# Abelian Categories and the Freyd-Mitchell Embedding Theorem

Geillan Aly

May 15, 2008

## 1 Categories

### 1.1 Introduction to Category Theory

The field of category theory may be the closest a mathematician can get to a Meta-Theory. By understanding the foundation of mathematical structures, similarities between seemingly disparate fields of study can be observed. While abstracting the abstract may seem counter-productive, studying the underlying structures and relationships between mathematical objects has opened up new and fascinating fields in mathematics.

**Definition 1.** *A* ***category*** *$C$ consists of the following:*

- *a class $Obj(C)$ of* ***objects***
- *a set $Mor_C(A,B)$ of* ***morphisms*** *for every ordered pair (A, B) of objects in $C$. When the category in question is clear, we will say Mor(A,B).*
- *a* ***composition function*** *of morphisms for every ordered triple (A, B, C) of objects in $C$ i.e. a function Mor(A,C) such that*

  $$Mor(B,C) \times Mor(A,B) \to Mor(A,C)$$

  *Denote $f : A \to B$ to indicate that $f$ is a morphism in Mor(A, B), and $gf$ or $g \circ f$ for the composition of $f \in Mor(A,B)$ with $g \in Mor(B,C)$.*

*A category must satisfy the following properties:*

- *For each object A in a category $C$, there exists an* ***identity morphism*** *$id_A \in Mor(A, A)$ with the property that $id_B \circ f = f = f \circ id_A$ for $f : A \to B$.*
- *The Associativity Axiom: $(hg)f = h(gf)$ for $f : A \to B$, $g : B \to C$, $h : C \to D$.*

We begin by discussing examples of categories and their morphisms. The collection of all sets forms the category **Sets**, where the set of morphisms in **Sets** are set maps. The collection of all groups forms the category **Groups**, where the morphisms are group

homomorphisms. Of special interest will be our final two examples. The category $\mathcal{G}$ of Abelian groups where, again, morphisms are group homomorphisms. Our final example will be **R-mod** (**mod-R**), the category of left (right) R-modules, where $R$ is a ring with unity and the morphisms are left (right) module homomorphisms.

**Definition 2.** *A **subcategory** $\overline{C}$ of a category $C$ is a subclass of objects and morphisms in $C$ such that the collections of morphisms $Mor_{\overline{C}}(A, B)$ for $A,\ B$ in $\overline{C}$ are closed under composition, and include the identity morphism $id_A$ for every object $A \in \overline{C}$.*

**Definition 3.** *A subcategory $\overline{C}$ in which $Mor_C(A, A') = Mor_{\overline{C}}(A, A')$ for all $A$ and $A'$ in $\overline{C}$ is a **full subcategory**.*

Clearly the category $\mathcal{G}$ is a subcategory of **Groups**.

**Definition 4.** *Let $\{C_i\}_{i\in I}$ be a set of objects in $C$. A **product** of $\{C_i\}_{i\in I}$ is an object $C$ together with a set of morphisms $\{p_i : C \to C_i\}_{i\in I}$ such that for any set $\{\tilde{p}_i : C' \to C_i\}_{i\in I}$ there exists a unique morphism $\tilde{f} : C' \to C$ such that $p_i\tilde{f} = \tilde{p}_i$.*

If $A$ is a product of $\{A_i\}$ then, for all $A' \in C$ the set of morphisms $Mor(A', A)$ is in one-to-one correspondence with the set $\prod_{i\in I} Mor(A', A_i)$.

**Definition 5.** *Let $\{C_i\}_{i\in I}$ be a set of objects in $C$. A **coproduct** of $\{C_i\}_{i\in I}$ is an object $C$ together with a set of morphisms $\{p_i : C_i \to C\}_{i\in I}$ such that for any set $\{\tilde{p}_i : C_i \to C'\}_{i\in I}$ there is a unique morphism $\tilde{f} : C \to C'$ such that $\tilde{f}p_i = \tilde{p}_i$.*

## 1.2 Morphisms Between Categories

**Definition 6.** *A **covariant functor** (or just **functor**) $F : C \to \mathcal{D}$ from a category $C$ to a category $\mathcal{D}$ is a relation such that every object $A \in Obj(C)$ is associated to an object $F(A)$ in $\mathcal{D}$ and every morphism $f : A \to B$ in $C$ is associated to a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$. A functor $F$ must satisfy the following properties:*

- *preserve identity morphisms, $F(id_A) = id_{F(A)}$*
- *preserve composition, $F(gf) = F(g)F(f)$ for $gf$ a composition of morphisms in $C$ where $g : B \to C$ in $C$.*

*Observe that $F$ induces set maps $Mor_C(A, B) \to Mor_{\mathcal{D}}(F(A), F(B))$ for every $A, B$ in $C$. Lastly, if $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, the composite $GF : C \to \mathcal{E}$ is defined in the following manner: $(GF)(A) = G(F(A))$ and $(GF)(f) = G(F(f))$ for all objects $A$ in $C$.*

*Similarly, a **contravariant functor** is a relation $F : C \to \mathcal{D}$ defined such that every morphism $f : A \to B$ is associated to $F(f) : F(B) \to F(A)$. Furthermore, $F(gf) = F(f)F(g)$ in $\mathcal{D}$ for $gf$ a composition of morphisms in $C$.*[1]

[1] Every category $C$ induces a naturally defined opposite category $C^{opp}$. Both categories have the same objects, while the morphisms in $C^{opp}$ are reversed i.e. $Mor_{C^{opp}}(A, B) = Mor_C(B, A)$ with composition in $C^{opp}$ similarly reversed. In this context, a covariant functor $F : C \to \mathcal{D}$ constructs a contravariant functor $G : C^{opp} \to \mathcal{D}$.

**Definition 7.** *Consider the elements $A, B, C, D$ and the maps $\alpha : A \to B$, $\varphi : A \to C$, $\psi : B \to D$ and $\beta : C \to D$. If in the following diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & B \arrow[d, "\psi"] \\ C \arrow[r, "\beta"] & D \end{tikzcd}$$

*$\psi\alpha = \beta\varphi : A \to D$ then the **diagram commutes**. Note that $A, B, C, D$ may either be objects or categories so long as the maps and elements in the diagram are well defined.*

**Definition 8.** *Given the following commutative diagram i.e. $\psi\gamma = \varphi$, the morphism $\varphi$ **factors through** $B$.*

$$\begin{tikzcd} & B \arrow[dr, "\psi"] & \\ A \arrow[ur, "\gamma"] \arrow[rr, "\varphi"] & & C \end{tikzcd}$$

**Definition 9.** *A **natural transformation** between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a relation $\rho : Obj(\mathcal{C}) \to Obj(\mathcal{D})$ such that*

- *for $A \in Obj(\mathcal{C})$, $\rho(A) \in Mor(F(A), G(A))$*
- *for any $f \in Mor_{\mathcal{C}}(A, B)$ the following diagram commutes:*

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\rho(A)"'] & F(B) \arrow[d, "\rho(B)"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

*If each $\rho$ is an isomorphism then $\rho$ is a **natural isomorphism** and the functors $F$ and $G$ are isomorphic.*

**Definition 10.** *A category $\mathcal{C}$ is **small** if Obj($\mathcal{C}$), the set of objects in $\mathcal{C}$ and $Morph(\mathcal{C})$, the collection of morphisms in $\mathcal{C}$ is a set, not just a class.*

The motivation for defining a small category lies in the foundation of set theory. One may be tempted to define the set of all functors $F : \mathcal{C} \to \mathcal{D}$ as the objects of a new category, with their natural transformations as the morphisms. Yet, the natural transformations of functors may not form a set. This difficulty can be circumvented by requiring that $\mathcal{C}$ be a small category.

An example of a small category is the category of isomorphic groups $\mathcal{IG}$, which is created from the category **Groups** as follows. Define an equivalence class on objects in **Groups** where two objects A and B are equivalent if and only if they are isomorphic groups. Each equivalence class therefore is an object in $\mathcal{IG}$. The morphisms

between the objects $A$, $B$ in $\mathcal{IG}$ are likewise a set of equivalent classes of morphisms between any two representatives of $A$ and $B$. Choose $a, a', b, b' \in$ Obj(**Groups**) and let $\varphi : a \to a'$ and $\psi : b \to b'$ be two isomorphisms so that $A = [a] \cong [a']$ and $B = [b] \cong [b']$ for $A, B \in Obj(\mathcal{IG})$. Then $f, g \in Mor(A, B)$ are equivalent if and only if the following diagram commutes.

$$\begin{tikzcd} a \arrow[r, leftrightarrow, "\varphi"] \arrow[d, "f"'] & a' \arrow[d, "g"] \\ b \arrow[r, leftrightarrow, "\psi"] & b' \end{tikzcd}$$

**Definition 11.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is an* ***equivalence of categories*** *if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that natural isomorphisms $GF \cong id_{\mathcal{C}}$ and $FG \cong id_{\mathcal{D}}$ exist. Two categories are therefore* ***equivalent*** *if there exist such functors that satisfy these relations.*

The categories **R-mod** and **mod-R** are naturally equivalent when R is a commutative ring using a quasi-identity functor $F$ : **R-mod** $\to$ **mod-R** such that $F(A) = A$ and $F(f) = f$.

**Definition 12.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is called a* ***faithful functor*** *if the set maps $Mor_{\mathcal{C}}(A, A') \to Mor_{\mathcal{D}}(F(A), F(A'))$ are injections for all $A, A' \in \mathcal{C}$. That is, if $f_1$ and $f_2$ are distinct maps from A to $A'$ in $\mathcal{C}$, then $F(f_1) \neq F(f_2)$.*

A subcategory $\overline{\mathcal{C}}$ of a category $\mathcal{C}$ gives rise to a natural inclusion functor $F : \overline{\mathcal{C}} \to \mathcal{C}$. The inclusion functor is clearly faithful.

**Definition 13.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is* ***full*** *if the maps $Mor_{\mathcal{C}}(A, A') \to Mor_{\mathcal{D}}(F(A), F(A'))$ are all surjections. That is, every $g : F(A) \to F(A')$ in $\mathcal{D}$ is of the form $g = F(f)$ for some $f : A \to A'$.*

**Definition 14.** *A functor $F : \mathcal{C} \to \mathcal{D}$ that is both full and faithful is* ***fully faithful***.

# 2 Abelian Categories

In an Abelian category $\mathcal{A}$ the set of morphisms between two objects $A, B$ form an Abelian group. The properties of Abelian categories make them very suitable for study. The final result of this paper, the Freyd-Mitchell Embedding Theorem allows for a concrete approach to understanding Abelian categories.

**Definition 15.** *A category $\mathcal{A}$ is an* ***Ab-category*** *if every set of morphisms $Mor_{\mathcal{A}}(C, D)$ in $\mathcal{A}$ is given the structure of an Abelian group in such a way that composition distributes over addition. For example, given a diagram $\mathcal{A}$ of the form*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g'"] \arrow[r, shift right, "g"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

*we have $h(g + g')f = hgf + hg'f$ in Mor(A, D).*

Letting A=B=C=D of objects A, B, C, D in $C$, we see that each $Mor(A, A)$ is an associative ring since by definition elements in $Mor(A, A)$ possess the characteristics of a ring. Therefore an **Ab**-category with one object is the same as a ring. Conversely, **R-mod** is an **Ab**-category for every ring $R$ because the sum of $R$-module homomorphisms is closed.

**Definition 16.** *A **zero object** is an object 0 in an **Ab-category** $\mathcal{A}$ such that for every object A in $\mathcal{A}$, there is a unique morphism from 0 to A and a unique morphism from A to 0. A zero object also defines a 0 morphism, which is a morphism $A \to B$ that factors through 0.*

**Definition 17.** *A category $\mathcal{A}$ is an **additive category** if it is an **Ab**-category with a zero object, 0, containing a product $A \times B$ for every pair A, B of objects of $\mathcal{A}$. In an additive category the zero morphism is the additive identity in $Mor(A, B)$.*

In an additive category, the finite products are the same as finite coproducts, where $A \times B$ is denoted $A \bigoplus B$. This follows directly from the following stronger statement about a coproduct. Observe that in an additive category, a product is defined, but not a coproduct; it's existence is determined in the following theorem.

**Theorem 1.** *In an additive category $\mathcal{A}$ let $i_1\{1, 0\} : A_1 \to A_1 \bigoplus A_2$ and $i_2\{0, 1\} : A_1 \to A_2 \bigoplus A_2$ Then $(A_1 \bigoplus A_2; i_1,\ i_2)$ is the coproduct of $A_1$ and $A_2$.*

**Proof: [HS], Thm 9.1, 9.2** First we need to show that $i_1p_1 + i_2p_2 = 1_{A_1 \bigoplus A_2}$, where $p_i : A_1 \bigoplus A_2 \to A_i$ such that $p_i|_{A_i} \cong 1_{A_i}$. This is true since

$$p_1(i_1p_1 + i_2p_2) = p_1i_1p_1 + p_1i_2p_2 = p_1$$

since $p_1i_1 = 1_{A_1}$ and $p_1i_2 = 0$. Similarly $p_2(i_1p_1 + i_2p_2) = p_2$ by the uniqueness property of the product $i_1p_1 + i_2p_2 = 1_{A_1 \bigoplus A_2}$. Now given $\varphi_i : A_i \to B$, for $i = 1, 2$ define $\langle\varphi_1, \varphi_2\rangle = \varphi_1p_1 + \varphi_2p_2 : A_1 \bigoplus A_2 \to B$. Then

$$\langle\varphi_1, \varphi_2\rangle i_1 = (\varphi_1p_1 + \varphi_2p_2)i_1 = \varphi_1p_1i_1 + \varphi_2p_1i_1 = \varphi_1$$

. Similarly $\langle\varphi_1, \varphi_2\rangle i_2 = \varphi_2$. $\langle\varphi_1, \varphi_2\rangle$ is also unique. If $\theta i_1 = \varphi_1,\ \theta i_2 = \varphi_2$ then

$$\theta = \theta(i_1p_1 + i_2p_2) = \theta i_1p_1 + \theta i_2p_2 = \varphi_1p_1 + \varphi_2p_2 = \langle\varphi_1, \varphi_2\rangle.$$

**Definition 18.** *In any additive category $\mathcal{A}$, a **kernel** of a morphism $f : B \to C$ is defined to be a map $i : A \to B$ such that $fi = 0$ which is universal with respect to this property. The kernel of a map is not necessarily unique. A **cokernel** of $f$ is a map $e : C \to D$ which is universal with respect to having $ef = 0$.*

**Definition 19.** *In an additive category $\mathcal{A}$, a morphism $i : A \to B$ is **monic** in $\mathcal{A}$ if $ig = 0$ implies $g = 0$ for every morphism $g : A' \to A$. Similarly, a morphism $e : C \to D$ is called **epic** in $\mathcal{A}$ if $he = 0$ implies $h = 0$ for all maps $h : D \to D'$.*[2]

[2] The definition of monic and epic in a non-Abelian category differs from the one presented here. Since the focus will be exclusively on Abelian categories, this specific definition works best for the purposes at hand.

In **Sets, Ab, R-mod** in which objects have an underlying set, the monic morphisms are precisely the morphisms that are set injections (monic morphisms) in the usual sense. In **Sets, Ab, R-mod** the epic morphisms are precisely the surjective maps (epimorphisms).

**Definition 20.** *In a category $C$, if $\alpha : C' \to C$ is a monic morphism, then $C'$ is a **subobject** of $C$.*

**Definition 21.** *In a category $C$, if $\{u_i : C_i \to C\}_{i \in I}$ is a set of subobjects of $C$, then a morphism $u : C' \to C$ is the **intersection** of the family if for each $i \in I$, $u$ can be written as $u = u_i v_i$ for some unique morphism $v_i : C' \to C_i$. Furthermore, every morphism $B \to C$ which factors through each $u_i$ factors uniquely through $u$.*

**Definition 22.** *An **essential extension** is a monic morphism $M \to E$ such that for every nonzero monic morphism $M' \to E$ the intersection of the images of $M \to E$ and $M' \to E$ are nonzero.*

**Definition 23.** *An **Abelian category** is an additive category $\mathcal{A}$ such that*

- *every map in $\mathcal{A}$ has a kernel and a cokernel.*
- *every monic in $\mathcal{A}$ is the kernel of its cokernel.*
- *every epic in $\mathcal{A}$ is the cokernel of its kernel.*

Thus by considering the definitions in an Abelian category, a monic morphism is a kernel and an epic morphism is a cokernel .

**Definition 24.** *A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of maps in an Abelian category is **exact** at $B$ if $ker(g) = im(f)$. This implies in particular that the composition $gf : A \to C$ is zero. A sequence of maps is exact of it is exact at every object in the sequence that is both the domain and codomain of a map in the sequence. A left-exact sequence is a sequence of the form $0 \to A_1 \to A_2 \to A_3$ where each object in the sequence is exact. Similarly, a right-exact sequence is a sequence of the form $A_1 \to A_2 \to A_3 \to 0$ where each object in the sequence in exact.*

**Definition 25.** *A **left (right)-exact functor** is a functor between Abelian categories that carries left (right)-exact sequences to left (right)-exact sequences.*

We can now understand another set of examples of a category; the ***functor category***. Let $C$ be any small category and $\mathcal{A}$ an Abelian category. Then the functor category $\mathcal{A}^C$ is the category whose objects are functors $F : C \to \mathcal{A}$. The morphisms in $\mathcal{A}^C$ are natural transformations. This category is an Abelian category where the group operation is the sum of natural transformations as follows: Let $F_1,\ F_2 \in Obj(\mathcal{A}^C)$, then $(F_1 \bigoplus F_2)(C) = F_1(C) \bigoplus F_2(C)$. Since $\mathcal{A}$ is Abelian, this is well defined. The identity is the functor $0 : C \to 0 \in \mathcal{A}$. It is easy to check that this structure forms a group.

Next, let $\mathcal{A}$ be a small Abelian category and $\mathcal{G}$ the category of Abelian groups, define $(\mathcal{A}, \mathcal{G})$ to be the category of additive functors[3] from $\mathcal{A}$ to $\mathcal{G}$.

[3] a functor whose domain and codomain are both additive categories

Finally, we provide an example of a functor that will be quite useful in proving our result. The *representation functor* is the contravariant functor $H : \mathcal{A} \to (\mathcal{A}, \mathcal{G})$ where $H(A) = Mor(A, -) \in (\mathcal{A}, \mathcal{G})$ and $H(f : A \to B) = (f, -) : Mor(B, -) \to Mor(A, -)$ . When $Mor(A, -)$ is being considered as an object in $(\mathcal{A}, \mathcal{G})$, we shall denote it by $H^A$. Given $f : A \to B \in \mathcal{A}$, it is convenient to denote the corresponding transformation by $H^f : H^B \to H^A$.

**Definition 26.** *An object $P$ in an Abelian category $\mathcal{A}$ is a **projective generator** if the functor $Mor(P, -) : \mathcal{A} \to \mathcal{G}$ is exact and faithful.*

**Definition 27.** *An object $C$ is an **injective cogenerator** if the contravariant functor $Mor(-, C) : \mathcal{A} \to \mathcal{G}$ is faithful and maps exact sequences into exact sequences.*

# 3 The Freyd-Mitchell Embedding Theorem

**Theorem 2.** ***The Freyd-Mitchell Embedding Theorem**: If $\mathcal{A}$ is an Abelian category, then for every small Abelian subcategory $\overline{\mathcal{A}} \subset \mathcal{A}$ there is a ring $R$ and an exact, fully faithful functor from $\mathcal{A}$ into **R-mod** which embeds $\overline{\mathcal{A}}$ as a full subcategory in the sense that $Mor_{\mathcal{A}}(M, N) \cong Mor_R(iM, iN)$, where $i$ is the embedding map.*

Assuming the proof of Freyd-Mitchell, the implications of the result need reflection. If any Abelian category can be embedded into a category of R-modules, the existence of such an embedding allows for a more hands-on study of the structures of any Abelian Category. This is possible since in a **R-mod**, there exist more practical forms of obtuse definitions and diagram chasing proofs. The results therefore trickle down to the original category.
Before the main result of this paper can be addressed, several necessary results will need to be proven. Note that the proof of Freyd-Mitchell can almost entirely be found in [F].

**Theorem 3.** ***Yoneda Lemma**: Let $\tau$ be a natural transformation from the functor $Mor_C(A, -)$ to the functor $F$ from **C** to **Sets** and $\tau_A$ be a map from $Mor_C(A, A)$ to the functor $F(A)$. Then $\tau \mapsto \tau_A(1_A)$ sets up a one-to-one correspondence between the set $Mor(Mor_C(A, -), F)$ of natural transformations from $Mor_C(A, -)$ to $F$ and the set $F(A)$.*

**Proof: [HS], Thm 2.4.1** First, we need to show that $\tau$ is determined by the element $\tau_A(1_A) \in F(A)$. Let $\varphi : A \to B$ and consider the diagram:

$$\begin{tikzcd} Mor_C(A, A) \arrow[r, "\varphi_*"] \arrow[d, "\tau_A"] & Mor_C(A, B) \arrow[d, "\tau_B"] \\ F(A) \arrow[r, "F(\varphi)"] & F(B) \end{tikzcd}$$

The diagram commutes since $\tau$ is a natural transformation. Therefore, by following the diagram,

$$\tau_B(\varphi) = (\tau_\beta)(\varphi_*)(1_A) = (F\varphi)(\tau_A)(1_A)$$

This proves that $\tau$ is determined by $\tau_A(1_A)$.

To show that the correspondence set by $\tau \mapsto \tau_A(1_A)$ is surjective, it is necessary to show that for all $k \in FA$, the rule $\tau_B(\varphi) = (F\varphi)(k)$, for $\varphi \in Mor_C(A, B)$ defines a natural transformation from $Mor_C(A, -)$ to $F$. Let $\theta : B_1 \to B_2$ and consider the diagram:

$$\begin{tikzcd} Mor_C(A, B_1) \arrow[r, "\theta_*"] \arrow[d, "\tau_{B_1}"] & Mor_C(A, B_2) \arrow[d, "\tau_{B_2}"] \\ F(B_1) \arrow[r, "F\theta"] & F(B_2) \end{tikzcd}$$

This diagram commutes if $\tau_{B_1}, \tau_{B_2}$ are defined by the necessary rule $\tau_\beta(\varphi) = (F\varphi)(k)$, for $\varphi \in Mor_C(A, B)$. We therefore verify the equation as follows:

$$(\tau_{B_2})\theta_*(\varphi) = (\tau_{B_2})(\theta\varphi) = F(\theta\varphi)(k) = F(\theta)F(\varphi)(k) = F(\theta)\tau_{B_1}(\varphi)$$

for $\varphi : A \to B_1$. Thus our result is proven.

**Theorem 4.** *The Yoneda Lemma implies that a sequence*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

*is exact, if for every $M$ in $C$ the following sequence is exact.*

$$Mor_C(M, A) \xrightarrow{\alpha_*} Mor_C(M, B) \xrightarrow{\beta_*} Mor_C(M, C)$$

**Proof: [W], Thm 1.6.11** Let $M = A$, then $\beta\alpha = \beta^*\alpha^*(1_A) = 0$. Letting $M = Ker(\beta)$, the inlcusion $i : Ker(\beta) \to B$ satisfies $\beta^*(i) - \beta i = 0$. Therefore there is an $f \in Mor(M, A)$ such that $i = \alpha^*(f) = \alpha(f)$ and $Ker(\beta) = Im(i) \subset Im(\alpha)$.

**Theorem 5.** *If an object $E \in (\mathcal{A}, \mathcal{G})$ is faithful, and preserves monic morphisms (carries monic morphisms to monic morphisms), then it is exact.*

**Proof: [F], Thm 7.11** We start with the exact sequence $A' \to A \to A'' \to 0$ in $\mathcal{A}$. The representation functor yields the exact sequence $0 \to H^{A''} \to H^A \to H^{A'}$ in $(\mathcal{A}, \mathcal{G})$. Since the functor $(-, E) : (\mathcal{A}, \mathcal{G}) \to \mathcal{G}$ is an exact functor (see [F] section 5.1), we can form the sequence $(H^{A'}, E) \to (H^A, E) \to (H^{A''}, E) \to 0$ in $\mathcal{G}$. The Yoneda Lemma shows this sequence is isomorphic to $E(A') \to E(A) \to E(A'') \to 0$. Thus we have shown that $E$ is a right exact sequence. Finally, a right-exact functor is exact if and only if it preserves monic morphisms. Assuming that a functor is exact shows that a monic morphisms are preserved. Conversely, a right exact functor that preserves monic morphisms is exact since a zero object can be inserted at the left end of a diagram, thus making the sequence exact.

**Definition 28.** *Consider the following commutative diagram in a category $C$.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & P \end{tikzcd}$$

*then $P$ is the pushout of $A \to B$ and $A \to C$ if for every pair of maps $B \to X$ and $C \to X$ such that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & X \end{tikzcd}$$

*commutes, there is a unique $P \to X$ such that $B \to P \to X = B \to X$ and $C \to P \to X = C \to X$.*

**Theorem 6.** *Let $M \to E$ be an essential extension in $(\mathcal{A}, \mathcal{G})$. Then If $M$ preserves monic morphisms, so does $E$.*

**Proof: [F], Thm 7.12** We will prove this by contradiction, supposing that E does not always preserve monic morphisms, i.e., there exists a monic morphism $f : A' \to A \in \mathcal{A}$ such that $E(A') \to E(A)$ is not a monic morphism in $\mathcal{G}$. Now choose $x$ such that $0 \neq x \in E(A')$ and $E(f)(x) = 0$. We construct a subfunctor $F \subset E$ generated by $x$ where $F(B) = \{y \in E(B) \big| \exists A' \to B \in \mathcal{A}$ where $(E(A') \to E(B))(x) = y\}$ Since $x \in F(A') \subset E(A')$, $F \neq 0$. For $B' \to B$, $\big(E(B') \to E(B)\big)\big(F(B')\big) \subset F(B)$. $F(B' \to B)$ can be defined by restricting $E$. So $F$ is a set-valued functor; it is also a group-valued functor as $F(B)$ is a subgroup of $E(B)$, since $F$ is the image of the transformation $\eta : H^A \to E$ such that $\eta(1_A) = x$.

Since $M \subset E$ is essential (induces an essential extension), $F \cap M \neq 0$. Therefore, there exists an object $B$ such that $F(B) \cap M(B) \neq \emptyset$. Let $0 \neq y \in F(B) \cap M(B)$. By the construction of $F$, there is a map $A' \to B$ such that $y = \big(E(A') \to E(B)\big)(x)$.

Now consider the commutative pushout diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] & \\ B \arrow[r] & P \arrow[r] & D \end{tikzcd}$$

By the pushout theorem (see [F], Thm. 2.54), it is clear that since $A' \to A$ is a monic morphism, $B \to P$ is also monic, where $P$ is the pushout of $A$ and $A'$. Finally since $M$ preserves monic morphisms, $(M(B) \to M(P))(y) \neq 0$.
Since $0 \neq (M(B) \to M(P))(y) \subset (E(B) \to E(P))(y)$ then

$$\begin{aligned} 0 \neq (E(B) \to E(P))(y) &= (E(B) \to E(P))(E(A') \to E(B))(x) \\ &= (E(A') \to E(P))(x) \\ &= (E(A) \to E(P))(E(A') \to E(A))(x) \\ &= 0 \end{aligned}$$

which is a contradiction.

**Theorem 7.** *Let $\mathcal{L}(\mathcal{A}) \subset Mor(\mathcal{A}, \mathcal{G})$ be the full sub-category of left exact functors. Then $\mathcal{L}(\mathcal{A})$ is Abelian and $H : \mathcal{A} \to \mathcal{L}(\mathcal{A})$ is an exact and fully faithful for $H$ the representation functor.*

**Proof [F], Thm 7.33, 7.31** An axiomatic check that $\mathcal{L}(\mathcal{A})$ is Abelian can be found in [F] Thm 7.31. When $H$ was initially defined, we saw that $H$ is a fully faithful functor. It suffices to show that $H$ is exact. Let

$$0 \to A' \to A \to A'' \to 0$$

be exact in $\mathcal{A}$. Next we need to show

$$0 \to H^{A''} \to H^{A} \to H^{A'} \to 0$$

is exact in $\mathcal{L}(\mathcal{A})$. In order to do this, we must show that

$$0 \to Mor(H^{A'}, E) \to Mor(H^{A}, E) \to Mor(H^{A''}, E) \to 0$$

is exact for $E$ an injective cogenerator in $\mathcal{L}(\mathcal{A})$ as found in Theorem 5. By the Yoneda Lemma,

$$0 \to Mor(H^{A'}, E) \to Mor(H^{A}, E) \to Mor(H^{A''}, E) \to 0$$

is isomorphic to

$$G \to E(A') \to E(A) \to E(A'') \to 0$$

The sequence is always exact if and only if $E$ is an exact functor. In fact, the previous theorem shows that since G preserves monic morphisms, $E$ is exact since it is an injective cogenerator and preserves monic morphisms.

**Theorem 8.** *$\mathcal{L}(\mathcal{A})$ is complete and has an injective cogenerator.*

**Proof: [F] Thm 7.32** The construction of products and sums in $\mathcal{L}(\mathcal{A})$ is straightforward. Given a family of left-exact functors $\{F_i\}$ their sum as defined in $(\mathcal{A}, \mathcal{G})$ is already left-exact and is the sum defined in $\mathcal{L}(\mathcal{A})$. The product of all functors $\{H^A\}_{A \in \mathcal{A}}$ is also left-exact and a generator for $\mathcal{L}(\mathcal{A})$ by construction. Therefore by referring to [F], Theorems 3.35, 3.36 and 3.37, $\mathcal{L}(\mathcal{A})$ has an injective cogenerator.

**Definition 29.** *Given two maps $f : A \to B$ and $g : A \to B$ for $A, B \in Obj(C)$, a map $K \to A$ is a **difference kernel** of $f$ and $g$ which fails to distinguish $f$ and $g$ and is universal to that respect. A **difference cokernel** is the dual notion of a difference kernel.*

**Definition 30.** *An Abelian category is **complete** if every pair of maps has a difference kernel and difference cokernel and every indexed set of objects has a product and a sum (a sum being a coproduct in an Abelian category).*

**Theorem 9.** *Let $\mathcal{A}$ be a complete Abelian category with a projective generator. Then for every small Abelian subcategory $\overline{\mathcal{A}} \subset \mathcal{A}$, there is a ring $R$ and an exact fully faithful functor of $\overline{\mathcal{A}}$ into the category of $R$-modules.*

**Proof: [F] Thm 4.44** Let $\overline{\mathcal{A}}$ be a small full exact subcategory of a complete Abelian category $\mathcal{A}$ and $\overline{P}$ a projective generator for $\mathcal{A}$. For each $A \in \overline{\mathcal{A}}$ consider the epimorphism $\bigoplus_{Mor(\overline{P}, A)} \overline{P} \to A$. Since $\overline{P}$ is a generator in a complete Abelian category, the

map $\bigoplus_{Mor(\overline{P},A)} \overline{P} \to A$ is epimorphic by way of the map defined for all $f \in Mor(\overline{P}, A)$

$$\overline{P} \xrightarrow{i_f} \Big( \bigoplus_{Mor(\overline{P},A)} \overline{P} \to A \Big) = \overline{P} \xrightarrow{f} A$$ [4]

Let $I = \bigcup_{A \in \overline{\mathcal{A}}} Mor(\overline{P}, A)$ and define $P = \bigoplus_I \overline{P}$ to obtain a projective generator $P$ such that for each $A \in \overline{\mathcal{A}}$ there is an epimorphism $P \to A$. We can see that $P$ is projective in the following manner. Given that each $P_i$ is projective for each $i$, let $A \to A''$ be an epimorphism. Then a morphism $P \to A''$ is determined by a family of morphisms $P_i \to A''$ for each of which we can write $P_i \to A'' = P_i \to A \to A''$. The morphisms $P_i \to A$ give us a morphism $P \to A$ with the desired property.

Next, let R be the ring of endomorphisms of $P$. For every $A \in \mathcal{A}$, the Abelian group $Mor(P, A)$ has a canonical R-module structure: for $x : P \to A \in Mor(P, A)$ and $r : P \to P \in R$ where $rx \in Mor(P, A)$ is defined as follows: $P \xrightarrow{r} P \xrightarrow{x} A$ .

Given a map $y : A \to B \in \mathcal{A}$, the induced map $\overline{y} : Mor(P, A) \to Mor(P, B)$ is an R-homomorphism since $\overline{y}(rx) = P \xrightarrow{r} P \xrightarrow{x} A \xrightarrow{y} B = r(\overline{y}(x))$. We can therefore define $F : \mathcal{A} \to \mathcal{G}^R$ by $F(A) = Mor(P, A)$ with the canonical R-module structure. By definition, $F$ is exact and faithful since $P$ is a projective generator. So it suffices to show that $F\big|_{\overline{\mathcal{A}}}$is full to prove that $F|_{\overline{A}}$ is exact and fully faithful. Given $A, B \in \overline{\mathcal{A}}$ and a map $\overline{y} : F(A) \to F(B) \in \mathcal{G}^R$ we need to find a map $y : A \to B \in \overline{\mathcal{A}}$ such that $F(y) = \overline{y}$. Let $0 \to K \to P \to A \to 0$ and $P \to B \to 0$ be exact sequences in $\mathcal{A}$. Observe that $F(P) = R$. We obtain the commutative diagram in **R-mod**:

$$\begin{tikzcd} 0 \arrow[r] & F(K) \arrow[r] & R \arrow[r] \arrow[d, "f"] & F(A) \arrow[r] \arrow[d, "\overline{y}"] & 0 \\ & & R \arrow[r] & F(B) \arrow[r] & 0 \end{tikzcd}$$

The existence of the map $f$ is insured by the projectiveness of R in **R-mod**. Since R is a ring, any automorphism on R must be equivalent to multiplication on the right by an R-element. We assume then that $f(s) = sr$ for all $s \in R$, where $r : P \to P \in R$.

Returning to $\mathcal{A}$, the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d, "r"'] & A \arrow[r] & 0 \\ & & P \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

is such that $K \longrightarrow P \xrightarrow{r} P \longrightarrow B = 0$ since $F(K) \longrightarrow R \xrightarrow{f} R \longrightarrow F(B) = 0$

[4] [F], Prop. 3.36

and $F$ is faithful. Hence there is a map $y : A \to B$ such that

$$\begin{tikzcd} P \arrow[r] \arrow[d, "r"'] & A \arrow[d, "y"] \\ P \arrow[r] & B \end{tikzcd}$$

commutes. Hence

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"'] & F(A) \arrow[d, "F(y)"] \\ R \arrow[r] & F(B) \end{tikzcd}$$

commutes. and since $R \to F(A)$ is epimorphic, $F(y) = \overline{y}$.

**Proof of Freyd-Mitchell: [F] Thm 7.34** We have seen by Theorem 7 that for every small Abelian category $\overline{\mathcal{A}}$ there exists an exact fully faithful contravariant functor into a complete Abelian category $\mathcal{A}$ with an injective cogenerator. By considering $\mathcal{A}^{opp}$, we obtain for every small Abelian category a fully faithful exact contravariant functor into a complete Abelian category with a projective generator. Lastly by Theorem 9, for every small Abelian category, there is an exact full embedding into a category of modules. Thus we have proved our main result.

# 4 Application: The Snake Lemma

The beauty and ease of using the Snake Lemma (once it is proven) comes from the structure of an R-module. In an R-module, we can diagram chase elements to prove the theorem. The structure of an R-module is also understood and well defined; it is easier to work and manipulate a theorem in this setting by utilizing the categorical definitions as applied to an R-module. Hence the terms kernel, cokernel, injective and surjective are well understood and applied in the traditional sense. For example, if we have $f : A \to A$ then the cokernel of $f$ would be the sub-module $S \subset A$ such that $S = A/(f(A'))$. Once the Snake Lemma is proven in **R-mod**, the Freyd-Mitchell Embedding Theorem states that the Snake Lemma can be applied to any arbitrary Abelian category.

**Theorem 10.** ***The Snake Lemma:*** *Consider the following commutative diagram of R-modules*

$$\begin{tikzcd} & A' \arrow[r, "i'"] \arrow[d, "f"'] & B' \arrow[r, "j'"] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "j"] & C & \end{tikzcd}$$

*where each row is exact. Then there is an exact sequence*

$$ker(f) \xrightarrow{i'_*} ker(g) \xrightarrow{j'_*} ker(h) \xrightarrow{\partial} coker(f) \xrightarrow{i_*} coker(g) \xrightarrow{j_*} coker(h)$$

*where* $\partial(c') = i^{-1}g(j')^{-1}(c')$ *for* $c' \in ker(h)$

Essentially the Snake Lemma can be visualized in the following manner:

$$\begin{tikzcd}
 & ker(f) \arrow[r, "i'_*"] \arrow[d] & ker(g) \arrow[r, "j'_*"] \arrow[d] & ker(h) \arrow[d] \arrow[dddll, dotted, "\partial"] & \\
 & A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[r, "j'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "j"] \arrow[d] & C \arrow[d] & \\
 & coker(f) \arrow[r, "i_*"] & coker(g) \arrow[r, "j_*"] & coker(h) &
\end{tikzcd}$$

**Proof**: We begin by showing that the induced maps between kernels and cokernels as well as the connecting morphism are well defined.

Show that the maps $i'_* : ker(f) \to ker(g)$, $j'_* : ker(g) \to ker(h)$ are well defined:
Choose $a' \in ker(f) \leq A'$. By mapping $a'$ through the commutative diagram, we can see that $gi'(a') = if(a') = 0$ showing that $i(a')$ must be in the kernel of $g$.

$$\begin{tikzcd}
a' \arrow[r, mapsto, "i'"] \arrow[d, mapsto, "f"] & i(a') \arrow[d, mapsto, dotted, "g"] \\
0 \in A \arrow[r, mapsto, "i"] & 0 \in B
\end{tikzcd}$$

The proof for $j'_*$ is similar.

Show that the maps $i_* : coker(f) \to coker(g)$, $j_* : coker(g) \to coker(h)$ are well defined:
Choose $[a] \in coker(f)$ and show the map $i_* : coker(f) \to coker(g)$ is well defined irrespective of the representative of the equivalence class. Let $a_1$ and $a_2$ be representatives of $[a]$ so that $a_1 + f(A') = a_2 + f(A')$. It follows that $a_1 - a_2 \in f(A')$. Following the diagram once again we see that

$$\begin{aligned}
a_1 + f(A') &= a_2 + f(A') \\
i(a_1 + f(A')) &= i(a_2 + f(A')) \\
i(a_1) + if(A') &= i(a_2) + if(A') \\
i(a_1) + gi'(A') &= i(a_2) + gi'(A') \\
[i(a_1)] &= [i(a_2)]
\end{aligned}$$

The proof for $j_*$ is likewise similar.

Show that $\partial : ker(h) \to coker(f)$ is well defined:
Choose $c' \in ker(h)$. Then by following the commuting diagram, $h(c') = j(b)$ for some $b \in ker(j)$, we see that the diagram exists since $j'$ is surjective by exactness. It remains to show that $i^{-1}(b)$ are in the same equivalence class in $coker(f)$.

$$\begin{tikzcd}
b' \arrow[r, mapsto] \arrow[d, mapsto] & c' \arrow[d, mapsto] \\
b \in ker(j) \arrow[r, mapsto] & 0
\end{tikzcd}$$

Let $a_1, a_2 \in i^{-1}(b)$, where $\partial(c') = a_1,\ a_2$. Then $i(a_1 - a_2) = i(a_1) - i(a_2) = 0$, so again, we follow the diagram to see that $a_1 - a_2 \in Im(f)$: $j \circ i(a_1 - a_2) = 0$ so $h^{-1}(j \circ i(a_1 - a_2)) = c' - c' = 0 \in ker(h)$. Thus by exactness, there exists an $a' \in A'$ such that $j' \circ i'(a') = h^{-1}(j \circ a(a_1 - a_2)$. Since the following diagram commutes, $a_1 - a_2 \in Im(f)$

$$\begin{tikzcd}
a' \arrow[rr, mapsto, bend left, "j'\circ i'"] \arrow[d, mapsto, dotted] & B' & c' - c' = 0 \in ker(h) \arrow[d, mapsto] \\
a_1 - a_2 \arrow[rr, mapsto, bend right, "j\circ i"'] & B & 0
\end{tikzcd}$$

We also need to consider what happens at $(j')^{-1}(c')$. Let $b'_1, b'_2 \in (j')^{-1}(c')$ Then $j'(b'_1 - b'_2) = j'(b'_1) - j'(b'_2) = c - c = 0$. Therefore, there exists an $a' \in A'$ such that $i'(a') = b'_1 - b'_2$. Therefore $i^{-1}(g(b'_1 - b'_2)) \in Im(f)$ which puts $b'_1$ and $b'_2$ in the same equivalence class, thus showing that $\partial$ is well defined.

$$\begin{tikzcd}
a' \arrow[r, mapsto] \arrow[d, mapsto] & b'_1 - b'_2 \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
f(a') \arrow[r, mapsto] & g(b'_1 - b'_2) \arrow[r, mapsto] & 0
\end{tikzcd}$$

Next, there are eight parts to the proof:

1. Show that $Im(i'_*) \subset ker(j'_*)$:
   By exactness, $j' \circ i' = 0\ \forall a' \in A'$ or $j' \circ Im(i'_*) = 0$ so it is clear that $j'_* \circ i'_* = 0$ since $j'_*$ and $i'_*$ are just restrictions of $i'$ and $j'$.

2. Show that $ker(j'_*) \subset Im(i'_*)$:
   For $b' \in ker(j'_*)$, show that there exists an $a' \in ker(f)$ such that $i'_*(a') = b'$.
   Choose $b' \in ker(j'_*)$. Then $g(b') = 0$, but note that since the bottom row is exact, $ker(i) = 0$. Since the following diagram commutes, we know that if there exists an $a'$ such that $i'_*(a') = b'$, then $a' \in ker(i'_*)$. Next we observe that since $b' \in ker(j'_*)$, and $Im(i') = ker(j')$ there is such an $a'$ that maps to $b' \in ker(j'_*)$.

$$\begin{tikzcd}
a' \arrow[r, mapsto, dotted] \arrow[rr, mapsto, bend left] \arrow[d, mapsto] & b' \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

3. Show that $Im(j'_*) \subset ker(\partial)$:
   Choose $b' \in ker(g)$ then $i^{-1}g(j')^{-1}(j'(c')) = i^{-1}(0) = 0$ since the sequence is exact, $Im(0) = 0 = ker(i)$.

4. Show that $ker(\partial) \subset Im(j'_*)$:
   $ker(\partial)$ are the elements that map to $Im(f)$. So these elements have an element $a' \in A'$ that map to $ker(\partial)$. By exactness, $ker(\partial) = 0 \in Im(j'_*)$

5. Show that $Im(\partial) \subset ker(i_*)$:
   The kernel of $i_*$ is defined as $ker(i_*) = \{a \in A | i(a) \in Im(g)\}$. The image of $\partial$ are members of an equivalence class $[a] \in coker(f)$. Choose a representative $a \in A$ for each class. By definition of $\partial$, $a$ maps to $g(b')$ under $i$ for some $b' \in B'$. Therefore $i_*(a) = 0$

6. Show that $ker(i_*) \subset Im(\partial)$:
   $ker(i_*) = 0 \subset Im(\partial)$

7. Show that $Im(i_*) \subset ker(j_*)$:
   By exactness, $j \circ i(A) = 0$ or $j \circ (Im(i)) = 0$ so $Im(i) \subset ker(j)$ which proves the claim since $i_*$ and $j_*$ are restrictions of $i$ and $j$ respectively.

8. Show that $ker(j_*) \subset Im(i_*)$:
   Choose $b + g(B') \in ker(j_*)$, where $j_*(b_g(B')) = 0$. Since $Im(i) = ker(j)$ and $i$ is injective, there exists an $a + f(A')$ such that $i_*(a) = b + g(B')$

   The Freyd-Mitchell Embedding theorem applies the Snake Lemma to any small Abelian category. Thus applications that rely on the Snake Lemma can be applied to any small Abelian category.

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