Fund of Funds, Portable Alpha, and Portfolio Optimization

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A common practice in the investment industry is for an investor to set an asset allocation benchmark portfolio and then to hire a portfolio manager to make the investment decisions; this approach is often referred to as a fund of funds or multi-manager program. The portfolio manager's objective is to maximize the fund-of-funds portfolio excess return over the benchmark portfolio subject to given tracking errors. The funds included in the managed portfolio or fund of funds should not only achieve an optimal portfolio return, but they also need to implement the target benchmark. This situation is common in the investment industry due to the investor's desire to maximize investment returns, while concurrently controlling portfolio risk through the asset allocation benchmark. According to Cerulli Associates [2007], at year-end 2007, $1,952 billion was invested globally in multi-manager funds of funds, a 19% increase compared to year-end 2006 and almost triple the $665 billion market value at year-end 2003.

In this article, we examine the portfolio optimization problem that results when a style constraint is introduced; that is, when an investor sets the benchmark style as an asset allocation policy. The individual fund styles are specified according to the approach taken by Sharpe [1992]. The advantage of using a style constraint is that the benchmark portfolios fall along the efficient frontier of the style-factor space. We measure a portfolio manager's active management skill relative to the benchmark style, and show that only a portfolio manager with active alpha–seeking skill can construct an efficient frontier that dominates the investor's efficient frontier.

We first focus on the portfolio optimization problem with asset allocation benchmarks in the excess return space for the portfolio manager. With the analytical solution of portfolio weights, we examine the efficient frontier of the portfolio excess return against the tracking error. We decompose the total tracking error into two components, namely, the deviation from the benchmark within the style-factor space and the additional risk factors associated with fund alphas. By quantifying the contribution of each component to the portfolio excess return, we show that the portfolio weights are determined by the active management ability of the fund manager or the fund alpha.

We then compare the performance of the fund of funds with that of the benchmark. We show that the active alpha–seeking skill of the portfolio manager is necessary in order to enhance the performance of the asset allocation portfolio. Specifically, the information ratios of funds included in the active portfolio determine whether the managed portfolio is more efficient than the benchmark. On the one hand, a portfolio manager with superior active alpha–seeking skill can identify funds that have a high information ratio, that is, funds with high values of alpha relative to the specific fund risk. In this case, the portfolio excess return over the benchmark is mainly due to additional risk factors. On the other hand, a portfolio manager with no active alpha–seeking skill can only rely on deviating from the benchmark portfolio to achieve portfolio excess return. Our results also suggest that a short-sale constraint has a direct effect on the performance of the managed portfolio.

Furthermore, we show that when index funds are included, the style (asset class) loadings of additional funds have no impact on the weights of these funds in the optimally managed portfolio. Only the alphas of these additional funds affect their weights and the portfolio excess returns. In other words, alphas of additional funds can be separated from their style loadings and are "portable" in the spirit of Kung and Pohlman [2004]. Our results provide theoretical guidelines for implementing portable alpha strategies in managing asset allocation portfolios.

The rest of the article is organized as follows. We begin with notations followed by the solution to the optimal fund portfolio problem in the excess return space. We then compare the benchmark portfolios and managed portfolios, and show that, in our framework, alphas of additional funds are portable. The last section concludes. Throughout the article, we use numerical examples to illustrate the properties of the managed portfolio in both total return space and excess return space.

STYLE CONSTRAINT AND PORTFOLIO OPTIMIZATION

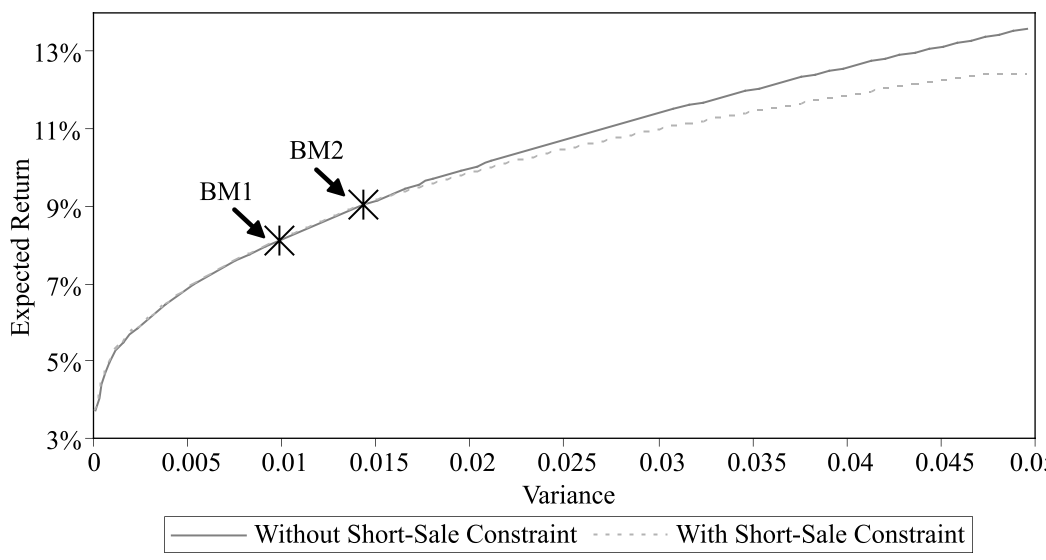
Sharpe [1992] proposed a factor model that helps determine the exposures of each component of an investor's overall portfolio to asset class factors. Sharpe's work is often referred to in academia as returns-based style analysis and is widely used by practitioners. ter Horst, Nijman, and de Roon [2004] asserted the usefulness of returns-based style analysis in estimating the relevant factor exposures of a fund, measuring fund performance, and predicting future fund returns.

In our study, the investor's efficient frontier is determined by a number of style (asset class) factors that can be viewed simply as various market indices, such as indices for large-cap, mid-cap, and small-cap equities; bonds; and T-bills or money markets. An investor essentially engages in passive investment strategies, setting a target benchmark style for the investment with a target return and variance. Exhibit 1 illustrates the conventional mean-variance frontier of investor portfolios, where BM1 and BM2 are examples of benchmark portfolios.

The portfolio manager invests in a universe of funds—each having its own style weights or loadings to

EXHIBIT 1

Efficient Benchmark Portfolio Frontier with and without a Short-Sale Constraint



the style factors—with the challenge of choosing the optimal weight for each fund in order for the fund of funds to beat the benchmark, while maintaining the target benchmark style within the range of tracking error. Clearly, to construct portfolios that will dominate the benchmark frontier, the portfolio manager of the fund of funds has to extend the asset class beyond the current style factors or identify funds (in combination with benchmark styles) that can achieve higher returns at a given level of risk. The ability to identify such funds thus measures the active alpha–seeking skill of the portfolio manager.

Notations

The following notations are used throughout the article:

- m = number of style factors
- r_F = return of style factors ($m \times 1$ vector)
- V = covariance matrix of r_F ($m \times m$ matrix)
- b_B = target benchmark style of the investor ($m \times 1$ vector)
- n = number of funds allowed in portfolio holdings
- r = return of n funds ($n \times 1$ vector)
- b = style loadings of n funds with respect to m style factors ($n \times m$ matrix)
- w = weights of the n funds in a portfolio ($n \times 1$ vector)
- l = a column vector with all elements equal to 1

Managed Portfolio: Excess Return versus Tracking Error

Now we consider a portfolio manager who is given the task of beating the benchmark portfolio subject to the style constraints. The portfolio manager has n number of managers (or funds) to be potential holdings in the managed portfolio or the fund of funds. The objective of the portfolio manager is to choose the fund weights so to enhance the portfolio return and at the same time match the target benchmark style within the range of tracking error e_T.

Under the style analysis, the return of the n funds can be expressed as

$$r = a + br_F + e \quad (1)$$

where α is the fund alpha and e is the tracking error, which is uncorrelated with the style factors. The variance-covariance matrix of funds can be written as

$$\Omega = bVb' + V_e$$

where V_e is the variance-covariance of e (i.e., V_e = Cov(e, e′)). For any portfolio with weight w on the n funds, its return is given by

$$r_p = w'\alpha + w'br_F + w'e$$

where $w'b$ is the portfolio style weight, $w'\alpha$ is the portfolio alpha, and $w'e$ is the portfolio style tracking error. The expected return and variance of the portfolio are

$$E[r_p] = w'\alpha + w'bE[r_F] \quad (2)$$

and

$$Var(r_p) = (w'b)V(w'b)' + w'V_e w \quad (3)$$

Given the target benchmark style set by the investors, say b_B, the return of the benchmark portfolio is then $b_B' r_F$. The expected portfolio excess return is

$$\mu_{EX} = (w'b - b_B')E[r_F] + w'a \quad (4)$$

and the corresponding portfolio tracking error is

$$e_T^2 = (w'b - b_B')V(w'b - b_B')' + w'V_e w \quad (5)$$

As pointed out in Roll [1992], Waring et al. [2000], and Jorion [2003], portfolio managers often choose the weights of funds by solving the following optimization problem in the excess return space:

$$\max_w [(w'b - b_B')E[r_F] + w'\alpha] \quad (6)$$

subject to

$$(w'b - b_B')V(w'b - b_B')' + w'V_e w = e_r^2 \quad (7)$$

and

$$w'l = 1$$

That is, the portfolio manager maximizes the portfolio excess return for the given tracking error; this problem is parallel to the Markowitz frontier framework.

To preserve the linearity of portfolio weights and to derive analytical solutions, we allow for short positions on funds in the managed portfolio or fund of funds (i.e., w_i of fund i in the weight vector w can be negative). The solution of the problem is given in Equation (A11) in Appendix A.

It is obvious that the first term in Equation (5), or the tracking error constraint, represents misfit or deviation within the style-factor space between the portfolio style weight and the target benchmark style weight. This component corresponds to the first term of the portfolio excess return in Equation (4). The second term in Equation (5) represents extra risk or variance from the individual fund style tracking error, and this component corresponds to the portfolio alpha, or the second term of the portfolio excess return in Equation (4). The tradeoff between the increased portfolio variance $(w'b - b'_B)V(w'b - b'_B)'$ due to deviation from the target benchmark style and the corresponding portfolio excess return $(w'b - b'_B)E[r_F]$ is obvious. Because all risk factors are the style factors in the investor's original portfolio frontier, the return variance trade-off is given by the investor's optimization problem as shown in the previous section. More interesting is the contribution of the portfolio tracking error, $w'V_e w$, to the portfolio excess return, $w'\alpha$. The portfolio tracking error represents additional risk factors beyond style factors, and the corresponding portfolio alpha is a measure of the portfolio manager's active alpha–seeking skill; that is, a more skillful portfolio manager would be able to identify a fund with a higher alpha given the level of tracking error. In the next section, we use examples to illustrate the contribution of fund alphas to portfolio excess returns.

Fund Alpha and Its Contribution to Excess Return

In this section, we use numerical examples to illustrate the contribution of fund alpha, or the fund manager's active management skill, to the excess return of the fund-of-funds portfolio. We consider four funds (Funds 1 to 4) in our analysis and assume that three (Funds 1 to 3) are index funds, which exactly match the style factors. These funds or style factors correspond to the efficient frontier in Exhibit 1 and can perfectly implement the investor's target benchmark style (for example, BM1 in Exhibit 1). Fund 4 is an additional fund in the portfolio manager's investment universe and has its own loadings. We also assume that Fund 4 has its own alpha with a nonzero style tracking error. The style loadings and the tracking error of Fund 4 are given in Appendix C. In Panels A and B of Exhibit 2, each portfolio excess return and tracking error frontier is generated by the three index funds plus Fund 4 with different values for the information ratio (IR). Note that Fund 4 can be regarded as any actively managed portfolio that is expected to generate alpha. In other words, although we just use one fund to illustrate the fund alpha and its contribution to excess return, the results hold in general.

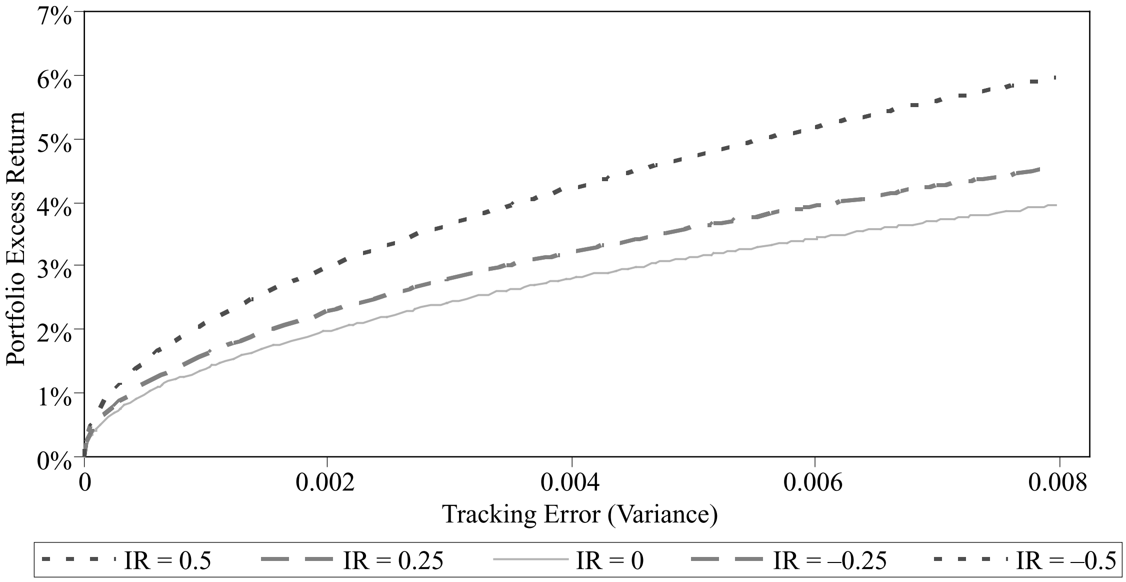
Both panels of Exhibit 2 show the portfolio excess return and tracking error frontiers that result from each of the four information ratio values of Fund 4. In Exhibit 2, Panel A, short positions are allowed. Equation (A11) in Appendix A is used to generate the portfolio excess return and tracking error frontiers in Panel A. In particular, we solve for a set of portfolio excess returns through Equation (A11) for a given set of portfolio tracking errors.

Exhibit 2, Panel A (without short-sale constraint), shows that the higher the information ratio (in absolute value), the better the portfolio excess return and tracking error efficient frontier. The portfolio efficient frontiers for the same information ratios in absolute values (but opposite signs) overlap each other because portfolio managers can short funds with negative information ratios. For example, a portfolio manager can obtain a positive alpha by taking a short position on a fund that is expected to have a negative alpha.

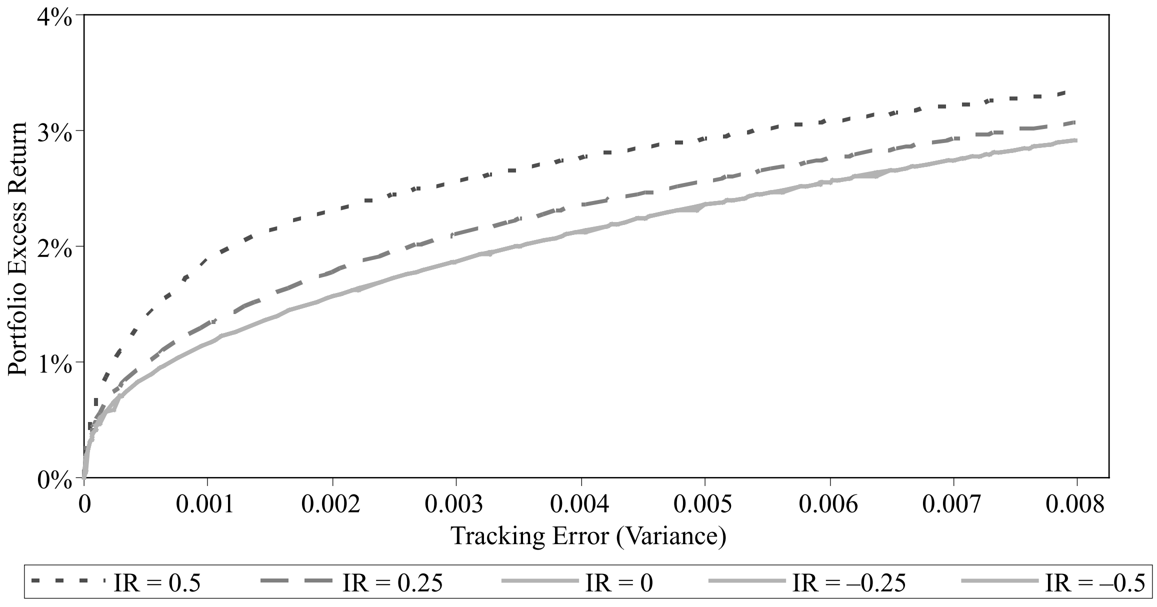
Exhibit 2, Panel B, shows the portfolio excess return and tracking error frontiers for which short positions are not allowed. The portfolio excess return and tracking error frontiers are generated by using a quadratic optimization. In Panel B (with a short-sale constraint), the frontiers with the information ratios –0.50, –0.25, and 0 are overlapping, and all are below the frontiers with the information ratios 0.50 and 0.25. When the information ratio of the additional Fund 4 is negative, the best a portfolio manager can do is to take a zero position in Fund 4, because he is not allowed to short it. This contrasts with the case without a short-sale constraint that is illustrated in Exhibit 2, Panel A, in which the portfolio excess return and tracking error efficient frontier are always positive, even for funds that have negative alphas. Therefore, the short-sale constraint limits the maximum excess return of the portfolio, but relaxing the short-sale constraint can increase the portfolio's

EXHIBIT 2

Panel A: Excess Return vs. Tracking Error with Different Information Ratio Values (without Short-Sale Constraint)



Panel B: Excess Return vs. Tracking Error with Different Information Ratio Values (with Short-Sale Constraint)



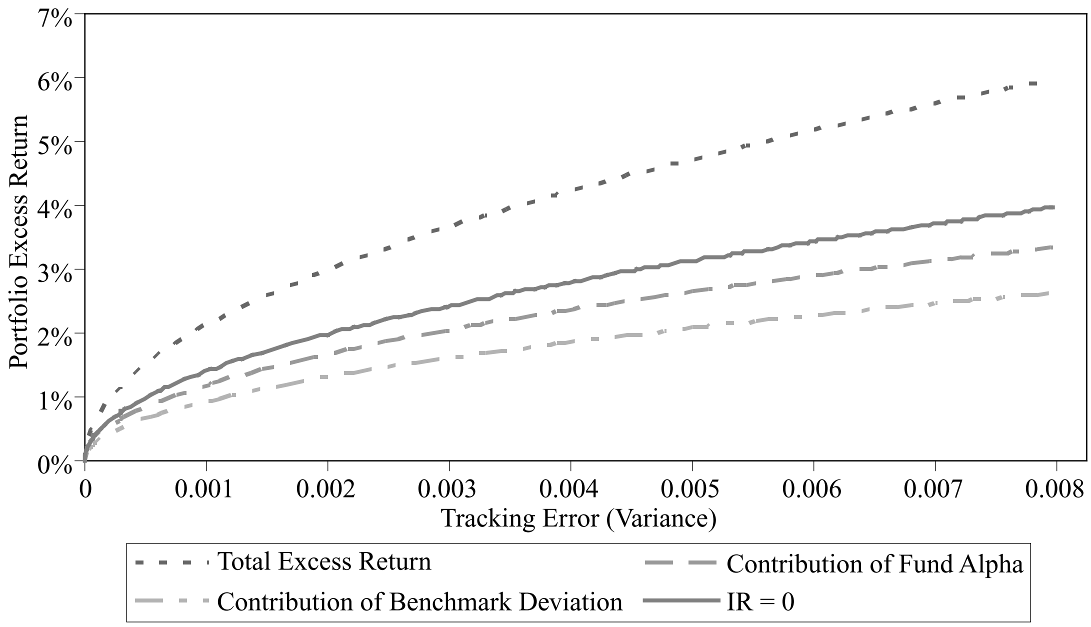
excess return by taking higher tracking error. Furthermore, the lowest portfolio excess return and tracking error efficient frontiers in both Exhibits 1 and 2 are associated with a zero information ratio. These efficient frontiers are achievable by a "naïve" investor who takes positions in the index funds. Therefore, the returns associated with these efficient frontiers should not be attributed to a portfolio manager's performance or skill. We discuss this further in the next section.

In Panels A and B of Exhibit 3, we plot the portfolio excess return components: the contribution of benchmark deviation, or $(w^{*\prime} b - b'_B)E[r_F]$, and the contribution of fund alpha, or $w^{*\prime}\alpha$, where w^* is the solution of the optimal problem defined in Equation (6). In both panels, short positions are allowed. The values of these two components are obtained by first solving the optimal problem in Equation (6) to get w^*. Then we substitute w^* into $(w^{*\prime}b - b'_B)E[r_F]$ and $w^{*\prime}\alpha$ to get the values of the two components. Exhibit 3, Panel A, plots the total portfolio excess return and the returns of its two components when the information ratio of Fund 4 is set to 0.50. For comparison, we also add the total portfolio excess return for the case in which the information ratio of Fund 4 is 0.

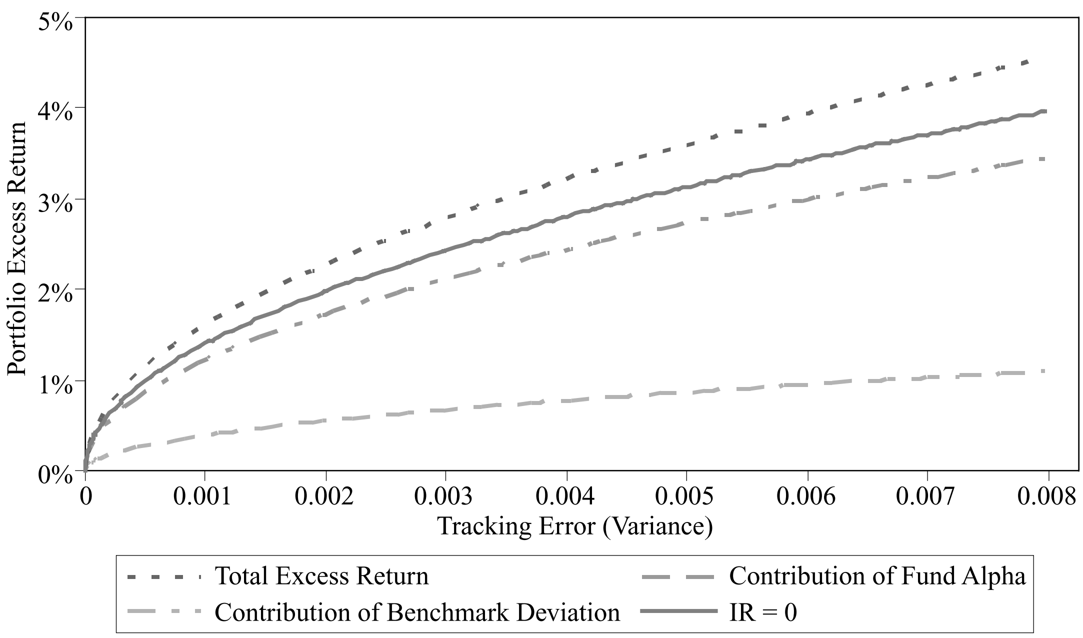
Panel A shows that the fund alpha contributes more return to the total excess return than the benchmark deviation, and that the total excess return is about 1.5 times

EXHIBIT 3

Panel A: Portfolio Excess Return Components, IR = 0.5 vs. IR = 0 (without Short-Sale Constraint)



Panel B: Portfolio Excess Return Components, IR = 0.25 vs. IR = 0 (without Short-Sale Constraint)



greater than when IR = 0. Panel B plots the case in which the information ratio of Fund 4 is set to 0.25. Compared to the case shown in Panel A, the case shown in Panel B indicates that the fund alpha contributes much less to total excess return than the benchmark deviation. In particular, the total excess return is not substantially higher than when IR = 0. Because the only difference between the cases in Panels A and B of Exhibit 3 is the information ratio of Fund 4, it suggests that portfolio manager skill at selecting a good fund is a "dominant" factor in the portfolio excess return.

More importantly, however, the different cases reveal a trade-off between the contributions of benchmark deviation and fund alpha. In particular, the larger the contribution of fund alpha, the less the contribution of benchmark deviation to the total excess return in terms of percentage. Note that the contribution of benchmark deviation in the case plotted in Exhibit 3, Panel B, is much higher than the contribution of benchmark deviation in the case shown in Exhibit 3, Panel A. For example, if the ex post fund alphas in both cases are 0, then the portfolio ex post excess returns will be completely determined by the benchmark deviations, which will cause the portfolio in Panel B to have a higher portfolio ex post excess return than the portfolio in Panel A. Therefore, investors need to take into account not only the risk due to portfolio tracking error, but also due to the trade-off between the benchmark deviation and the fund alpha.

MANAGED PORTFOLIOS: TOTAL RETURN AND RISK

In this section, we compare the total return of the managed portfolio with the investor's benchmark portfolio. That is, instead of focusing on the excess return and tracking error frontier of the portfolio manager as illustrated in the previous section, we are interested in the total return of the portfolio versus the total risk or variance. Ultimately, it is the total return and total risk that matter to an investor. Note that in our framework, the benchmark portfolios are along the efficient frontier of the style-factor space. In addition, the style loadings of any fund distinguish the returns associated with the style factors from the excess fund return (or the fund alpha) associated with additional risk factors. This excess fund return corresponds to the style tracking error that is orthogonal to the style factors. In this sense, the managed portfolio cannot do worse than the investor's benchmark efficient frontier. In addition, the further advantage of style constraint is that both investor and portfolio manager can use the style analysis as a tool to decide whether certain funds should be included in the portfolio. In other words, the benchmark style constraints can be used to set certain criteria for fund selection.

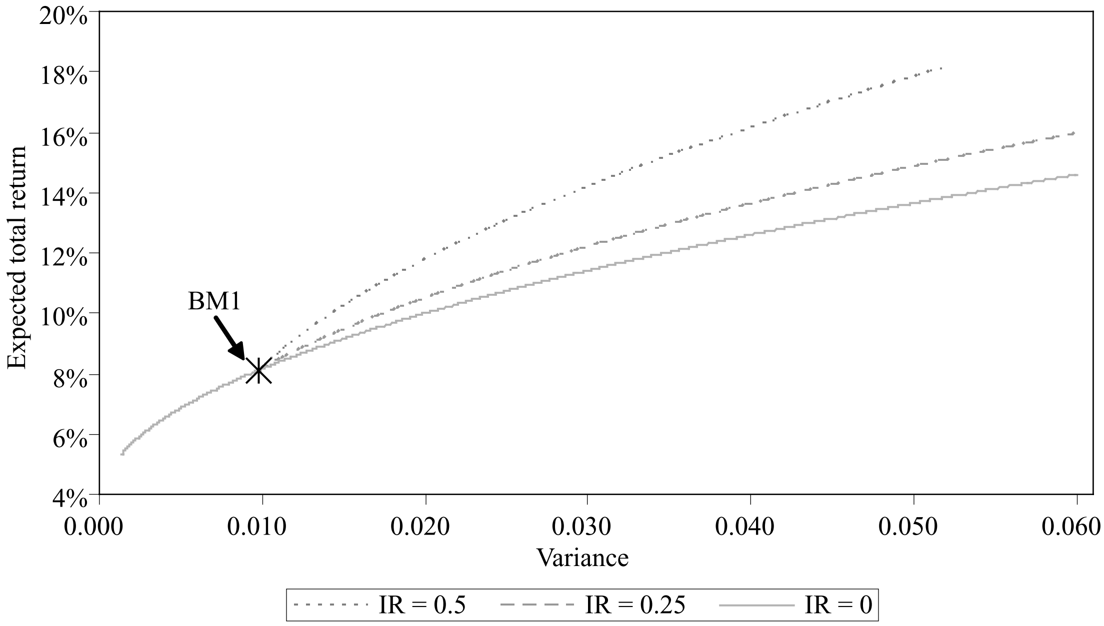
In the following discussion, we again use numerical examples to illustrate the difference between an investor's efficient frontier and a portfolio manager's efficient frontier. We plot different efficient frontiers generated by Funds 1 to 4 when Fund 4 has different information ratios, which result in different managed portfolio frontiers. The managed portfolio expected return and variance are calculated using Equations (2) and (3), but the weights in the two equations are the solutions of the optimal problem defined in Equation (6). That is, the investor's portfolio is managed by a portfolio manager who constructs the portfolio through the portfolio excess return space. Note that the portfolio frontier below the benchmark BM1 is only determined by the style factors because the portfolio manager only adds value to the portfolio when the portfolio variance is beyond that of BM1. Exhibit 4, Panel A, shows the managed portfolio frontiers when short-sale positions are allowed.

The efficient frontier when IR = 0 is the investor's efficient frontier generated from the benchmark factors. Remember that Funds 1 to 3 are index funds that mimic the three style factors. When the information ratio of Fund 4 is 0, the manager does not add any value to the investor's portfolio, which results in the same efficient frontiers for the portfolio manager and the investor. When the information ratio of Fund 4 is larger than 0, the portfolio manager adds value for the investor and the managed portfolio efficient frontier is above the investor's efficient frontier. In other words, only portfolio managers with active alpha-seeking skills can construct a portfolio frontier that dominates the investor's efficient frontier. Panel A also shows that a portfolio manager's active management skills should be measured relative to the style factors (i.e., only the returns beyond the investor's efficient frontier should be counted as the manager's performance). In Panel B, we illustrate the impact of a short-sale constraint on both the investor's and the portfolio manager's efficient frontiers.

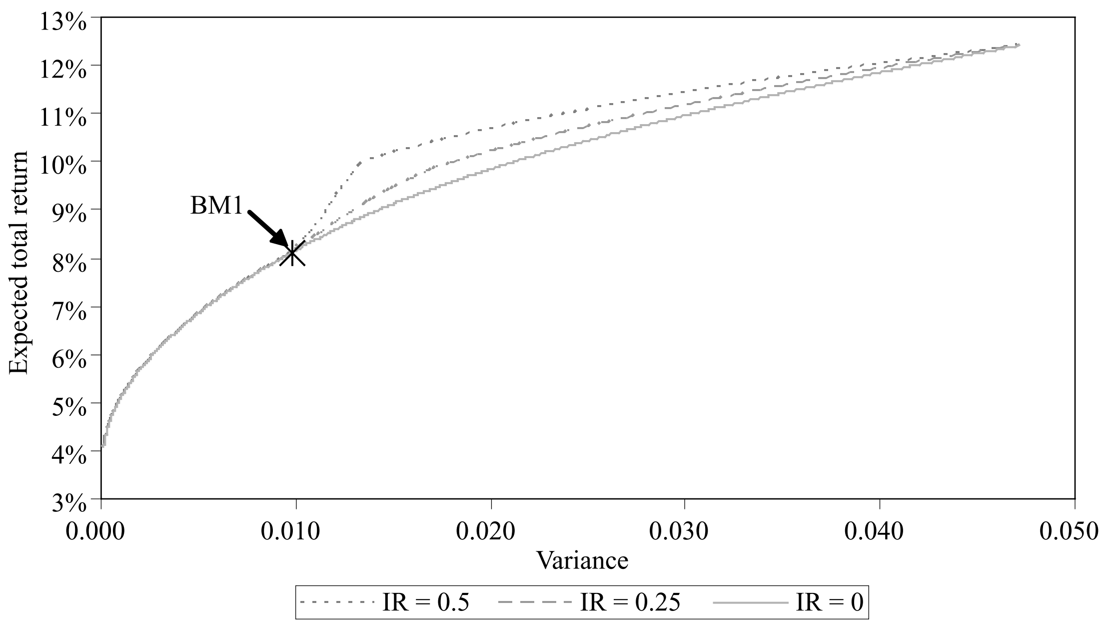
A short-sale constraint limits both the investor's and the manager's efficient frontiers. Compared to Panel A of Exhibit 4, the constraint reduces the portfolio manager's ability to add value to the investor's portfolio. Although an index or ETF can be shorted, it is often impossible to short active managers who are expected to have negative alpha. Therefore, the framework

EXHIBIT 4

Panel A: Managed Portfolio Frontier at Different Alpha Values (without Short-Sale Constraint)



Panel B: Managed Portfolio Frontier at Different Alpha Values (with Short-Sale Constraint)



presented in Panel B of Exhibit 4 is more suited to implementing this portfolio optimization technique in practice. Since it is quite easy to short ETF or indices, this framework can be used to construct "portable alpha" portfolios. In the following section, we discuss this in more detail.

PORTABLE ALPHA

As just discussed, according to Sharpe's 1992 returns-based style analysis, portfolio managers provide two types of return: the return from the deviation within the style-factor space between the benchmark and the managed portfolio, and the return from fund selection skill (alpha). When an investor determines his style benchmark, he not only determines the specific benchmark, but also the frontier of the benchmark. This efficient frontier is achievable by an investor through investments in index funds. Because the return from the benchmark deviation is contributed by the benchmark style factors, it does not improve the style benchmark frontier. Thus, to outperform the style benchmark, portfolio manager fund selection skill is essential.

In a related framework, Kung and Pohlman [2004] suggested that active investment managers provide two types of return: the return from the market exposure (beta risk) and the return from fund selection skill (alpha). They further argued that the return from fund alpha can be achieved by separating alpha from beta using a beta-neutral portfolio that is implemented through an overlay or by strategic asset allocation. Therefore, the fund alpha is portable in the sense that it is independent of the strategic asset allocation and can be applied to another portfolio.

In the returns-based style approach, the return from the benchmark deviation can be naturally separated from the return from fund selection skill through index funds. Here we show that the same property of portable alpha can be achieved.

Portable Alpha. Assume m index funds exist with a nonsingular matrix of style loadings and zero alphas. Further, there are $n - m$ number of funds with nonzero alphas (their variance-covariance matrix is nonsingular) and the weights of these $n - m$ number of funds in the managed portfolio, subject to tracking error constraint, are independent of their style loadings. In other words, the weights of these $n - m$ numbers of funds are totally determined by the fund alphas.

This result suggests that in the style investment, when index funds (or any funds with zero tracking errors and zero alphas) are included in a portfolio, the style loadings of additional funds are irrelevant in determining the weights of these funds in the managed portfolio. In other words, the overall portfolio return—the return from the benchmark deviation and the return from fund selection—is not affected by the style loadings; that is, the fund's style loadings and its alpha are separated. As a result, the fund alphas are portable and can be applied to any portfolio as long as the portfolio includes index funds to implement its benchmark style. This conclusion is drawn from Propositions 1 and 2 in Appendix B, where the proof is provided for each proposition.

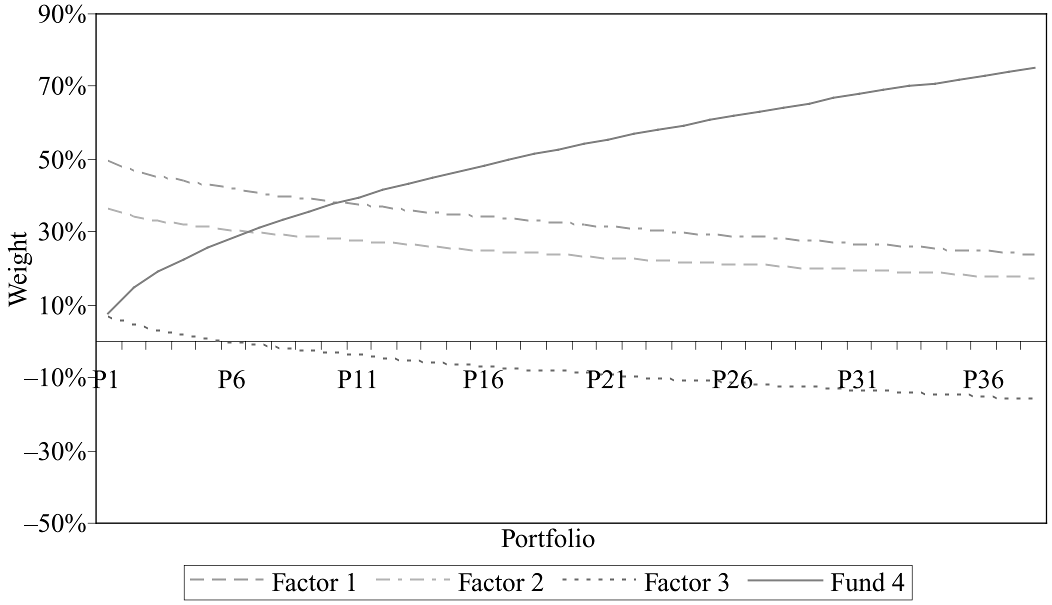
A simple and special case of the proposition is that the m index funds simply represent the m uncorrelated style factors. The intuition is that for any style deviation from the benchmark in a nonzero alpha fund, the difference can be minimized through adjusting the weights in the index funds. Thus, the nonsingularity of the style loading matrix of the index fund is necessary and sufficient for alpha to be portable. Panels A and B of Exhibit 5 illustrate the results of portable alpha when Funds 4 and 5 have the same alpha, 0.5, but different factor loadings.

Both panels of Exhibit 5 represent the portfolios on the efficient frontier formed by the style factors with Funds 4 and 5, respectively. In the two panels, the weights on Funds 4 and 5 are the same for a given portfolio, consistent with the expectation based on the proposition that the two funds have the same alpha. The factor loadings in the two panels are different, however, because Funds 4 and 5 have different factor loadings. To achieve the optimal return, the portfolio factor loadings need to be adjusted accordingly with the style loadings of the two funds. Theoretically, using this portfolio optimization framework to implement portable alpha can take advantage of funds with positive alpha and funds with negative alpha. To take advantage of negative alpha funds, the manager would short funds with negative alpha and go long index funds or ETFs. As we pointed out in the previous section, however, it is often impossible to short active managers, regardless of their expected alpha. Thus, we have illustrated this framework with only positive alpha funds, for which portable alpha is created by going long active funds and shorting index funds or ETFs.

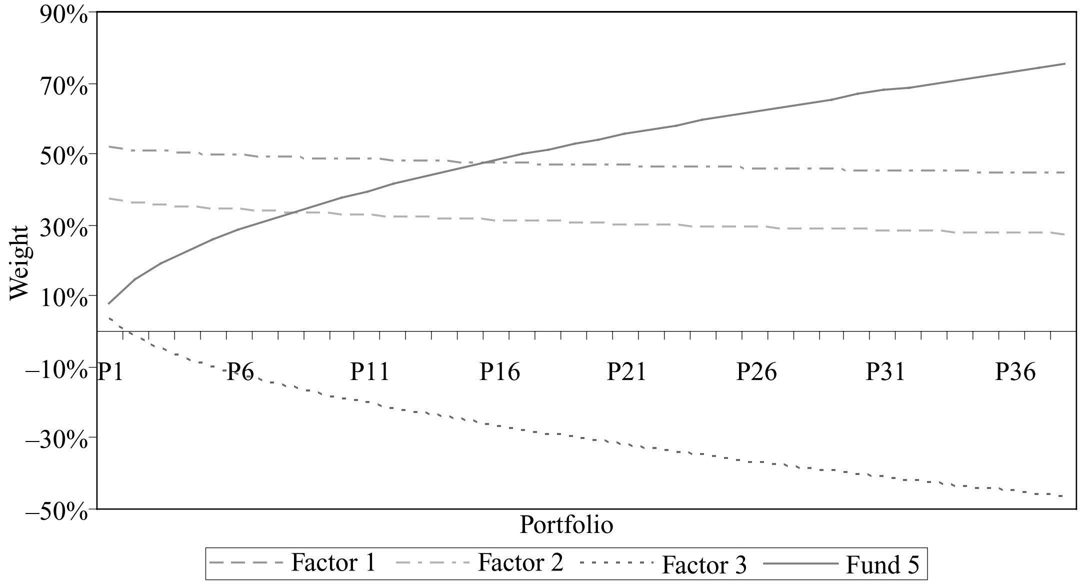
The proposition further suggests that a portfolio manager should only be rewarded for fund alpha selection, not style loadings. In terms of constructing a managed portfolio (a fund of funds), a portfolio manager can divide the portfolio construction process into two tasks: finding index funds for the style benchmarks and searching for funds that provide alpha to the portfolio excess return within the permitted tracking error. Based on the proposition, in principle, any investment strategy can be used in the portfolio construction as long as the portfolio tracking error is within the permitted range. For instance, portfolio managers can use regular funds, hedge funds, managed futures, or other derivative contracts, provided alpha is present in the strategy.

EXHIBIT 5

Panel A: Managed Portfolio Weight with Fund 4



Panel B: Managed Portfolio Weight with Fund 5



CONCLUSION

In this article, we examined the portfolio optimization problem when a portfolio manager manages a multi-manager program or fund of funds with style constraints (or asset allocation). The portfolio manager's job is to maximize the portfolio excess return over a benchmark subject to given tracking errors. We show that the performance of the style-constrained portfolio is determined by the active management skill of the portfolio manager. Specifically, the active management skill is measured by the portfolio manager's ability to identify funds with positive alpha. Thus, the key to maximizing portfolio return is for an investor to choose a portfolio manager with active alpha–seeking skill. We further showed that in the case when the style factors are market indices, the fund alpha is portable in the sense that the fund weights are totally determined by the fund alpha and independent of the style loadings. The results presented in this article will not only help investors set up optimal style constraints, but also will provide portfolio managers a portfolio optimization framework to manage a multi-manager program or fund of funds.

Appendix A

PORTFOLIO EXCESS RETURN TRACKING ERROR FRONTIER

The style-constraint optimization problem is

$$\max_{w}[(w'b - b'_B)E[r_F] + w'\alpha] \tag{A1}$$

subject to

$$(w'b - b'_B)V(w'b - b'_B)' + w'V_e w = e_T^2 \tag{A2}$$

and

$$w'\iota = 1 \tag{A3}$$

Denote $\mu_{FS} = b\mu_F + \alpha$ and $\mu_F = b'_B E[r_F]$. Since the variance-covariance matrix of funds is $\Omega = bVb' + V_e$, the Lagrangian can be written as follows:

$$L = w'\mu_{FS} - \mu_B + \lambda_1(w'\iota - 1) + 0.5\lambda_3 \times [w'\Omega w - 2b'_B Vb'w + b'_B Vb_B - e_T^2]$$

Taking partial derivatives with respect to w, we have the following first-order condition:

$$\mu_{FS} + \lambda_1\iota + \lambda_3[\Omega w - bVb_B] = 0$$

and the solution is of the form

$$w^* = \frac{\Omega^{-1}(\lambda_3 bVb_B - \mu_{FS} - \lambda_1\iota)}{\lambda_3} \tag{A4}$$

We next select the values of the λs so that the three constraints are satisfied. First, combining Equations (A4) and (A3) we have

$$\lambda_3(\iota'\Omega^{-1}bVb_B) - \iota'\Omega^{-1}\mu_{FS} - \lambda_1\iota'\Omega^{-1}\iota = \lambda_3 \tag{A5}$$

Denote $A_1 = \mu'_{FS}\Omega^{-1}\mu_{FS}$, $B_1 = \iota'\Omega^{-1}\mu_{FS}$, $C_1 = \iota'\Omega^{-1}\iota$ and $D_1 = A_1 - B_1^2/C_1$. Note that since Ω^{-1} is strictly positive definite, we have $A_1 > 0$ and $C_1 > 0$. In addition, we have

$$(B_1\mu_{FS} - A_1\iota)'\Omega^{-1}(B_1\mu_{FS} - A_1\iota) = A_1(A_1C_1 - B_1^2) \tag{A6}$$

The left side of the preceding equation is strictly positive. Therefore, $A_1C_1 - B_1^2 > 0$. As a result, $D_1 > 0$. Substituting these notations into Equation (A5), we have

$$\lambda_3(\iota'\Omega^{-1}bVb_B) - B_1 - \lambda_1 C_1 = \lambda_3 \tag{A7}$$

Second, from Equation (A4), we have

$$w'\Omega w = \frac{\lambda_3^2 b'_B Vb'\Omega^{-1}bVb_B - 2\lambda_3\mu'_{FS}\Omega^{-1}bVb_B - 2\lambda_1\lambda_3\iota'\Omega^{-1}bVb_B}{\lambda_3^2} + \frac{2\lambda_1\iota'\Omega^{-1}\mu_{FS} + \mu'_{FS}\Omega^{-1}\mu_{FS} + \lambda_1^2\iota'\Omega^{-1}\iota}{\lambda_3^2} \tag{A8}$$

Denote $H = \iota'\Omega^{-1}bVb_B$, $F = b'_B Vb'\Omega^{-1}bVb_B$, and $G = \mu'_{FS}\Omega^{-1}bVb_B$. Note that if b is an identity matrix and $\Omega = V$, then $H = 1$, $F = \sigma_B^2$, and $G = \mu_F + \alpha' b_B$. Using these notations and Equation (A8), Equation (A2) leads to

$$A_1 + 2B_1\lambda_1 + C_1\lambda_1^2 = (e_T^2 - \sigma_B^2 + F)\lambda_3^2 \tag{A9}$$

Solving for λ_1 and λ_3 using Equations (A7) and (A9), we have

$$\lambda_3 = \pm\sqrt{\frac{A_1C_1 - B_1^2}{C_1(e_T^2 - \sigma_B^2 + F) - (H-1)^2}}$$

and

$$\lambda_1 = \frac{(H-1)}{C_1}\lambda_3 - \frac{B_1}{C_1}$$

Now, substituting λ_1 and λ_3 into Equation (A4) and calculating the portfolio expected excess return (i.e., μ_{EX}), we have

$$\begin{aligned}\mu_{EX} &= (w'b - b'_B)E[r_F] + w'\alpha \\ &= -\frac{D_1}{\lambda_3} + G - \frac{(H-1)B_1}{C_1} - b'_B\mu_F\end{aligned} \tag{A10}$$

Finally, we have

$$e_T^2 - \frac{\left[\mu_{EX} + \frac{(H-1)B_1}{C_1} + \mu_B - G\right]^2}{D_1} = \sigma_B^2 - F + \frac{(H-1)^2}{C_1} \tag{A11}$$

Furthermore, for a given benchmark b_B and a tracking error e_T^2, the portfolio volatility with the style investment can be expressed as

$$Var(r_p) = 2F - \frac{2H(H-1)}{C_1} - \sigma_B^2 + e_T^2 + \left[\frac{2HB_1}{C_1} - 2G\right]\frac{1}{\lambda_3} \quad \text{(A12)}$$

and the expected return as

$$E[r_p] = G - \frac{(H-1)B_1}{C_1} - \frac{D_1}{\lambda_3} \quad \text{(A13)}$$

APPENDIX B

PORTABLE ALPHA

Proposition 1 Suppose that w* is the solution of the following optimal problem:

$$\max_w[(w'b - b'_B)E[r_F] + w'\alpha] \quad \text{(B1)}$$

subject to

$$(w'b - b'_B)V(w'b - b'_B)' + w'V_e w = e_T^2 \quad \text{(B2)}$$

and

$$w'\iota = 1 \quad \text{(B3)}$$

Let $b = \binom{b_0}{b_1}\iota$ *and* $\iota = \binom{\iota_0}{\iota_1}$*, where* b_0 *is an* $m \times m$ *matrix and nonsingular, representing style loadings of m funds that have zero-alphas (index funds, for the sake of convenience);* b_1 *is an* $(n-m) \times m$ *matrix representing style loadings of* $n - m$ funds in addition *to m index funds; and* ι_0 *and* ι_1 *are* $m \times 1$ *and* $(n-m) \times 1$ *vectors with element 1s. Further assume* $V_e = \begin{pmatrix} 0 & 0 \\ 0 & V_\alpha \end{pmatrix}$ *and* $\alpha = \binom{0}{\alpha_1}$*, where* V_α *is the* $(n-m) \times (n-m)$ *matrix and nonsingular, representing the variance-covariance matrix of the additional fund alphas, and* α_1 *is a* $(n-m) \times 1$ *vector of the non-index-fund alphas. Then, the portfolio alpha and tracking error frontier defined by Equations (B1), (B2), and (B3) does not depend on the style loadings of non-index-funds; that is,*

$$[(w^{*\prime}b - b'_B)E[r_F] + W^{*\prime}\alpha] \quad \text{(B4)}$$

independent of b_1.

Proof: From Equation (A4), the optimal weight defined by Equations (B1), (B2), and (B3) is

$$w^* = \frac{\Omega^{-1}(\lambda_3 bVb_B - \mu_{FS} - \lambda_1\iota)}{\lambda_3}$$

where,

$$\lambda_3 = \pm\sqrt{\frac{A_1C_1 - B_1^2}{C_1(e_T^2 - \sigma_B^2 + F) - (H-1)^2}}$$

$$\lambda_1 = \frac{(H-1)}{C_1}\lambda_3 - \frac{B_1}{C_1}$$

and $A_1 = \mu'_{FS}\Omega^{-1}\mu_{FS}$, $B_1 = \iota'\Omega^{-1}\mu_{FS}$, $C_1 = \iota'\Omega^{-1}\iota$, $H = \iota'\Omega^{-1}bVb_B$, $F = b'_BVb'\Omega^{-1}bVb_B$, and $G = \mu'_{FS}\Omega^{-1}bVb_B$, and where $\Omega = bVb' + V_e$ and $\mu_{FS} = bE[r_F]$. Substituting w^*, λ_1, and λ_3 into Equation (B4), we have

$$(w^{*\prime}b - b'_B)E[r_F] + w^{*\prime}\alpha = -\frac{A_1 - B_1^2/C_1}{\lambda_3} + G - \frac{(H-1)B_1}{C_1} - b'_BE[r_F] \quad \text{(B5)}$$

We want to show that the right side of Equation (B5) is independent of b_1. Since λ_3 is the function of A_1, B_1, C_1, H, and F, and e_T^2 and σ_B^2 are given, we need to show that A_1, B_1, C_1, H, and F are independent of b_1.

First, since the rank of b is m, and b_0 is full rank, there is linear dependency between b_0 and b_1. Therefore, a set of real numbers, denoted as β, is such that

$$b_1 = \beta b_0 \quad \text{(B6)}$$

where β is the $(n-m) \times m$ matrix. Since b_1 and b_0 are style loadings in b, it can be shown that

$$\beta\iota_0 = \iota_1 \quad \text{(B7)}$$

Second, since

$$\Omega = bVb' + V_e = \begin{pmatrix} b_0Vb_0' & b_0Vb_1' \\ b_1Vb_0' & b_1Vb_1' + V_\alpha \end{pmatrix}$$

we have

$$\Omega^{-1} = \begin{pmatrix} (b'_0)^{-1}[V^{-1} + b'_1V_\alpha^{-1}b_1](b_0)^{-1} & -(b'_0)^{-1}b'_1V_\alpha^{-1} \\ -V_\alpha^{-1}b_1(b_0)^{-1} & V_\alpha^{-1} \end{pmatrix}$$

Using Equations (B6) and (B7), we have

$$\Omega^{-1} = \begin{pmatrix} (b_0')^{-1}V^{-1}(b_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \beta' V_\alpha^{-1}\beta & -\beta V_\alpha^{-1} \\ -V_\alpha^{-1}\beta & V_\alpha^{-1} \end{pmatrix} \quad \text{(B8)}$$

Furthermore, using Equations (B6), (B7), and (B8), we have the following results:

$$A_1 = \mu_F' V^{-1}\mu_F + \alpha_1' V_\alpha^{-1}\alpha_1$$

$$B_1 = \mu_F' V^{-1}(b_0)^{-1}\iota_0$$

$$C_1 = \iota_0'(b_0')^{-1}V^{-1}(b_0)^{-1}\iota_0$$

$$H = \iota_0'(b_0')^{-1}b_B$$

$$F = b_B' V b_B$$

and

$$G = b_B'\mu_F$$

The right sides of the preceding equations are not related to b_1; therefore the right side of Equation (B5) is independent of b_1.

Proposition 2 Given the assumptions in Proposition A, the portfolio return at the mean-variance space can be expressed as

$$E[r_p] = b_B' E[r_F] + (w*'b - b_B')E[r_F] + w*'\alpha \quad \text{(B9)}$$

and both $(w*'b - b_B')E[r_F]$ and $w*'\alpha$ are independent of b_1. As a result, $E[r_p]$ is independent of b_1 as well. Here, $b_B' E[r_F]$ is the benchmark return, $(w*'b - b_B')E[r_F]$ is the return due to the deviation from the benchmark, $w*'\alpha$ is the return from funds, and $w*'$ is the optimal weights given by the optimization problems expressed in Equations (B1), (B2), and (B3).

Proof: Denote $w* = \binom{w_0^*}{w_1^*}$, where w_0^* is the $m \times 1$ matrix of the weights associated with the index funds and w_1^* is the $(n-m) \times 1$ matrix of the weights associated with the funds that are additional to the index funds. From Equation (A4), the optimal weight defined by the optimal problems expressed in Equations (B1), (B2), and (B3) is

$$w^* = \frac{\Omega^{-1}(\lambda_3 bVb_B - \mu_{FS} - \lambda_1\iota)}{\lambda_3}$$

Similar to Proposition 1, we can derive following results:

$$\mu_{FS}'\Omega^{-1} = (\mu_F' V^{-1}(b_0)^{-1} - \alpha_1' V_\alpha^{-1}\beta \quad \alpha_1' V_\alpha^{-1})$$

$$\iota'\Omega^{-1} = (\iota_0'(b_0')^{-1}V^{-1}(b_0)^{-1} \quad 0)$$

and

$$b'\Omega^{-1} = (V^{-1}(b_0)^{-1} \quad 0)$$

Using these results, we have

$$(w*'b - b_B')E[r_F] = \left(-\frac{1}{\lambda_3}\mu_F' V^{-1} - \frac{\lambda_1}{\lambda_3}\iota_0'(b_0')^{-1}V^{-1}\right)E[r_F] \quad \text{(B10)}$$

and

$$W*'\alpha = w_1^{*'}\alpha_1 = -\frac{1}{\lambda_3}\alpha_1' V_\alpha^{-1}\alpha_1 \quad \text{(B11)}$$

both of which are independent of b_1.

Appendix C

NUMERICAL EXAMPLES

The Exhibits 6, 7, 8, and 9 provide the numbers we used for factors, benchmarks, and funds. These numbers are unchanged across all examples used in the article except the information ratio for Fund 4, which is specified in each case.

Exhibit 6

Factor Return, Standard Deviation, and Correlation (1926–2004)

Factor	ER	STD	Correlation		
S&P 500 TR	12%	22%	1	0.1415	–0.0190
U.S. LT Govt TR	6%	8%	0.1415	1	0.1078
U.S. 30-Day T-Bill TR	4%	1%	–0.0190	0.1078	1

EXHIBIT 7
Two Asset Allocation Benchmarks

Factor	BM 1	BM 2
Factor 1	0.383	0.462
Factor 2	0.525	0.635
Factor 3	0.092	–0.097

EXHIBIT 8
Benchmark Expected Return and Standard Deviation

	BM 1	BM 2
Expected Return	8.12%	9.03%
Standard Deviation	9.92%	11.97%

EXHIBIT 9
Fund Factor Loadings, Alphas, and Tracking Errors

	Factor 1	Factor 2	Factor 3	Alpha	Tracking Error
Fund 1	1	0	0	0	0
Fund 2	0	1	0	0	0
Fund 3	0	0	1	0	0
Fund 4	0.383	0.525	0.092	Vary*	3%
Fund 5	0.25	0.25	0.50	0.5	3%

Note: The information ratios for Fund 4 are specified in the cases we discussed in the article. The alpha in each case can be calculated from the information ratio and the tracking error.

ENDNOTE

We would like to thank Roger Ibbotson and Grant Wang for helpful comments and suggestions.

Data in Exhibit 6 is provided by Ibbotson Associates. The expected return (ER) and standard deviation (STD) are calculated from the monthly returns of the indices over the time period from January 1926 to December 2004.

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