reports the GMM estimation results for the three alternative term structure models, namely the SV, SVJ and SVJT models. The moment conditions used for the GMM estimation are the daily changes of yields of different maturities, with the lagged yields as instrumental variable. Standard errors are reported in parenthesis next to the parameter estimates. The blank cell indicates that the parameter is restricted to be zero. The Hansen *J*-test statistic is also reported for each model together with the degree of freedom and the *p*-value.

exhibits more curvature than the SVJ model with a term structure closer to that of empirical data (see Panel B of Table 2). This shows that jumps with stochastic intensity allows for potentially large variations of volatility across yields of different maturities.

5. Further analysis

5.1. Stochastic volatility, jumps, and macroeconomic shocks

Recent works of Ang and Piazzesi (2003) and Duffee (2006) consider the joint dynamics of bond yields and macroeconomic variables. They find that the macroeconomic variables have strong power in explaining the variations in bond yields. In particular, Ang and Piazzesi (2003) find that macroeconomic variables primarily explain movements at the short end and middle part of the yield curve while unobservable factors account for most of the movements at the long end of the yield curve. Note that the models used in Ang and Piazzesi (2003) are of affine structure with no jumps, while Duffee (2006) integrates out latent variables of the term structure model. These studies focus on changes of yield curve at monthly frequency since macroeconomic variables are only observed at monthly intervals.

Our analysis extends the current literature on two fronts. First, our model generalizes the linear-quadratic term structure by incorporating jumps of stochastic intensity in the short rate. Second and more importantly, we obtain empirically daily estimates of stochastic volatility and jump intensity, allowing us to directly investigate the relation between these variables and macroeconomic shocks.

Table 4
Term structure dynamics and macroeconomic variables

	Inflation (IN)	Real activity (RA)	Volatility (SV)	Jump volatility (JV)
Constant	1.838*(0.721)	0.849 (0.746)	4.811*(1.477)	1.314 (0.710)
IN_0			0.188** (0.101)	0.028 (0.105)
IN_{-1}	0.357* (0.069)	0.008 (0.072)	−0.040 (0.109)	−0.023 (0.113)
IN_{-2}	−0.106 (0.073)	−0.313* (0.075)	0.116 (0.113)	0.082 (0.117)
IN_{-3}	0.149* (0.070)	0.119 (0.072)	0.031 (0.112)	−0.018 (0.116)
RA_0			−0.028 (0.130)	0.020 (0.124)
RA_{-1}	0.074 (0.058)	0.154* (0.061)	−0.025 (0.112)	−0.004 (0.107)
RA_{-2}	0.118* (0.057)	0.223* (0.059)	−0.115 (0.112)	0.054 (0.107)
RA_{-3}	−0.014 (0.058)	0.474* (0.060)	0.005 (0.132)	0.009 (0.126)
SV_{-1}	0.070** (0.038)	−0.006 (0.047)	−0.108 (0.095)	−0.090 (0.091)
SV_{-2}	−0.038 (0.036)	−0.078 (0.049)	−0.082 (0.088)	0.005 (0.084)
SV_{-3}	−0.010 (0.035)	0.019 (0.048)	0.186** (0.088)	0.087 (0.084)
JV_{-1}	−0.029 (0.051)	0.003 (0.052)	0.318* (0.100)	0.172** (0.095)
JV_{-2}	−0.019 (0.052)	0.033 (0.054)	0.080 (0.097)	−0.004 (0.093)
JV_{-3}	0.017 (0.052)	−0.007 (0.054)	0.033 (0.097)	0.290* (0.094)
Adj. R^2	0.120	0.297	0.197	0.071

This table reports the VAR estimation results for the two groups of macroeconomic variables, namely inflation (IN) and real activity (RA), stochastic volatility (SV), and jump volatility (JV). The results are based on monthly observations from January, 1984 to February, 2002. Standard errors are reported in parenthesis next to the parameter estimates. The * and ** indicate the coefficient significantly different from 0 at the 5% and 10% critical levels, respectively.

We follow Ang and Piazzesi (2003) and choose two groups of macroeconomic variables. The first group consists of inflation measures: the consumer price index (CPI) and the producer price index (PPI), and the second group contains real activity variables: the index of help wanted advertising in newspapers (HELP), the unemployment rate (UE), the growth rate of employment (EMPLOY), and the growth rate of industrial production (IP). All growth rates (including inflation) are measured as the difference in logs of the index from month $t-1$ to month t. The monthly time series data of macro variables are obtained from the Bureau of Labor Statistics of US Department of Labor for the sample period of January, 1984–February, 2002. As in Ang and Piazzesi (2003), we extract the first principal component from the inflation group and the first principal component from the real activity group, which are denoted as "inflation" or IN and "real activity" or RA, respectively. The variations accounted by the principal components in the inflation group are 0.842 and 0.158, and the variations accounted by the principal components in the real activity group are 0.654, 0.327, 0.017 and 0.001. This suggests that it is reasonable to use the first principal component in our analysis. The correlation between the first principal component of the inflation group (IN) and that of the real activity group (RA) is low (4.05%).

We try to answer two empirical questions. One is whether the state variables that drive the term structure dynamics have predictive power of future economic activities, and the other is whether the changes of volatility and in particular jump intensity are due to shocks in macroeconomic variables. To answer the above questions, we estimate standard VAR models for the macroeconomic variables, using the first principal components of inflation (IN) and real activity (RA), as well as stochastic volatility and jump volatility. To capture the response of stochastic volatility and jump volatility to the current information in macroeconomic variables, we also include the contemporaneous macroeconomic variables in the stochastic volatility and jump volatility equations. We include lags up to three months in the VAR model. The VAR estimation results reported in Table 4 suggest that while stochastic volatility is predictive of next month's inflation with a coefficient significant at the 10% critical level, it is not predictive of future real activity. Interestingly, random jump is predictive of neither inflation nor real activity. The results also show a significant coefficient for the contemporaneous inflation in the SV equation, that is, an increase in inflation tends to drive up stochastic volatility in the short rate. A jump in the short rate also tends to drive up the level of volatility. The estimates for the jump volatility equation suggest that other than the autoregressive structure, no variables are predictive of random jumps. In other words, jumps are not triggered by informational shocks in the macroeconomic variables.

5.2. What drives jumps in interest rates?

The results of the previous subsection and those in Johannes (2004) suggest that to understand what drives jumps in the short rate and thus the term structure dynamics, we need to consider a finer information set at a frequency higher than the monthly level. While the aggregation of daily information over a month period can reduce noise component, it also smoothes out day-to-day informational shocks in the market. In this subsection, we examine jumps at daily frequency and relate them to macroeconomic news releases and other events. In particular, we are interested in how often jumps in the short rate are associated with scheduled macroeconomic news release and events. Again, we focus on the two groups of macro variables, inflation and real activity. We collect the historical announcement dates for CPI, PPI, industrial production, and employment. The dates of PPI and industrial production cover the entire time period of our sample from 1984 to 2002, while those of CPI and employment only go back to the beginning of 1994. We also collect the historical meeting dates of Federal Open Market Committee (FOMC). As reported in Table 5, there are 218 announcement dates for PPI and industrial production, and 97 announcement dates for CPI and employment. In addition to the 145 scheduled FOMC meetings, there are 3 unscheduled FOMC meetings.[8]

Similar to Johannes (2004), we use the estimates of jump intensity to identify the days when jumps occur. In particular, we focus on the days with jump intensity estimate higher than the sample mean. This results in 105 days with jumps. A couple of interesting patterns are noted. First, there are more jumps toward the beginning of the sample period during the 80's than in the second half of the sample period. For example, during the period of 1994–2002, there are only 26 jumps. Second, jumps in the short rate are clustered, that is, jumps often happen in consecutive days.

Panel A of Table 5 reports the number of jumps on the scheduled announcement or FOMC meeting days. Out of the 729 unique announcement or meeting days, there are 32 unique days with jumps. Note that we are not attempting to identify any confounding news or information that causes the jump. Instead, we simply document the patterns of jumps associated with the scheduled

[8] Fleming and Remolona (1997) document that the employment, producer price index (PPI), federal (fed) funds target rate, and consumer price index (CPI) announcements have the most pronounced effect on the bond market.

Table 5
News that drives the jumps in interest rates

	January 1984–Februry 2002			January 1994–Februry 2002	
	PPI	Industrial production	FOMC	CPI	Employment
Panel A: Jumps associated with scheduled news announcements or FOMC meetings					
Total announcements[a]	218	218	145	97	97
Jumps on the day[b]	17	13	8	2	4
Positive	5	3	4	0	1
Negative	12	10	4	2	3
Jumps on the day before[c]	10	8	5	0	1
Positive	5	2	2	0	0
Negative	5	6	3	0	1
Jumps on the day after[d]	3	5	7	2	0
Positive	0	2	2	0	0
Negative	3	3	5	2	0

Date	Event
Panel B: A sample of jumps associated with unscheduled events (January 1998–February 2002)	
10/08/1998	Federal Reserve Chairman Alan Greenspan warns the economy is slowing.
10/28/1998	US reports $70 billion budget surplus for fiscal'98, dwarfing May estimate
12/29/1999	Possibly a millennium effect
12/05/2000	Election 2000: state court denies Al Gore's challenge to Florida's vote tally
01/03/2001	Unscheduled FOMC meeting (cut overnight rate by 50 bps)
04/18/2001[e]	Unscheduled FOMC meeting (cut Fed Funds rate by 50 bps)
09/13/2001[e]	Post September 11, 2001, bond market reopens in attack aftermath

[a] There are total 729 unique announcement or meeting dates. For example, IP and PPI share 44 dates, employment and PPI share 2 dates, there are 9 FOMC meetings dates overlap with IP dates, and 2 FOMC meetings dates overlap with PPI dates.

[b] There are total 32 unique jumps on the announcement or meeting dates. For example, IP and PPI share 9 dates, and IP and CPI share 1 date.

[c] There are total 20 unique jumps on the day before announcement or meeting dates, of which 5 overlap with those on the announcement dates of other news.

[d] There are total 15 unique jumps on the day after announcement or meeting dates, of which 4 overlap with those on the announcement dates of other news, and 2 overlap with those the day before the announcement of other news. This results in total unique 56 jumps associated with scheduled news announcements or FOMC meetings, including one day lead or lag.

[e] These dates are also one day before the announcement of other news. However, there are no jumps on the news announcement dates.

news release or FOMC meeting dates. A couple of interesting observations are noted. First, there are 9 common days of PPI and industrial production announcements that are associated with jumps. Taking away these common dates, jump seems to be more likely associated with PPI information than with other information, with additional 8 days of PPI announcements associated with jumps and only 4 days of industrial production announcements associated with jumps. This is consistent with the results in the previous subsection that inflation has more impact on the interest rate dynamics. Second, among all jumps associated with news announcement dates, there are dominantly more negative ones than positive ones, where the sign of jumps is based on the skewness of intraday T-bill future yield changes. The only exception are the FOMC meeting dates where the shocks appear to be symmetric. To take into account of possible pre-announcement anticipation or post-announcement reaction, we also consider jumps one day before or after the scheduled announcement or FOMC meeting dates. Out of the 20 unique jumps one day ahead of the announcement or meeting dates, 5 overlap with the announcement dates of other macroeconomic news or FOMC meeting dates. The shocks appear to be less asymmetric for the pre-announcement anticipation than the announcement day reaction. There are also total 15 unique jumps on the day after announcement or meeting dates. The asymmetry shows up again for the post-announcement jumps.

In total, there are 56 unique jumps associated with scheduled news announcements or FOMC meetings, including one day lead and lag. They only account for roughly half of the jumps identified in our sample. The other half could be associated with other scheduled news release not included in our study. There is, however, evidence that a significant portion of the jumps may not be associated with scheduled news release at all. Instead, they are caused by unanticipated economic news or geopolitical development. Panel B of Table 5 provides a short list of such events during the period of January 1998–February 2002. These events suggest that the uncertainty of timing is an important component of jump risk. To account for unanticipated daily informational shocks, it is important to model jumps with stochastic intensity.

6. Conclusion

We study a unifying class of linear-quadratic term structure models (LQTSMs) under the jump-diffusion framework, which extends most existing term structure models. The model incorporate random jumps of stochastic intensity in the short rate process. We propose an IS-GMM approach for the estimation of the linear-quadratic term structure models. The estimation results show that the jump intensity process is negatively correlated with interest rate changes. Negative jumps have, on average, larger size than positive jumps.

The time series estimates of latent state variables, namely stochastic volatility and jump intensity, allow us to investigate their relation with macroeconomic variables. Our empirical results suggest that at monthly frequency while stochastic volatility is related to inflation, jumps are neither predictive of nor triggered by informational shocks in macroeconomic variables. Once focusing on daily frequency, however, we document interesting patterns for jumps associated with various events in the financial market. We find that jumps associated with informational shocks in macroeconomic news are more negative than positive. In addition, while a large number of jumps are related to scheduled news announcement dates, a significant portion of jumps are driven by unanticipated economic news or geopolitical development.

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Appendix A

This appendix provides technical details of our results. In particular, we derive the equilibrium interest rate, and the ODEs that determine the bond price and conditional characteristic function formulae for the general model and special cases.

A.1. Change of probability measure and interest rate process in equilibrium

We specify the parameters of model (3) as follows:

$$\mu_{X_1} = \kappa_{10} + \kappa_{11}X_1 + \kappa_{12}X_2 + \zeta(X_2),$$
$$\text{for } (\kappa_{10}, \kappa_{11}, \kappa_{12}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_1\times n_1} \times \mathbb{R}^{n_1\times n_2},$$
$$\zeta(X_2) = \left(X_2^\top \zeta_1 X_2, \ldots, X_2^\top \zeta_{n_1} X_2\right)^\top, \quad \text{for } \zeta_i \in \mathbb{R}^{n_2\times n_2},$$
$$\mu_{X_2} = \kappa_{20} + \kappa_{22}X_2, \quad \text{for } (\kappa_{20}, \kappa_{22}) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_2\times n_2},$$
$$(\Omega_{11})_{ij} = (\omega_{10})_{ij} + (\omega_{11})_{ij}^\top X_1 + (\omega_{12})_{ij}^\top X_2 + X_2^\top v_{ij} X_2,$$
$$\text{for } \left((\omega_{10})_{ij}, (\omega_{11})_{ij}, (\omega_{12})_{ij}, v_{ij}\right) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2\times n_2},$$
$$(\Omega_{12})_{ij} = (\omega_{20})_{ij} + (\omega_{22})_{ij}^\top X_2, \quad \text{for } \left((\omega_{20})_{ij}, (\omega_{22})_{ij}\right) \in \mathbb{R} \times \mathbb{R}^{n_2},$$
$$\lambda = \varphi + \phi^\top X_1 + \psi^\top X_2 + X_2^\top \Psi X_2,$$
$$\text{for } (\varphi, \phi, \psi, \Psi) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2\times n_2}.$$

Moreover, the square matrices $\Omega_{22} \in \mathbb{R}^{n_2\times n_2}$, ζ_i, v_{ij}, and Ψ are symmetric.

Following the general equilibrium approach of Cox et al. (1985a), and Ahn and Thompson (1988), we first show that the instantaneous interest rate, $r(t)$, is endogenously determined as a linear-quadratic function of the state variables. Consider an economy in which there is a competitive market for instantaneous borrowing and lending at the spot interest rate $r(t)$ that is to be determined. For simplicity, we assume that there is a single non-dividend-paying risky asset, which one can think of as an equity market index, and a representative utility maximizing investor. The risky asset price follows the jump-diffusion process:

$$dS = \left(\mu_S - \lambda\mu_{Q_S}\right)S dt + \sigma_S^\top S dZ + Q_S S dN, \tag{A.1}$$

where Q_S is the jump size with mean μ_{Q_S}, and independent of N, Q_X, and Z.[9] The drift term μ_S and diffusion term σ_S are specified as:

$$\mu_S = \alpha_1 + \beta_1^\top X_1 + \gamma_1^\top X_2 + X_2^\top \Phi_1 X_2, \tag{A.2}$$
$$\sigma_S^\top \sigma_S = \alpha_2 + \beta_2^\top X_1 + \gamma_2^\top X_2 + X_2^\top \Phi_2 X_2, \tag{A.3}$$
$$\Sigma_X \sigma_S = \begin{pmatrix} \alpha_3 + \beta_3^\top X_1 + \gamma_3^\top X_2 + \xi(X_2) \\ \alpha_4 + \gamma_4^\top X_2 \end{pmatrix}, \tag{A.4}$$
$$\xi(X_2) = \left(X_2^\top \xi_1 X_2, \ldots, X_2^\top \xi_{n_1} X_2\right)^\top,$$

where for $i = 1$ and 2, $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}^{n_1\times 1}$, $\gamma_i \in \mathbb{R}^{n_2\times 1}$, $\Phi_i \in \mathbb{R}^{n_2\times n_2}$ are symmetric matrices, $\alpha_3 \in \mathbb{R}^{n_1\times 1}$, $\alpha_4 \in \mathbb{R}^{n_2\times 1}$, $\beta_3 \in \mathbb{R}^{n_1\times n_1}$, $\gamma_3 \in \mathbb{R}^{n_2\times n_1}$, $\gamma_4 \in \mathbb{R}^{n_2\times n_2}$, and ξ_i/s $\in \mathbb{R}^{n_2\times n_2}$ are symmetric matrices with $1 \leqslant i \leqslant n_1$.

The investor has wealth W and allocates it between the risky asset and the risk-free security. For simplicity, we assume that there is no intermediate consumption and the investor chooses an optimal portfolio to maximize the power utility on her wealth at a terminal date T:

$$\max_w \mathbf{E}_t\left[\frac{W_T^{1-\rho}}{1-\rho}\right],$$

where $\mathbf{E}_t$ is the conditional expectation operator and $\rho > 1$ is the risk aversion coefficient. Following the method of Cox et al. (1985a), and Ahn and Thompson (1988), the spot interest rate can be found as

$$r = \mu_S + \sigma_S^\top \sigma_S W\frac{J_{WW}}{J_W} + \sigma_S^\top \Sigma_X^\top \frac{J_{WX}}{J_W} + \lambda\mathbf{E}\left[\frac{\Delta J_W}{J_W} Q_S\right], \tag{A.5}$$

where $\Delta J_W = J_W(W(1+Q_S), X+Q_X, t) - J_W(W, X, t)$. We then solve investor's indirect utility function as[10]

$$J(W, X, t) = g(X, t)\frac{W^{1-\rho}}{1-\rho}, \tag{A.6}$$

where g has the following form:

$$g(X, \tau) = e^{a(\tau)+b(\tau)^\top X_1 + c(\tau)^\top X_2 + X_2^\top d(\tau) X_2}, \tag{A.7}$$

where $\tau \equiv T - t$, $a \in \mathbb{R}$, $b \in \mathbb{R}^{n_1\times 1}$, $c \in \mathbb{R}^{n_2\times 1}$, and $d \in \mathbb{R}^{n_2\times n_2}$ is a symmetric matrix, and a, b, c, and d satisfy certain ODEs with zero initial conditions. Now by combining (A.5), (A.6), (A.7), (A.2), (A.3), and (A.4), we can see that r is a linear-quadratic function of the state variables.

We next consider Proposition 1. Following the argument in Cheng and Scaillet (2007) and using (A.6) and (A.7), the price of any security, $F(W, X, t)$, satisfies the PDE:

$$\begin{aligned} rF = {} & F_t + \left(r - \lambda\mathbf{E}\left[(1+Q_S)^{1-\rho} e^{b^\top Q_{X_1}} Q_S\right]\right) W F_W \\ & + \left[\mu_X - \lambda\mu_{Q_X} - \rho\Sigma_X\sigma_S + \Sigma_X\Sigma_X^\top (bc + 2dX_2)\right]^\top F_X \\ & + \frac{1}{2}\sigma_S^\top \sigma_S W^2 F_{WW} + \sigma_S^\top \Sigma_X^\top F_{WX} + \frac{1}{2}\mathrm{tr}(\Sigma_X\Sigma_X^\top F_{XX^\top}) \\ & + \lambda\mathbf{E}[(1+Q_S)^{1-\rho} e^{b^\top Q_{X_1}} \Delta F]. \end{aligned}$$

where $\Delta F = (F(W(1+Q_S), X+Q_X, t) - F(W, X, t))$. Direct computation can show that this equation can also be obtained if the state variables follow the process defined by (4) and the following equations:

$$\mathrm{Prob}(dN^* = 1) = \lambda^* dt,$$
$$\lambda^* = \varphi^* + \phi^{*\top} X_1 + \psi^{*\top} X_2 + X_2^\top \Psi^* X_2,$$
$$\mathbf{E}[e^{k^\top Q^*_{X_1}}] = \delta_X^*(k).$$
$$\mu^*_{X_1} = \kappa^*_{10} + \kappa^*_{11}X_1 + \kappa^*_{12}X_2 + \zeta^*(X_2),$$
$$\mu^*_{X_2} = \kappa^*_{20} + \kappa^*_{22}X_2,$$
$$\zeta(X_2)^* = (X_2^\top \zeta_1^* X_2, \ldots, X_2^\top \zeta^*_{n_1} X_2)^\top,$$

where $\mu_X^* \equiv \mu_X - \rho\Sigma_X\sigma_S + \Sigma_X\Sigma_X^\top \begin{pmatrix} b \\ c + 2dX_2 \end{pmatrix}$, $\lambda^* \equiv \lambda\chi\delta_X(b)$, $\delta_X^*(k) \equiv \frac{\delta_X(b+k)}{\delta_X(b)}$, $h^*(q) \equiv \frac{e^{b^\top q}}{\delta_X(b)} h(q)$ with h and h^* being the probability density functions of Q_{X_1} and $Q^*_{X_1}$ respectively. Hence, we have proved Proposition 1.

A.2. Bond price and conditional characteristic function

In this subsection, we present first the bond price formula and then formula for the characteristic function under the risk-neutral probability measure. In the absence of arbitrage the bond price $P(X_1, X_2, \tau)$ satisfies the partial differential equation (PDE):

$$\begin{aligned} 0 = {} & P_t - rP + \left[\kappa^*_{10} + \kappa^*_{11}X_1 + \kappa^*_{12}X_2 + \zeta^*(X_2) - \lambda^*\mu^*_{Q_{X_1}}\right]^\top P_{X_1} \\ & + \left(\kappa^*_{20} + \kappa^*_{22}X_2\right)^\top P_{X_2} + \frac{1}{2}\mathrm{tr}\left[\Omega_{11}P_{X_1X_1^\top} + 2\Omega_{12}^\top P_{X_1X_2^\top} + \Omega_{22}P_{X_2X_2^\top}\right] \\ & + \lambda^*\mathbf{E}\left[P(X_1 + Q^*_{X_1}, X_2, \tau) - P(X_1, X_2, \tau)\right], \end{aligned} \tag{A.8}$$

with initial condition $P(X_1, X_2, 0) = 1$. We guess a solution of the form $P = e^{A(\tau)+B(\tau)^\top X_1 + C(\tau)^\top X_2 + X_2^\top D(\tau) X_2}$ with initial values $A(0) = 0$, $B(0) = 0_{n_1\times 1}$, $C(0) = 0_{n_2\times 1}$, $D(0) = 0_{n_2\times n_2}$. Take partial derivatives of P and substitute them into the PDE to derive:

[9] For brevity, we do not consider the case where Q_S and Q_X are correlated in this paper although it can be done in a straightforward way.

[10] Details are omitted for brevity and available upon request.

$$0=\frac{dA}{d\tau}-\frac{dB^\top}{d\tau}X_1-\frac{dC^\top}{d\tau}X_2-X_2^\top\frac{dD}{d\tau}X_2$$
$$-\left(\alpha+\beta^\top X_1+\gamma^\top X_2+X_2^\top\Phi X_2\right)+\left[\kappa_{10}^*+\kappa_{11}^*X_1+\kappa_{12}^*X_2+\zeta^*(X_2)\right.$$
$$\left.-\left(\varphi^*+\phi^{*\top}X_1+\psi^{*\top}X_2+X_2^\top\Psi^*X_2\right)\mu^*_{Q_{X_1}}\right]^\top B$$
$$+\left(\kappa_{20}^*+\kappa_{22}^*X_2\right)^\top(C+2DX_2)+\frac{1}{2}\mathrm{tr}\{\Omega_{11}BB^\top+2\Omega_{12}^\top B(C+2DX_2)^\top$$
$$+\Omega_{22}\left[(C+2DX_2)(C+2DX_2)^\top+2D\right]\}$$
$$+\left(\varphi^*+\phi^{*\top}X_1+\psi^{*\top}X_2+X_2^\top\Psi^*X_2\right)\left(\delta_X^*(B)-1\right).$$

We collect the terms with the same powers of the state variables, and define $B\equiv(B_1,\ldots,B_{n_1})^\top$, $C\equiv(C_1,\ldots,C_{n_2})^\top$, and $D\equiv(D_1,\ldots,D_{n_2})$. Since the PDE holds for any values of X_1 and X_2, these coefficients must be zero. And that leads to the following ODEs:

$$\frac{dA}{d\tau}=-\alpha+\left(\kappa_{10}^{*\top}-\varphi^*\mu^{*\top}_{Q_{X_1}}\right)B+\kappa_{20}^{*\top}C+\frac{1}{2}B^\top\omega_{10}B$$
$$+B^\top\omega_{20}C+\frac{1}{2}C^\top\Omega_{22}C+\mathrm{tr}(\Omega_{22}D)+\varphi^*\left(\delta_X^*(B)-1\right),$$
$$\frac{dB}{d\tau}=-\beta+\left(\kappa_{11}^{*\top}-\phi^*\mu^{*\top}_{Q_{X_1}}\right)B+\frac{1}{2}\sum_{i,j}B_iB_j(\omega_{11})_{ij}+\phi^*\left(\delta_X^*(B)-1\right),$$
$$\frac{dC}{d\tau}=-\gamma+\left(\kappa_{12}^{*\top}-\psi^*\mu^{*\top}_{Q_{X_1}}\right)B+\kappa_{22}^{*\top}C+2D\kappa_{20}^*+2D\omega_{20}^\top B+2D\Omega_{22}C$$
$$+\frac{1}{2}\sum_{i,j}B_iB_j(\omega_{12})_{ij}+\sum_{i,j}B_iC_j(\omega_{22})_{ij}+\psi^*\left(\delta_X^*(B)-1\right),$$
$$\frac{dD}{d\tau}=-\Phi+\sum_i B_i\zeta_i^*-\Psi^*\mu^{*\top}_{Q_{X_1}}B+\left(D\kappa_{22}^*+\kappa_{22}^{*\top}D\right)+2D\Omega_{22}D$$
$$+\frac{1}{2}\sum_{i,j}B_iB_jv_{ij}+\sum_{i,j}B_i\left(D_j(\omega_{22})_{ij}^\top+(\omega_{22})_{ij}D_j^\top\right)+\Psi^*\left(\delta_X^*(B)-1\right).$$

This concludes the proof of Proposition 2.

To derive the characteristic function $f(s,X,\tau)$, observe that f is a martingale and hence it satisfies a PDE similar to (A.8):

$$0=f_t+\left(\kappa_{10}^*+\kappa_{11}^*X_1+\kappa_{12}^*X_2+\zeta^*(X_2)-\lambda^*\mu^*_{Q_{X_1}}\right)^\top f_{X_1}$$
$$+\left(\kappa_{20}^*+\kappa_{22}^*X_2\right)^\top f_{X_2}$$
$$+\frac{1}{2}\mathrm{tr}\left[\Omega_{11}f_{X_1X_1^\top}+2\Omega_{12}^\top f_{X_1X_2^\top}+\Omega_{22}f_{X_2X_2^\top}\right]$$
$$+\lambda^*\mathbf{E}\left[f(s,X_1+Q^*_{X_1},X_2,\tau)-f(s,X_1,X_2,\tau)\right],$$

with the initial condition $f(s,X,0)=e^{is^\top X}$. We again guess a solution of the functional form $f=e^{i[A^*(s,\tau)+B^*(s,\tau)^\top X_1+C^*(s,\tau)^\top X_2+X_2^\top D^*(s,\tau)X_2]}$ with initial conditions $A^*(s,0)=0$, $B^*(s,0)=s_1$, $C^*(s,0)=s_2$, $D^*(s,0)=0$. Similar to the derivation of bond prices, A^*, B^*, C^*, and D^* are solutions of respective ODEs. This proves Proposition 3.

A.3. SV and SVJ models

For the SVJ model, the bond price as $P=e^{A+B^\top X}$, where $X=\begin{pmatrix}r\\ v\end{pmatrix}$, and A and $B=\begin{pmatrix}B_r\\ B_v\end{pmatrix}$ satisfy the ODEs:

$$\frac{dA}{d\tau}=\Pi^{*\top}B+\lambda^*\left(pe^{\mu_+^*B_r}+(1-p)e^{\mu_-^*B_r}-1\right),$$
$$\frac{dB}{d\tau}=\begin{pmatrix}-1\\ 0\end{pmatrix}+\Lambda^{*\top}B+\begin{pmatrix}0\\ \frac{1}{2}B^\top\Gamma^*B\end{pmatrix},$$

with initial values $A(0)=0$, $B(0)=\begin{pmatrix}0\\ 0\end{pmatrix}$, and $\Pi^*\equiv\begin{pmatrix}\mu_r^*\\ \mu_v^*\end{pmatrix}$, $\Lambda^*\equiv\begin{pmatrix}\kappa_{rr}^* & \kappa_{rv}^*\\ \kappa_{vr}^* & \kappa_{vv}^*\end{pmatrix}$, and $\Gamma^*\equiv\begin{pmatrix}1 & \rho\sigma_v\\ \rho\sigma_v & \sigma_v^2\end{pmatrix}$. Using the notation $s=\begin{pmatrix}s_r\\ s_v\end{pmatrix}$, the CCF has the form: $f=e^{i(A^*+B^{*\top}X)}$ where A^* and $B^*=\begin{pmatrix}B_r^*\\ B_v^*\end{pmatrix}$satisfy the ODEs:

$$\frac{dA^*}{d\tau}=\Pi^{*\top}B^*+\lambda^*\left(pe^{\mu_+^*B_r}+(1-p)e^{\mu_-^*B_r}-1\right),$$
$$\frac{dB^*}{d\tau}=\Lambda^{*\top}B^*+\begin{pmatrix}0\\ \frac{1}{2}B^{*\top}\Gamma^*B^*\end{pmatrix},$$

with initial values $A^*(0)=0$ and $B^*(0)=\begin{pmatrix}s_r\\ s_v\end{pmatrix}$. The bond price and CCF formulas for the SV model can be obtained by setting $\lambda=0$ in the above solutions for the SVJ model.

A.4. SVJT Model

For this model, $X_1=r$ and $X_2=\begin{pmatrix}\sqrt{v}\\ \sqrt{\lambda^*}\end{pmatrix}$.

Writing the bond price as $P=e^{A+B^\top X_1+C^\top X_2+X_2^\top DX_2}$, then A, B, $C=\begin{pmatrix}C_v\\ C_\lambda\end{pmatrix}$, and $D=\begin{pmatrix}D_{vv} & D_{v\lambda}\\ D_{v\lambda} & D_{\lambda\lambda}\end{pmatrix}$ satisfy the ODEs:

$$\frac{dA}{d\tau}=\mu_r^*B+\Pi^{*\top}C+\frac{1}{2}C^\top\Gamma^*C+\mathrm{tr}(\Gamma^*D),$$
$$\frac{dB}{d\tau}=-1+\kappa_{rr}^*B,$$
$$\frac{dC}{d\tau}=B\Xi^*+2D\Pi^*+\left(\Lambda^{*\top}+B\Upsilon^*+2D\Gamma^*\right)C,$$
$$\frac{dD}{d\tau}=\Theta^*+B\zeta^*+\left(\Lambda^{*\top}+B\Upsilon^*\right)D+D\left(\Lambda^*+B\Upsilon^{*\top}\right)+2D\Gamma^*D,$$

with initial values $A(0)=0$, $B(0)=0$, $C(0)=\begin{pmatrix}0\\ 0\end{pmatrix}$, and $D(0)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, and $\zeta^*\equiv\begin{pmatrix}\zeta_{vv}^* & \zeta_{v\lambda}^*\\ \zeta_{v\lambda}^* & \zeta_{\lambda\lambda}^*\end{pmatrix}$, $\Pi^*\equiv\begin{pmatrix}\mu_v^*\\ \mu_\lambda^*\end{pmatrix}$, $\Xi^*\equiv\begin{pmatrix}\kappa_{rv}^*\\ \kappa_{r\lambda}^*\end{pmatrix}$, $\Lambda^*\equiv\begin{pmatrix}\kappa_{vv}^* & \kappa_{v\lambda}^*\\ \kappa_{\lambda v}^* & \kappa_{\lambda\lambda}^*\end{pmatrix}$, $\Theta^*\equiv\begin{pmatrix}\frac{1}{2}B^2 & 0\\ 0 & -\mu^*_{Q_r}B+pe^{\mu_+^*B_r}+(1-p)e^{\mu_-^*B_r}-1\end{pmatrix}$, $\Upsilon^*\equiv\begin{pmatrix}\rho_{rv}\sigma_v & \rho_{r\lambda}\sigma_\lambda\\ 0 & 0\end{pmatrix}$, and $\Gamma^*\equiv\begin{pmatrix}\sigma_v^2 & \rho_{v\lambda}\sigma_v\sigma_\lambda\\ \rho_{v\lambda}\sigma_v\sigma_\lambda & \sigma_\lambda^2\end{pmatrix}$.

Using the notation $s=\begin{pmatrix}s_r\\ s_v\\ s_\lambda\end{pmatrix}$, the CCF has the form:

$$f=e^{i\left(A^*+B^{*\top}X_1+C^{*\top}X_2+X_2^\top D^*X_2\right)},$$

where A^*, B^*, $C^*=\begin{pmatrix}C_v^*\\ C_\lambda^*\end{pmatrix}$, and $D^*=\begin{pmatrix}D_{vv}^* & D_{v\lambda}^*\\ D_{v\lambda}^* & D_{\lambda\lambda}^*\end{pmatrix}$ satisfy the ODEs:

$$\frac{dA^*}{d\tau}=\mu_r^*B^*+\Pi^{*\top}C^*+\frac{1}{2}C^{*\top}\Gamma^*C^*+\mathrm{tr}(\Gamma^*D^*),$$
$$\frac{dB^*}{d\tau}=\kappa_{rr}^*B^*,$$
$$\frac{dC^*}{d\tau}=B\Xi^*+2D^*\Pi^*+\left(\Lambda^{*\top}+B^*\Upsilon^*+2D^*\Gamma^*\right)C^*,$$
$$\frac{dD^*}{d\tau}=\Theta^*+B^*\zeta^*+\left(\Lambda^{*\top}+B^*\Upsilon^*\right)D^*+D^*\left(\Lambda^*+B^*\Upsilon^{*\top}\right)+2D^*\Gamma^*D^*,$$

with initial values $A^*(0)=0$, $B^*(0)=s_r$, $C^*(0)=\begin{pmatrix}s_v\\ s_\lambda\end{pmatrix}$, and $D^*(0)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.

A.5. Numerical solutions of bond prices

In this paper, we employ a procedure using the implicit backward-difference method for the numerical solutions of ODEs. The details can be found in, e.g., (Aiken, 1985). The following results illustrates the effectiveness and accuracy of the numerical procedure for the CIR model and the SV model. The parameter values of the CIR model are set as $\mu=0.04,\kappa=1,\sigma=0.1$, and $r_0=0.04$. The analytical solutions are based on the closed-form bond pricing formula. The parameter values of the SV model are set equal to the parameter estimates as reported in Table 3 with $r_0=0.06$. The comparison for the SV model is based on bond prices calculated from Monte Carlo simulations, where the numbers in the brackets under bond price estimates are standard errors of the simulated prices with 5000 replications.

Maturity	1-m	3-m	6-m	1-y	2-y	5-y	10-y	30-y
Panel A: bond prices of the CIR model								
Analytical solutions	0.9967	0.9900	0.9802	0.9608	0.9231	0.8187	0.6703	0.3012
Numerical solutions	0.9967	0.9901	0.9802	0.9609	0.9234	0.8194	0.6711	0.3018
Panel B: bond prices of the SV model								
Maturity	1-m	3-m	6-m	1-y	2-y	5-y	10-y	30-y
MC simulations	0.9948	0.9839	0.9664	0.9298	0.8598	0.6868	0.4741	0.1077
Standard error	0.075	0.0089	0.0094	0.0101	0.0122	0.0097	0.0081	0.0061
Numerical solutions	0.9931	0.9822	0.9659	0.9301	0.8610	0.6899	0.4790	0.1101

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