Mass-Radius Relation: Hydrogen Burning Stars

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Abstract. The purpose of this work is to show the mass-radius relationship of hydrogen burning stars on the lower level of the main sequence by using the Runge-Kutta code. This paper covers the theoretical predictions that go into finding this ratio. Since the radius of a star cannot always be measured directly, being able to predict it is important. Stellar interiors are part of the fundamental properties of stars that is vital to understand in astronomy. The results of the computational prediction were compared to observations and the known relationship from the Hertzsprung-Russel (H-R) diagram. We found the relationship for mass and radius to be nearly linear with a slight hump as you travel from the center of the star to the outer edge of the star. We also found that the radius and pressure relationship was the inverse. For stars of different pressures, we found that as the pressure increases the mass is proportional to

$$\frac{1}{r^4}$$

(1)

Keywords: (HB stars: hydrogen-burning stars · RK: Runge-Kutta · Mass-Radius Ratio.

1 Introduction

The sequence of a star’s life is something that is well known in astronomy today. Stars begin their life burning hydrogen, which is the fusion of four hydrogen nuclei into a single helium nucleus. Since there is nuclear fusion, we can assume the internal structure of every star is in hydro static equilibrium. The reaction depends on the mass of the star, and the mass depends on the core temperature and density. As the fusion rate increases, and hydrogen begins to run out, the radius of the star will increase.

In the 1930s, it was discovered that there is a relationship between the mass of a star and its luminosity and radius. The solution for determining a star’s luminosity and radius is known as the Vogt-Russel theorem. This theorem breaks down for stars that are not uniform, so for the purpose of this paper we will be assuming a spherically symmetric star. There are other relationships that can be used to find the mass-radius relationship. The Stefan-Boltzman law relates luminosity, radius and temperature, while the mass-luminosity relation relates luminosity and mass.
The relationship between mass and radius is known to be nearly linear. The relationship between mass and radius depends on how the nuclear reaction relates to the temperature. The mass-radius ratio increases by a factor of three over 2.5 orders of magnitude of mass. For the purposes of this paper we expect to find that $R$ is proportional to $M$ to the $3/7$ since we are only considering the P-P Chain portion of the main sequence.

Takenori Nakano found that there is a lower limit to the mass of main-sequence stars called the hydrogen-burning minimum mass. At certain masses the steady state of a hydrogen star cannot be maintained. Stars about 80 times the mass of Jupiter cannot sustain hydrogen fusion. Since sun like stars are not convective in their cores, hydrogen is limited to what it was at that radius when the star was formed. We will focus on the lower portion of the main sequence. This is because throughout the lower portion, the energy comes from the proton-proton chain, while the upper main sequence uses the CNO cycle. The stars in the lower main sequence maintain a steady state for an extended amount of time since they have a large amount of nuclear energy. The energy released from the core will maintain a high pressure, which keeps the weight of the star from collapsing in on itself due to gravity.

The particles in the inner structure of a star have their own energy levels as well as additional energy from the vibrational and rotational motion. Since the star is in equilibrium, all motion is excited, and speeds follow the Maxwell-Boltzmann distribution. This, coupled with the fact that the gravity of a star makes the gravity of individual particles insignificant, allows us to use the ideal gas law for our derivations. The motion of the molecules are non-relativistic since the speeds are so great. The ideal gas law suffices for high temperature and low pressures with minimal collisions. The cores of hydrogen burning stars are an ideal gas, and therefore, we can use the ideal gas law.

The paper is organized in the following way. In Sect. 2 we present the methods. We demonstrate the equations and how they were used. We introduce the method we used to write the code to find the mass-radius relationship, and we make the equations dimensionless in order to use them in the code. In Sect. 3, we calculate our error and discuss the effects of the error. In Sect. 4, we present our numerical results, and Sect. 5 is the mass-radius relation where we discuss our results and how they relate to the HR diagram. Sect 6 is the discussion and implications of these results. Finally, Sect. 7 is the code for Runge-Kutta used to find the relationships.

2 Methods

2.1 Equations Used

In order to model the internal structure of a spherically symmetric HB star, three main equations were used.

$$\frac{dP}{dr} = -g\rho$$ (2)
\[ dM(r) = 4\pi\rho r^2 dr \]  \hspace{1cm} (3)

\[ P = K\rho^\gamma \]  \hspace{1cm} (4)

Eq. (1) is the basic differential equation for a fluid in hydrostatic equilibrium. The pressure gradient is balanced by the external force of gravity within the internal structure of the HB stars. The effective gravitational acceleration for a spherically symmetric star is given by \[ g = \frac{GM(r)}{r^2} \]. Eq. (2) is the differential mass volume relationship, again, assuming a spherically symmetric star. The ideal gas law yields the polytropic equation of state which gives a relationship between the pressure and density of the internal structure of HB stars. Solving for \( \rho \) in Eq. (3) yields:

\[ \rho = \left( \frac{P}{K} \right)^\frac{1}{\gamma} \]  \hspace{1cm} (5)

Inserting Eq. (4) into both Eq. (2) and (1) yields differential equations written in terms of density, mass, pressure, radius, and a few constants that will be morphed into their appropriate terms when re-writing the ODEs in terms of dimensionless variables:

\[ \frac{dM(r)}{dr} = \left( \frac{P}{K} \right)^\frac{1}{\gamma} \left( 4\pi r^2 \right) \]  \hspace{1cm} (6)

\[ \frac{dP}{dr} = -\frac{GM(r)}{r^2} \left( \frac{P}{K} \right)^\frac{1}{\gamma} \]  \hspace{1cm} (7)

Eq. (5) and (6) will be used in subsection 2.3 to construct equations of dimensionless variables that will be suitable for performing computational analysis.

2.2 Runge-Kutta 4th Order

Eq. (5) and (6) are ordinary differential equations that can be solved using various numerical analysis approximations. For our purposes, we chose to approximate the radii of HB stars using the Runge-Kutta Second Order approximation. This method is based on finding slope approximations at multiple points between two times: \( t = t_0 \) and \( t = t_0 + \Delta t \). For RK4, four slopes are calculated:

\[ k_1 = f(x_0, t_0) \]  \hspace{1cm} (8)

\[ k_2 = f(x_0 + k_1 \frac{\Delta t}{2}, t_0 + \frac{\Delta t}{2}) \]  \hspace{1cm} (9)

\[ k_3 = f(x_0 + k_2 \frac{\Delta t}{2}, t_0 + \frac{\Delta t}{2}) \]  \hspace{1cm} (10)

\[ k_4 = f(x_0 + k_3 \Delta t, t_0 + \Delta t) \]  \hspace{1cm} (11)

The slopes \( k_1-k_4 \) are calculated and combined to produce an approximation of one average slope that is used to calculate the value of the solution to the ODE:

\[ \text{avg. slope} = \left( \frac{1}{6} \right) [k_1 + 2k_2 + 2k_3 + k_4] \]  \hspace{1cm} (12)
The method described above leads into a fourth order approximation of the ODE, however, for our purposes, we utilized the second order RK method, which is based on the midpoint method.

For RK2, only two slopes are calculated, given initial conditions, and used to find an approximated value for the function at a new time-step. The implemented method, given our initial conditions for the mass and pressure of a given HB star, takes the form:

\[ M(r_n) = M_c \]  
\[ P(r_n) = P_c \]  

(13) 

(14)

The initial conditions above represent the central mass and pressure for a given star. These are an essential part of the RK2 method used. Let Eq. (5) and (6) be represented by \( f(M/P, r) \) below:

\[ \frac{dM}{dr} = \left( \frac{P}{K} \right)^{\frac{1}{\gamma}} (4\pi r^2) = f(M, r) \] 
\[ \frac{dP}{dr} = -\frac{GM}{r^2} \left( \frac{P}{K} \right)^{\frac{1}{\gamma}} = f(P, r) \] 

(15) 

(16)

Finally, we have:

\[ k_1 = f(M_n/P_n, r_n) \]  
\[ k_2 = f(M_n/P_n + k_1 h/2, r_n + h/2) \]  
\[ M/P(n + 1) = M_n/P_n + h k_2 \] 

(17) 

(18) 

(19)

Eq. (16), (17), and (18) will be used for the numerical analysis, along with the dimensionless forms of Eq. (14) and (15).

### 2.3 Dimensionless Variables

Before solving the ODEs of Eq. (5) and (6), they must first be put into a dimensionless variable form:

\[ M'M_0 = M \]  
\[ P'P_0 = P \]  
\[ r'r_0 = r \] 

With the dimensionless variables above, Eq. (5) and (6) can be transformed as:

\[ \frac{dM'}{dr'} = (P')^{\frac{1}{\gamma}} (r')^2 \beta \] 
\[ \beta = \left( \frac{4\pi}{M_0} \right) \left( \frac{P_0}{K} \right)^{\frac{1}{\gamma}} (r_0)^3 = 1 \] 

(20)
\[ \frac{dP'}{dr'} = -\left( \frac{m'}{r'^2} \right) (P')^{\frac{1}{\gamma}} \alpha \]

\[ \alpha = \frac{\left( \frac{GM_0}{r_0} \right) (P_0)^{\frac{1-\gamma}{\gamma}}}{K^{\frac{1}{\gamma}}} = 1 \]

Eq. (19) and (20) are the equations that were used in the RK2 numerical analysis. Solving these equations shed light on the correlation between pressure, mass, and radii of HB stars.

### 3 Error Estimation

The numerical results were verified using a convergence plot. The convergence plot is used to determine the optimal time step (i.e. \( \Delta r \)) by plotting the error of our pressure as a function of the step-size, \( h \) (Note: \( h \) is analogous to our \( \Delta r \)).

The error estimation was found by setting \( \gamma = 1 \) which allows us to denote the mass of the sun as our reference: the radii and mass calculations will thus be in terms of solar masses.

The error estimation follows as:

\[ \epsilon = \frac{|P - P_0|}{P_0} \]

\( P_0 \) was found by integrating Eq. (20) with \( \gamma = 1 \). The expression found through integration by hand was:

\[ P_0 = \exp(-1) \]

The convergence plot was made using this value for \( P_0 \).

The minimum value of epsilon represents the smallest possible error for the pressure calculation. From the convergence plot, we were able to choose the specific value for our step-size that would optimize the numerical results for radial dependence of pressure and mass of HB stars.

Fig. 1).
Fig. 1. Convergence plot. Epsilon graphed on the vertical axis, step-size, h, on the horizontal axis. The peak (minimum value of $\epsilon$) is at a step-size value of $h = 0.605986$. 
4 Numerical Results

After using the dimensionless differential equations and using the convergence plot to find the right step size, $h = 0.61$. We assumed that the pressure at the center of the star was 1.0. We then ran the code (found in section 7) and found that the radius of this star was about 17.6 solar radii. The mass of this star was about 13.1 solar masses. We plotted how mass and radius change as $r$ increases and how pressure and radius change as $r$ increases. The following figure shows how mass changes with radius. Mass should increase as radius increases because as you get farther from the center of the star, there is more mass enclosed.

Fig. 2.

![Mass as a function of Radius](image)

**Fig. 2.** This graphs shows the relationship between the mass and radius of a star.
The following figure shows how the pressure of a given HB star changes with radius. It appears to be the inverse of mass and radius because as you get farther from the center of the star, pressure will decrease.

\text{Fig. 3).}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{pressure_radius_graph.png}
\caption{This graph shows the relationship between the Pressure and radius of the star.}
\end{figure}

5 Mass Radius Relation

The mass radius relationship is as follows,

\[ M \alpha \frac{1}{r^4} \]  \hspace{1cm} (24)

This relationship is shown by Figure 4. The more massive stars are found at the upper half of the HR diagram, but we focused on the lower half where the proton-proton chain is. As we increased the pressure variable in our code, we found that mass increased while the radius fluctuated. We expected this fluctuation because the HR diagram shows that as mass increase linearly, radius does not. it fluctuates as stars get more massive.

\text{Fig. 4).}
6 Discussion and Conclusion

After running the code, we found the relationship between mass and radius to be

\[ \frac{1}{r^4} \tag{25} \]

This proved that as radius increases, so does Mass. This is shown by Fig2. This makes sense because when you start at the center of the star, \( r = 0 \), there is no mass enclosed in a radius of zero. We can use Eq.(2) to verify that this is the case. As \( r \) increases, more mass is enclosed and continues to increase until you reach the edge of the star. Once you reach the edge of the star, there is no more mass to enclose. We defined the edge of the star as when pressure reaches zero. Since there is no more mass, pressure would be equal to zero. As radius increases pressure will decrease. We expected this because the entire star is being compressed by gravity and at the center the pressure is at a maximum. As you move away from the center pressure will decrease and continue to decrease until the edge of the star. After the edge of the star, pressure is zero because there is no more mass being compressed by gravity.

References

7 Appendix A: Code

Figure 5 below is the code that was used for numerically analyzing HB stars using the RK method.

Fig. 5.

```c
#include<stdio.h>
#include<math.h>

double RHB(int ieq, double r, double P, double M)
{
    double rpow=10.0-6.0;
    float gamma=0.6;
    if(ieq==0)
    {
        return -pow(P, gamma)*M/(pow(r, 2.0)+f);
    }
    else if(ieq==1)
    {
        return pow(r, 2.0)*pow(P, gamma);
    }

    int main(void)
    {
        double h=0.005;
        float var[2];
        float varmid[2];
        double r=0.0;
        float gamma=0.6;
        float P;M;
        float Pr;M;
        int ieq;
        float M, Eps, Mh;
        float Ni;

        //K=
        var[0]=Mh; //P
        var[1]=(1.333)*Mh*P; //pow(pr, 2.0)*pow(P, gamma); //M
        do
        {
            for(ieq0; ieq1; ieq++) //half step calculation/
            {
                varmid[ieq0]=var[ieq0] + (h+0.5)*RHB(ieq0, r, var[0], var[1]);
            }
            for(ieq0; ieq1; ieq++) //full step calc/
            {
                varmid[ieq0]=var[ieq0] + h*RHB(ieq0, r, varmid[0], varmid[1]);
                printf("be Wright", var[0], r);
            }
        }
        while (var[1]==0.0);
        // Eps=abs(M-Mh/Mh);
        //printf("be Wright, h,Eps
        return 0;
    }
```

Fig. 5. The numerical method/code utilized in order to analyze the mass, pressure, and radii of a variety of HB stars.