ON THE EXPONENTIAL FUNCTION
AND PÓLYA'S PROOF (1)

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Let \( a_1, \ldots, a_n \) be positive numbers. Denote by
\[
A = \frac{1}{n} \sum_{i=1}^{n} a_i
\]
and
\[
G = \left[ \prod_{i=1}^{n} a_i \right]^{1/n}.
\]
Pólya [1, pp. 108] has given an elementary proof that
\( A \geq G \) as follows
\[
1 = e^0 = \exp \left[ \sum_{i=1}^{n} \left( \frac{a_i}{A} - 1 \right) \right] \geq \prod_{i=1}^{n} \frac{a_i}{A} = \frac{G^n}{A^n}.
\]

Wetzel [2] pointed out the two properties essential for the proof characterize the exponential function namely

(1) \( f(x + y) \geq f(x) f(y) \)

(2) \( f(x) \geq 1 + x \).

THEOREM (Wetzel). Let \( f \) be defined on an interval containing the origin and such that \( f \) satisfies (1) and (2) on \( I \) then \( f(x) = e^x \).

If we consider functions of several variables then the analogues of (1) and (2) are

(3) \( f(x + y) \geq f(x) f(y) \)

(4) \( f(x) \geq \prod_{i=1}^{n} (1 + x_i) \)

(1) Received April, 1974.
where \( x = (x_1, \cdots, x_n), \; y = (y_1, \cdots, y_n), \; x + y = (x_1 + y_1, \cdots, x_n + y_n) \). Unfortunately these are not sufficient for Pólya's proof and we replace (3) by

\[
(5) \quad f(x) = 1 \quad \text{for all} \quad x = (x_1, \cdots, x_n)
\]

with \( \sum_{i=1}^{n} x_i = 0 \).

From (5), with \( x_i = \frac{a_i}{A} - 1 \) we have \( 1 = f(x) \), then from (4) we have \( 1 \geq \prod_{i=1}^{n} \left( 1 + \frac{a_i}{A} - 1 \right) = \frac{G^n}{A^n} \). It seems natural to ask whether (4) and (5) characterize the \( n \)-variable exponential function \( \exp \left( \sum_{i=1}^{n} x_i \right) \). The answer is no as is shown by the following lemma.

**Lemma.** Let \( -1 < x_i < 1 \) for \( i = 1, 2, \cdots, n \) and

\[
1 - \left( \sum_{i=1}^{n-1} x_i \right) > 0
\]

then

\[
(4) \quad f(x) \geq \prod_{i=1}^{n} (1 + x_i)
\]

\[
(5) \quad f(x) = 1 \quad \text{for} \quad x_1 + \cdots + x_n = 0
\]

\[x = (x_1, \cdots, x_n)\]

where

\[
f(x) = \frac{1 + x_n}{1 - \sum_{i=1}^{n-1} x_i} \neq \exp \left( \sum_{i=1}^{n} x_i \right).
\]

**Proof.** We proceed by induction. Consider \( n = 2 \), since

\(-1 < x_1 < 1, \; 1 > 1 - x_1^2 > 0 \) and \( 1 + x_2 > 0 \) hence \( \frac{1}{1 - x_1} > 1 + x_1 \)
and \( \frac{1+x_2}{1-x_1} \geq (1+x_1)(1+x_2) \). If \( x_1 + x_2 = 0 \) then \( x_1 = -x_2 \) and \( 1-x_1 = 1+x_2 \) hence \( \frac{1+x_2}{1+x_2} = 1 \). Suppose now that
\[
\frac{1+x_k}{1-\sum_{i=1}^{k-1} x_i} \geq \prod_{i=1}^{k} (1+x_i) \quad \text{or equivalently} \quad \frac{1}{1-\sum_{i=1}^{k-1} x_i} \geq \prod_{i=1}^{k-1} (1+x_i)
\]
for all \( k \leq n \), all \(-1<x_i<1, i=1,2,\ldots,k \) and \( 1-\left(\sum_{i=1}^{k-1} x_i\right)>0 \).

Rewrite
\[
\frac{1}{1-\sum_{i=1}^{k} x_i} = \left(\frac{1}{1-x_k}\right) \left[1-\sum_{i=1}^{k-1} \frac{x_i}{1-x_k}\right].
\]

Since
\[
1-\sum_{i=1}^{k-1} \frac{x_i}{1-x_k} > 0, \quad \text{i.e.} \quad 1-\sum_{i=1}^{k} x_i > 0
\]
by the induction hypothesis
\[
\frac{1}{1-\sum_{i=1}^{k} x_i} \geq \frac{1}{1-x_k} \cdot \prod_{i=1}^{k-1} \left[1+\frac{x_i}{1-x_k}\right].
\]

But \( \frac{x_i}{1-x_k} \geq 1+x_i \) and \( \frac{1}{1-x_k} \geq 1+x_k \) hence
\[
\frac{1}{1-\sum_{i=1}^{k} x_i} \geq \prod_{i=1}^{k} (1+x_i) \quad \text{or} \quad \frac{1+x_{k+1}}{1-\sum_{i=1}^{k} x_i} \geq \prod_{i=1}^{k+1} (1+x_i).
\]

Clearly (5) is satisfied by \( f \) for \( n \geq 2 \).

We should note at this point that even (4) and (5) are stronger properties that are necessary for Pólya's proof. All that is really needed is the inequality
\[
1 \geq \prod_{i=1}^{n} (1+x_i)
\]
if \( x_1 + \ldots + x_n = 0 \). This of course is a simple consequence of

\[
1 = e^0 = \exp \sum_{i=1}^{n} x_i \geq \prod_{i=1}^{n} \exp x_i
\]

\[
\geq \prod_{i=1}^{n} (1 + x_i).
\]

However we can also give an elementary proof without using the exponential function.

**Lemma.** Let \( x_1, \ldots, x_n \) be real numbers such that \( x_1 + \ldots + x_n = 0 \) then \( \sum_{i=1}^{n} (1 + x_i) \geq 1 \).

**Proof.** Since \( x_1 + \ldots + x_n = 0 \) if not all \( x_i \)'s are zero, there is one such that \( x_i < 0 \). Without loss of generality assume \( x_n < 0 \). Consider \( n = 2 \) then \( (1 + x_1)(1 + x_2) = (1 - x_1)(1 + x_1) = 1 - x_1^2 \leq 1 \). Suppose then that for all \( k \leq n - 1 \) and \( x_1 + \ldots + x_k = 0 \), \( 1 \geq \prod_{i=1}^{k} (1 + x_i) \). Then \( 1 \geq \left( \prod_{i=1}^{n-2} (1 + x_i) \right) (1 + x_{n-1} + x_n) \) but \( 1 + x_{n-1} + x_n = (1 + x_n) \left( 1 + \frac{x_{n-1}}{1 + x_n} \right) \). Since we also assume without loss of generality that \( -1 < x_n \), we have \( \frac{x_{n-1}}{1 + x_n} > x_{n-1} \), hence \( 1 + x_{n-1} + x_n \geq (1 + x_n)(1 + x_{n-1}) \) and \( 1 \geq \prod_{i=1}^{n} (1 + x_i) \).

Hence we now could prove that \( A \geq G \) without using the properties of the exponential function.

In conclusion we return to the characterization of the exponential function.

**Theorem.** Let \( f \) be defined on an open rectangle \( I \) in \( \mathbb{R}^n \) such that the origin is in \( I \). Suppose further that

\[
f(0) = 1 \quad 0 = (0, \ldots, 0)
\]

\[
f(x + \hat{h}_i) \geq f(x)(1 + h_i)
\]
where \( \hat{h}_i = (0, \ldots, h_i, \ldots, 0) \) for \( i = 1, \ldots, n \) and \( x, \hat{h}_i, x + \hat{h}_i \) in 1. Then \( f(x) = \exp \sum_{i=1}^{n} x_i \).

**Proof.** We consider first the case where \( h_i > 0 \). From (7) we have

\[
f(x) \geq f(x + h_i)(1 - h_i)
\]

or

\[
f(x) \frac{1}{1 - h_i} \geq \frac{f(x + h_i) - f(x)}{h_i}
\]

and also \( \frac{f(x + h_i) - f(x)}{h_i} \geq f(x) \). If \( h_i < 0 \) both inequalities reverse. In either case we conclude that \( \frac{f(x)}{x_i} = f(x) \), hence \( f(x) = C \exp \sum_{i=1}^{n} x_i \). From (6) \( C = 1 \).

Note that (6) and (7) are «weaker» than (3) and (4).

**References**
