How the mass relationship with a White Dwarf Star

Introduction.

A white dwarf star (also known as a degenerate dwarf star) is a low-luminosity, high-density, high-temperature star. Because it is white in color and relatively small in size, it is named as a white dwarf star. The white dwarf is a star that has evolved to the end, consisting mainly of carbon, and is covered with a layer of hydrogen and helium. The white dwarf star gradually cools and darkens in billions of years. It is small in size and low in brightness, but high in density and high in quality.

According to the above principle, the mass of white dwarfs is obviously different from when it was stars. After the end of the hydrogen fusion reaction, the medium-low-mass star will carry out the helium fusion at the core, burn the helium into a carbon and oxygen process, and expand into a red giant star. If the red giant does not have enough mass to produce a higher temperature that will allow the carbon to coalesce, carbon and oxygen will build up at the core. After the gas that emits the outer layers becomes a planetary nebula, only the core part remains, and this wreck will eventually become a white dwarf star. Therefore, white dwarfs usually consist of carbon and oxygen. However, it is also possible that the core temperature can reach a high temperature that causes carbon fusion but is still insufficient for fusion, and a white dwarf star composed of oxygen, strontium and magnesium can be formed.

Under this circumstance, the force that mainly maintains the shape of the white dwarf is only the gravitation and degenerated force. Degenerated state is a when particles are extruded to together. The force is basically from electrons and protons but not usual thermal gas pressure.

Therefore, it is very interesting to study the mass distribution of a white dwarf star. It tells us the ground and underground structure of the white dwarf, and also tells us the weight on the white dwarf. This allows us to not approach a star close to death, and know what a white dwarf is like in a safe distance.
For mathematical way to know more details about the mass, radius, density information for white dwarf star, the professor and others astronomy website tell us:

\[
\frac{dP}{dr} = -GM(r) \times \frac{\rho}{r^2} \\
\frac{dP}{dm} = -Gm/4\pi r^4 \times \rho \\
\frac{dM(r)}{dr} = 4\pi r^2 \\
P = K\rho^\Gamma \\
(\Gamma = 1 + (1/n) and where P is pressure, \rho is density and K is a constant of proportionality.)
\]

**Assumption & chosen Numerical method.**

Combining the first two equation, we could have:

\[
\frac{d}{dr}[\frac{r^2}{\rho} \times (dP/dr)] = -G(\frac{dm}{dr}) \\
(1/r^2) \times \frac{d}{dr}(\frac{r^2}{\rho} \times dP/dr) = 4\pi G\rho
\]

Put the P function in to it, we could have:

\[
((n + 1)K/4\pi Gn) \times (1/r)d/dr \times (r^2/\rho^{(n - 1/n)})d\rho/dr = -\rho \\
\rho = \rho_c \times \theta^n \\
[(n + 1)K/4\pi G(\rho_c)^\Gamma] = \alpha^2 \\
(\alpha is the value for dimensionless) \\
According that, the radius function r equals to:
\]

\[
r = \alpha \xi \ and \ the \ radius \ value \ of \ the \ white \ dwarf \ star \ is \ R = \alpha \xi_1
\]

The next step is put all dimensionless variables including density and radius function back to substitute the differential equation we just get above, And we can have:

\[
1/\xi^2[d/d\xi(\xi^2 d\theta/d\xi)] = -\theta^n
\]

(as known as Lane–Emden equation)

P is the pressure, \(\rho\) is the density, and K is the proportional constant. The standard boundary condition is \(\theta(0) = 1\) and \(\theta' (0) = 0\). So the solution to this equation is to describe the relationship between star pressure and density and radius, and the given multi-index \(n\) is also the multi-party index \(n\) of the multi-ball. Hydrostatic equilibrium is related to potential energy, density, and pressure.
This differential equation explains the relationship between radius and density. Because of the dimensionless process, this differential equation describes the ratio of density and the relationship as the radius increases. Since we already have differential equations, we are going to use the ODE method to solve this problem by drawing pictures.

**Verification of the Numerical method.**

Before using the ODE method, we also need find out the right hand side of $d\theta/d\xi$ and right hand side of value of the changing of $d\theta/d\xi$. Just set the changing of $d\theta/d\xi = \varphi$. Then, we should have $d\theta/d\xi$ and $d\varphi/d\xi$.

\[
d\theta/d\xi = -\varphi/\xi^2
\]

\[
d\varphi/d\xi = \theta^n \times \xi^2
\]

From those two differential equations, it is easy to find value of $\theta$ and $\xi$. However, the density is the value associate with $\theta^n$ and $\xi$. So, I use the RK-4 which is Second order Runge-Kutta time stepping to approximate method finding $\theta$ first and calculate the $n$ times power of it to find the density distribute on radius. After that, we are going to find the ratio of density on different radius ratio. Also, for make the graph, the ratio of radius should by divided by the radii of the white dwarf star $R$ (in dimensionless variable is $\xi_1$). It cannot find the the value of $\xi_1$ directly. By using the boundary condition, the ratio of density is zero at the surface of the sphere. Then, for the program, we just let the density ratio to be zero first and make $\xi$ axis to divide the value we find $\xi_1$.

After that, we already know the density distribution on radius. It is easy to find how much mass inside the specific radii in side of one white dwarf star. For the integral of the value of the star sphere, we add each surface on a very small length on radius. By using non-dimensionless part of value which is $\alpha$ and others physics constant value like G and K, we could know the specific value of mass inside.

Finally, we will find the mass distribution and how much inside a specific radii.

**Results.**
Conclusions.

According the graph of density and radii, we could find out that the density will decrease along side of the radii of the white dwarf star. The central part get more mass of the whole star, and less mass distribute on the surface of the star. And also, it is not hard to find that the index value which is n in the function also effect the mass distribution of the star very much. When n goes to large value, the more density in the center rather than
distribute evenly. But smaller value of n will make the mass distribute the mass not that extremely. The density function of radii make the mass graph like what the graph shows. It not simply shows the relationship with mass and radii cube, but have changes on the changing of mass inside of one radius includes.

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And also, we could find the relationship with mass with different white dwarf stars according to the integral of one white dwarf star and the graph of density and radii.

\[ M = -4\pi a^3 \rho_c \xi^2 (d\theta/d\xi)(\xi_1) \]

Integral with the density from radii equals zero to the radius of the star.

Then we could know that \( R \propto (1/m)^{1/3} \) when index n is less than 3.

After determining the initial conditions
Solution, the relationship between density, mass and radius. We can find that white dwarfs have specific properties: the greater the mass, the volume on the contrary, the smaller the density, the greater the density, so that you can understand the structural characteristics of the white dwarf.

The stable structure of the white dwarf is caused by the balance of the outward electron degeneracy pressure and the inward gravitational force.
According the reading material shows me, This shows that the larger the white dwarf, the smaller the radius. As the quality of white dwarfs increases and the radius decreases, electrons being confined to a smaller spatial area, more electrons occupy a state of great momentum and become relativistic. at this time
The pressure formula of the relativistic degenerate electron gas is applied so that the pressure gradient is equal to the gravity and the radius R is about to go. This means It is impossible to achieve equilibrium by properly adjusting the radius of the white dwarf. Then the quality of the white dwarf exists on one Limit, beyond this limit, the electron's degenerative pressure can no longer compete with gravity.
C code I used.

code for calculate density and radii by using RK-4:

```c
#include<stdio.h>
#include<math.h>

float RHS(int ieq, float x, float dfx, float fx, float n)
{
    float ddfx;
    float newdfx;
    if(ieq == 0)
    {
        return -dfx / (x*x+0.00001);
    }
    else if(ieq == 1)
    {
        return x*x * pow(fx,n);
    }
}

int main(void)
{
    float h = 0.008;
    float x = 0.0;
    float k1[2],k2[2],k3[2],k4[2];
    float var[2];
    // var[0] standing for fx, var[1] standing for dfx
    var[0]=1.0;
    var[1]=0.0;
    float n = 1.5;
    int i = 0;
    float xc;
    while(pow(var[0],n)>0.0001)
    {
        printf("%f %f\n", x, pow(var[0],n));
        for(int j=0; j<2; j++)
        {
            k1[j]=var[j]+0.5*h*RHS(j,x,var[1],var[0],n);
        }
        for(int j=0; j<2; j++)
        {
            k2[j]=var[j]+0.5*h*RHS(j,x+0.5*h,k1[1],k1[0],n);
        }
        for(int j=0; j<2; j++)
        {
            k3[j]=var[j]+0.5*h*RHS(j,x+0.5*h,k2[1],k2[0],n);
        }
```
```c
{    
    k3[j]=var[j]+0.5*h*RHS(j,x+0.5*h,k2[1],k2[0],n);
}
for(int j=0; j<2; j++)
{
    k4[j]=var[j]+0.5*h*RHS(j,x+h,k3[1],k3[0],n);
    var[j]=k4[j];
}

x = x + h;
//
}

//
return 0;
}

Python graphing code for calculate density and radii by using RK- 4:
code for calculate mass and radii by using RK- 4:

```c
#include<stdio.h>
#include<math.h>

float RHS(int ieq, float x, float dfx, float fx, float n)
{
    float ddfx;
    float newdfx;
    if(ieq == 0)
    {
        return -dfx / (x*x+0.00001);
    }
    else if(ieq == 1)
    {
        return x*x * pow(fx,n);
    }
}
```
int main(void)
{
    float h = 0.008;
    float x = 0.0;
    float k1[2], k2[2], k3[2], k4[2];
    float var[2]; // var[0] standing for fx, var[1] standing for dfx
    var[0] = 1.0;
    var[1] = 0.0;
    float n = 1.5;
    int i = 0;
    float xc;

    float M = 0.0;
    float K = 4.27;
    float rolhCenter = 1.0 * pow(10, 9);
    float G = 6.674 * pow(10, -11);
    float alpha = sqrt(K * (n + 1) / G * 4 * 3.1416 * pow(rolhCenter, (n - 1) / n));

    while (pow(var[0], n) > 0.0001)
    {
        M = M + (4 * 3.1416 * x * x * pow(var[0], n));
        printf("%e %e \n", x * alpha, M * rolhCenter);
        for (int j = 0; j < 2; j++)
        {
            k1[j] = var[j] + 0.5 * h * RHS(j, x, var[1], var[0], n);
        }
        for (int j = 0; j < 2; j++)
        {
            k2[j] = var[j] + 0.5 * h * RHS(j, x + 0.5 * h, k1[1], k1[0], n);
        }
        for (int j = 0; j < 2; j++)
        {
            k3[j] = var[j] + 0.5 * h * RHS(j, x + 0.5 * h, k2[1], k2[0], n);
        }
        for (int j = 0; j < 2; j++)
        {
            k4[j] = var[j] + 0.5 * h * RHS(j, x + h, k3[1], k3[0], n);
            var[j] = k4[j];
        }
    }
    x = x + h;
    //
}
Python graphing code for calculate mass and radii by using RK-4:

```python
In [1]: import pandas as pd
    import numpy as np
In [2]: import matplotlib.pyplot as plt
In [3]: df = pd.read_csv('mnd.dat', skiprows = 0, header = None, names = ['r', 'M'], delim_whitespace=True)
In [4]: r = np.array(df.r)
In [5]: plt.subplot(111)
    plt.plot(r, df.M, color = 'b', label = 'white dwarf')
    plt.legend(loc = 'best')
    plt.title('Radius vs Mass')
    plt.xlabel('radius')
    plt.ylabel('mass')
    plt.grid(True)
    plt.savefig('figure2')
```

Reference.

An Introduction to the Theory o - Dina Prialnik Chapter 5

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