Mass-Radius Relation of White Dwarfs

PHYS 305 Project
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I. Introduction:

According to Nasa’s website\(^1\), the white dwarf is a star that exhausted all its nuclear fuel and most of its materials of the outer surface had been expelled. However, although the size of a white dwarf is very close to the size of the earth, it is 200,000 times as dense. Nonetheless, the radius of the white dwarf has an inverse relationship with its mass, and that is because throughout the losing of the outer materials, the radius of the star will decrease as expected, and yet, the value of the mass will be increasing relative to the increasing of the density. The white dwarf cools down after billions of years and becomes a massive cold rock wanderers in the space. However, in this paper we are trying to prove this mass-radius relationship for a white dwarf and This will be done by calculating the mass-radius using the analytic values of the two parameters K and Γ of Lane–Emden equation and writing the proper C- code that will gives us the more accurate results. Then, we will compare the results that we get with other measured results from other scientific papers on the observations of white dwarfs.

II. Theory:

The main goal for this section is to derive Gamma and Kapa to use them to determine the mass and the radius of a white dwarf\(^2\).

As we established in the introduction part, the white dwarf gets to a vary cold temperature and theoretically zero temperature. However, the only pressure that preventing a white dwarf from collapsing is the pressure associated with the electrons. First, let’s take the case of an ideal gas of electrons at temperature \(T=0\)˚ where we are ignoring the electrostatic interactions-. Nonetheless, the system’s thermodynamics will be taken in count in a non-relativistic frame with respecting to the fluid. However, we can also use the local frame of Lorentz and set \(n\) to be the number density of baryons (i.e. protons). By setting the total energy density of the system to \(\epsilon\), we will have the energy density per baryon as \(\frac{\epsilon}{n}\).

However, using the first law of thermodynamics we get

\[
\tilde{d}Q = d\left(\frac{\epsilon}{n}\right) + Pd\left(\frac{1}{n}\right) \tag{1}
\]

Where \(\tilde{d}Q\) is the energy gained by the baryon –the dash above it points to the fact that it is not a perfect differential-, \(P\) is the pressure and \(\frac{1}{n}\) is nothing but the volume per baryon.

Moreover, we also know since our investigation subject is a fluid segment, that is \(dQ\) will have the value of \(dQ = Ts\), where \(s\) is entropy.

By substituting the value of \(\tilde{d}Q\) into (1) and solve for the first term on the RHS, we get

\[
d\left(\frac{\epsilon}{n}\right) = -Pd\left(\frac{1}{n}\right) + Ts \tag{2}
\]

Going back to the white dwarf, we will define the concentration of its baryons of the \(i\)th type as

\[
Y_i \equiv \frac{n_i}{n} \tag{3}
\]
Where $n$ is the total baryonic density, putting in mind that we are ignoring the non-conserved cases of the baryonic which happens at very high energies of $10^{15}$ Gev or higher.

After that, we are setting $\epsilon = \epsilon(n,s,Y_i)$ and by using the general expression in (2), we get

$$d \left( \frac{\epsilon}{n} \right) = -P d \left( \frac{1}{n} \right) + T ds + \sum_i \mu_i dY_i \quad (4)$$

where $\mu_i$ is the chemical potential of the $i$ th type and it also represents the energy density change per a type of constant volume and entropy.

The pressure in this case can be expressed as following

$$P \equiv \frac{dE}{dv} = \frac{\partial \left( \frac{\epsilon}{n} \right)}{\partial \left( \frac{1}{n} \right)} = n^2 \frac{\partial \left( \frac{\epsilon}{n} \right)}{\partial (n)} \quad (5)$$

where we can express the temperature as

$$T \equiv \frac{\partial \left( \frac{\epsilon}{n} \right)}{\partial s} \quad (6)$$

We also have the value of $\mu_i$ given by

$$\mu_i \equiv \frac{\partial \left( \frac{\epsilon}{n} \right)}{\partial Y_i} = \frac{\partial \epsilon}{\partial n_i} \quad (7)$$

However, an isolated white dwarf has no heat transferred which means that $dQ = 0$, hence, $d \left( \frac{\epsilon}{n} \right) = 0$, and for this case only and from the second law of thermodynamics in equilibrium states, we have $ds = 0$, and finally, the volume will not be changing. Therefore, by substituting the values above, equation (4) becomes

$$\sum_i \mu_i dY_i = 0 \quad (8)$$

Still, we need to prove the following

$$dQ = d \left( \frac{\epsilon}{n} \right) + P d \left( \frac{1}{n} \right) \leq T ds \quad (9)$$

(second law of thermodynamics)

If we are at an equilibrium state where the temperature $T = 0$, $d \left( \frac{1}{n} \right) = 0$, thus $d \left( \frac{\epsilon}{n} \right) \leq 0$, and now we have the Helmholtz free energy per baryon given by

$$f \equiv \frac{\epsilon}{n} - Ts \quad (10)$$
By keeping $T$ and $\frac{1}{n}$ as constants, the second law of thermodynamics will become

$$df \leq 0 \quad (11)$$

And now, by keeping $P$ and $T$ as constants, we will have the quantity that is called Gibbs free energy per baryon as follow

$$g \equiv \frac{\epsilon + P}{n} - Ts \quad (12)$$

where for this quantity also, $dg \leq 0$

Now, after taking the derivative of equation (12) and using equation (4) to substitute the values, we will have the following equation

$$dg = \left(\frac{1}{n}\right) dP - sdT + \sum_i \mu_i dY_i \quad (13)$$

Now, we can get equation (8) for minimum $g$ and where $P$ and $T$ are constants. However, since all the extensive properties scale within the volume range, we will have

$$\sum_i \mu_i Y_i = g \quad (14)$$

And moving forward from this point, we can introduce another way to describe the density for each type as

$$n = \int \frac{g f d^3p}{h^3} \quad (15)$$

Where $h$ is the Planck constant, $f$ is the distribution function, $g^\dagger$ is a statistical weight, and $d$ is the number of states of a particle with a given value of momentum $p$.

For massive particles, we have $g^\dagger = 2s + 1$, $-s$ is the spin - where $g^\dagger = 2$ is for the photons and $g^\dagger = 1$ for neutrinos.

The function $f$ gives the average occupation number of a cell in phase space as

$$\epsilon = \int \frac{E g f d^3p}{h^3} \quad (16)$$

Where $E$ is the energy

However, for isotropic case we will have

$$p = \int \frac{pv g f d^3p}{3h^3} \quad (17)$$

And the velocity as

$$v = \frac{pc^2}{E} \quad (18)$$
Also, for the ideal gas at an equilibrium state we will have Fermi-Dirac expression as following

\[ f(E) = \frac{1}{e^{\frac{E-E_F}{kT}} \pm 1} \quad (19) \]

Where the plus sign is for the fermions and the minus one is for the bosons. However, the derivation of this won't be mentioned because it is not related to the case that we are investigating.

We have Fermi energy is defined as

\[ E_F \equiv (p_F^2c^2 + m^2c^4)^\frac{1}{2} \quad (20) \]

Where \( p_F \) is the Fermi momentum

By using the condition of \( f \) and applying it into equation (15), it will become as

\[ n = \frac{8\pi p_F^3}{3\hbar^3} \quad (21) \]

And by solving for \( p_F \) we will have

\[ p_F = \left( \frac{3\hbar^3}{8\pi} \right)^\frac{1}{3} \quad (22) \]

By using the value of equation (18) and substitute it back into (17) and also by set the integration of equation (17) from 0 to \( p_F \), it will become

\[ P = \frac{8\pi}{3\hbar^3} \int_{0}^{p_F} \frac{p^4c^2}{(m^2c^4+p^2c^2)^{\frac{3}{2}}} dp \quad (23) \]

For a non relativistic case, \( mc^2 >> pc \), thus, we can remove \( pc \) from the denominator and hence the expression in (23) will become

\[ P \approx \frac{8\pi}{3\hbar^3m} \int_{0}^{p_F} p^4 \ dp \]

\[ = \frac{8\pi}{15\hbar^3m} \left( \frac{3\hbar^3}{8\pi} \right)^{\frac{5}{3}} \quad (24) \]

In contrast, for the ultra relativistic case \( pc >> mc^2 \), therefore, the integration of (23) will become

\[ P \approx \left( \frac{3}{\pi} \right)^{\frac{1}{3}} \frac{hcn^3}{8} \quad (25) \]

We have the mass density \( \rho_0 \) defined as
\[ \rho_0 = \sum_i n_i m_i = 2 n_p m_p + n_e m_e \]  
\[ \approx 2 m_p n_p \]  
(26)

Where \( m_i \) is the mass of the type of matter we are considering. We also assumed that the protons and the neutrons they dominate the mass in a white dwarf and exist in an equal amount. That because in this case the gases that are most likely to exist are Oxygen and Carbon.

\[ \frac{\text{#protons}}{\text{#total nucleons}} \frac{\rho_0}{m_p} = \frac{1}{2} \frac{\rho_0}{m_p} = n_e \]  
(27)

Thus, we can write the pressure of the electrons as

\[ P_e = \frac{\frac{1}{2\pi^2}}{8} \left( \frac{\rho_0}{2 m_p} \right)^\frac{4}{3} \approx K_1 \rho_0^\frac{4}{3} \]  
(28)

which gives us the polytropic equation for the relativistic case, and by plugging the values and solving for \( K_1 \) we get

\[ K_1 = \frac{\left( \frac{3}{8} \right) \frac{1}{2\pi^2} \left( \frac{1}{2 m_p} \right)^\frac{4}{3}}{8} \approx 4.9 \times 10^{14} \text{ CGS units} \]

similarly, the polytropic equation for non-relativistic case will be derive from equation (5) and it will be

\[ P_e = \frac{8\pi}{15h^3 m_e} \left( \frac{3^2 \rho_0 h^3}{8\pi} \right)^\frac{5}{3} = K_2 \rho_0^\frac{5}{3} \]  
(29)

and again, by solving to \( K_2 \) we will get

\[ K_2 = \frac{8\pi}{15h^3 m_e} \left( \frac{3^2 \frac{1}{2 m_p h^3}}{8\pi} \right) \approx 4.1 \times 10^{12} \text{ CGS} \]

Thus, we have

\[ \Gamma = \begin{cases} 
\frac{4}{3} & \text{for extremely relativistic case} \\
\frac{5}{3} & \text{for non relativistic case}
\end{cases} \]
III. Computational work:

With a solid understanding of the theory of what governs white dwarfs, we can now begin to analyze the internal interactions of these stellar bodies. The two main forces at play are gravity and electron degenerate pressure. However, to determine the size and mass of a white dwarf, we must start within the star at a given core density \( \rho_c \) and core radius \( r_c \). Only then can we use equations (1) and (2) from project the sheet, and integrate incrementally until the pressure reaches 0, which will indicate that we have reached the surface of the white dwarf. We used the numerical methods of Trapezoidal integration to calculate and update the mass of the star and the Runge Kutta 4th order to differentiate the Pressure equation.

Derivation of dimensionless equations:

\[
\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}, \quad \text{and} \quad \frac{dM}{dr} = 4\pi r^2 \rho
\]

Let \( M(r) = M_0 M(r) \), \( \rho = \rho_0 \bar{\rho} \), \( P = P_0 \bar{P} \), and \( r = r_0 \bar{r} \)

Where \( P_0 \) is the pressure of the core of the sun, \( r_0 \) is the solar radius and \( M_0 \) is the solar mass we also have the density of the water as

\[\rho_0 = \text{water density} = \frac{1g}{cm^2}\]

Thus, we get

\[
\frac{d\bar{P}}{d\bar{r}} = -\frac{GM_0 \rho_0 M(r)\bar{\rho}}{r_0 P_0 \bar{r}^2}, \quad \frac{d\bar{M}}{d\bar{r}} = 4\pi r_0^3 \rho_0 \bar{r}^2 \bar{\rho}, \quad \alpha = -\frac{GM_0 \rho_0}{r_0 P_0}, \beta = \frac{4\pi r_0^3 \rho_0}{M_0}
\]

Results:

Radius vs Mass

[Graph image]
The graph above is the result for the relativistic case, $\Gamma = \frac{4}{3}$, and we have the mass increasing along with the radius. However, at some specific mass the radius continues to increase even though the mass is reaching a constant value, and as a result, the mass becomes independent from the radius.

![Radius vs Mass](image)

Similarly, for the non-realistic case, $\Gamma = \frac{5}{3}$, we have the mass increasing with the radius. However, in this case, at some specific radius, the radius becomes constant and independent from the mass.

**IV. Error analysis:**
Our graph does not satisfy the expected results (as shown in Graph.1), and that because we have the radius increasing along with the mass, and that is the opposite of the mass-radius relation of a white dwarf. However, there are possible errors that caused this problem; first, there is a possible error in the calculation of the constants of the dimensionless equations. Second, the fact that we chose a step size that did not work properly and cause a massive error that effected the results.

![Error vs Step Size](image)

These two graphs are showing the error for each step size for our results.
V. Comparison to the observations:

Graph 1. Shows an observational support for a white dwarf showing the mass-radius relationship via Hipparcos. The lines that are labeled as He, C, Mg, and Fe representing the mass-radius relationship of Hamada & Salpeter (1961) at zero-temperature. Where the dashed lines are representing the mass-radius relationship of Wood (1995) at 15 000 K and 8000 K temperatures, and Comparing this to our results we can see that this is the opposite of what happened in our results.

VI. Conclusion:

In this paper, the main focus was to find the mass-radius relation of a white dwarf and compare our theoretical results by the observational ones. Nonetheless, we used derivation to find the analytical values for the two parameters K and Γ of the Lane–Emden equation which we eventually used their values to calculate our results. Where we used two different Γ values for the cases of relativistic and non-relativistic. However, the results that we got were not as we expected and we listed some possible errors that caused this inconsistency. For relativistic case at some specific mass, the mass became constant and the radius kept increasing. However, for the non-relativistic case, the radius become constant and the mass kept increasing. In conclusion, although our results were not as expected, the paper served its purpose and the mass-radius relation was calculated and proved using both theoretical, and C-programing methods.
References


#include <stdio.h>
#include <math.h>

/*Description of Project*/

int main(void)
{
    // Initialize Variables
    FILE *out_file;
    out_file = fopen("Project.dat","w");
    FILE *stepOut = fopen("stepOut.dat","w");
    if (out_file == NULL)
    {
        printf("Error! Could not open file!
\n";
        exit(-1);
    }
    float pi = 3.14159;
    double G = 6.67e-11;
    float K = 3124819.7;
    float r_c, r, del_r, r1, r2;
    float rho_c, rho;
    double alpha = 0.1;
    double beta = 4.7;
    float eta = 1;
    float P_c, P;
    float M_c, M;
    float gamma = 4.0/3.0;
    float newM, newP, Pf;
    float M_r;
    int count, i;
    double epsilon;

    // Determine Initial Conditions

    // for (rho_c=.01; rho_c<=.5; rho_c+=.01)
    // {
        r_c = 0.0001;
        del_r = .000001;
        rho_c = 1;
        P_c = eta*pow(rho_c,gamma);
        M_c = (4.0/3.0)*pi*alpha*pow(r_c,3)*rho_c;

        M = M_c;
        P = P_c;
        rho = rho_c;

        fprintf(out_file,"%d\t%f\t%f\t%f\n", count, r_c, M, P,M/r_c);
        count = 1;
    
    while (P > 0) // run until we reach the outside of the star
    {
        // Increment r
        r1 = r_c + del_r*(count-1);
        r2 = r_c + del_r*count; // new r
        printf("%d\t%f\t%f\t%f\n",count,M,P,rho,r2);
        // Update Mass using Trapezoidal Integration
        newM = M + 2.0*pi*alpha*(pow(r1,2)+pow(r2,2))*rho*del_r;
    }
}