Nonseparable space-time covariance models: some parametric families

De Iaco S.[1] Myers D. E.[2] Posa D.[3,4]

[1] Università di Chieti, Pescara - ITALY; e-mail: sdeiaco@tiscalinet.it
[2] University of Arizona, Tucson AZ - USA; myers@math.arizona.edu
[3] Università di Lecce, Lecce - ITALY; posa@economia.unile.it
[4] IRMA-CNR, Bari - ITALY; irmadp08@area.ba.cnr.it

Abstract

As a consequence of one of the stability properties of the covariance function in $\Re^n$, new parametric families of spatio-temporal covariance functions have been obtained starting from the product and the product-sum covariance models (De Cesare et al., 2000). Moreover, some non-integrable space-time covariance models have been generated and these parametric families cannot be obtained, in general, from Cressie-Huang representation (Cressie and Huang, 1999). Practical problems have been addressed in order to estimate and model the covariance or variogram of spatial-temporal random fields from data and a case study has been presented.

Keywords: Product-Sum Models, Integrated models, Separability, Admissibility, Model Fitting

1 Introduction

The importance of simultaneously studying spatial and temporal aspects of processes is well-known. However the structural analysis of such processes is more difficult than for spatial processes or for temporal processes. Estimating and modeling the correlation of a space-time process are principal objectives in geostatistical analysis. The extension of geostatistical techniques to the space-time domain, in order to provide tools for joint analysis of the space and time components, is only apparent because of the certain characteristics of

spatial-temporal phenomena (Rouhani and Myers, 1990). A recent review of geostatistical space-time models was given by Kyriakidis and Journel (1999).
In order to estimate the correlation of a space-time process, the main questions are:

- Is it useful and does it make sense to define a spatio-temporal metric, such as

$$d(\mathbf{u_1}, \mathbf{u_2}) = (a(x_1 - x_2)^2 + b(y_1 - y_2)^2 + c(t_1 - t_2)^2)^{1/2},$$

  with $\mathbf{u_1} = (x_1, y_1, t_1), \mathbf{u_2} = (x_2, y_2, t_2)$ where (x_1, y_1), $(x_2, y_2) \in D \subseteq \Re^2$, and t_1, $t_2 \in T \subseteq \Re$, where D and T are the spatial and temporal domains, respectively. In general the units for space and time will be disparate, e.g., meters and hours.

- How to choose a space-time covariance or variogram model and how to choose parameters to ensure that the best fit to data is achieved?

One of the objectives of this paper is to furnish answers to the above questions; moreover, starting from the product-sum covariance model (De Cesare et al., 2000) and by using one of the fundamental theorems (well known as stability properties) regarding covariance functions in $\Re^n$ (Matern, 1960, p. 10; Chilès and Delfiner, 1999, p. 60), some non-separable parametric families of space-time covariance functions have been derived. It is important to point out that this new class of models cannot be obtained, in general, from Cressie-Huang representation (Cressie and Huang, 1999).
In section 2 there is a review of basic space-time concepts and main results used to obtain valid space-time covariance functions Examples of a wide range of valid models are given in section 4.1. Section 5 is concerned with some practical aspects, linked with the problems of fitting space-time covariance functions or variogram models to data. A case study using the preceding results is given in Section 6.

2 Review of basic space-time concepts

Let $Z(\mathbf{s}, t)$ be a random variable at the location $\mathbf{s}$, in space, and t, in time, and let

$$Z = \{Z(\mathbf{s}, t), (\mathbf{s}, t) \in D \times T, D \subset \Re^n, T \subset \Re_+\},$$

be a second order stationary spatial-temporal random field, with expected value, covariance and variogram, respectively:

$$E(Z(\mathbf{s}, t)) = 0, \tag{1}$$

$$C_{s,t}(\mathbf{h}_s, h_t) = Cov(Z(\mathbf{s} + \mathbf{h}_s, t + h_t), Z(\mathbf{s}, t)) =$$

$$= E(Z(\mathbf{s}+\mathbf{h}_s, t+h_t) \cdot Z(\mathbf{s}, t)), \tag{2}$$

$$\gamma_{s,t}(\mathbf{h}_s, h_t) = \frac{Var(Z(\mathbf{s}+\mathbf{h}_s, t+h_t) - Z(\mathbf{s}, t))}{2} =$$
$$= \frac{E(Z(\mathbf{s}+\mathbf{h}_s, t+h_t) - Z(\mathbf{s}, t))^2}{2}, \tag{3}$$

where $(\mathbf{s}, \mathbf{s}+\mathbf{h}_s) \in D^2$ and $(t, t+h_t) \in T^2$. The above probabilistic framework is often a preliminary remark for the space-time analysis. A slightly different approach is required for non-stationary spatial-temporal phenomena, but this aspect of the problem is not treated in this paper.

Structural analysis begins with estimating the space-time covariance or variogram. Given the set A of data locations in space-time

$$A = ((\mathbf{s}_i, t_j), i = 1, 2 \ldots, n_s, j = 1, 2, \ldots, n_t),$$

the sample space-time covariance and variogram, $\widehat{C}_{s,t}$ and $\widehat{\gamma}_{s,t}$ respectively, can be estimated as follows:

$$\widehat{C}_{s,t}(\mathbf{r}_s, r_t) = \frac{1}{|L(\mathbf{r}_s, r_t)|} \sum_{L(\mathbf{r}_s, r_t)} [Z(\mathbf{s}+\mathbf{h}_s, t+h_t) \cdot Z(\mathbf{s}, t)], \tag{4}$$

$$\widehat{\gamma}_{s,t}(\mathbf{r}_s, r_t) = \frac{1}{2|L(\mathbf{r}_s, r_t)|} \sum_{L(\mathbf{r}_s, r_t)} [Z(\mathbf{s}+\mathbf{h}_s, t+h_t) - Z(\mathbf{s}, t)]^2, \tag{5}$$

where $|L(\mathbf{r}_s, r_t)|$ is the cardinality of the set

$$L(\mathbf{r}_s, r_t) = \left\{(\mathbf{s}+\mathbf{h}_s, t+h_t) \in A, (\mathbf{s}, t) \in A : \mathbf{h}_s \in Tol(\mathbf{r}_s) \text{ and } h_t \in Tol(r_t)\right\},$$

and $Tol(\mathbf{r}_s), Tol(r_t)$ are, respectively, specified tolerance regions around $\mathbf{r}_s$ and r_t.
It is important to note that no space-time metric is required. In fact, the pairs of points separated by $(\mathbf{h}_s, h_t)$ are determined by computing, separately, the spatial and temporal distances; thus, the pairs of realizations, $z(\mathbf{s}, t)$ and $z(\mathbf{s}+\mathbf{h}_s, t+h_t)$, correspond to points that are simultaneously separated by $\mathbf{h}_s$, in space domain, and h_t, in time domain.

The second step of the structural analysis is to fit a theoretical valid model to the sample space-time covariance or variogram.
A covariance or variogram model is *admissible* (or *valid*) if and only if the non-negative definiteness conditions are satisfied. A space-time covariance function, $C_{s,t}$ in (2), is admissible if and only if it is non-negative definite, namely:

$$\sum_{i=1}^{n_s} \sum_{j=1}^{n_t} \sum_{k=1}^{n_s} \sum_{l=1}^{n_t} a_{ij} a_{kl} C_{s,t}(\mathbf{s}_i - \mathbf{s}_k, t_j - t_l) \geq 0, \tag{6}$$

for any $(\mathbf{s}_i, t_j) \in D \times T$, any $a_{ij} \in \Re, i = 1, \dots, n_s, j = 1, \dots, n_t$, and any positive integers n_s and n_t.
A space-time variogram, $\gamma_{s,t}$ in (3), is admissible if and only if $-\gamma_{s,t}$ is conditional non-negative definite, i.e.,

$$-\sum_{i=1}^{n_s}\sum_{j=1}^{n_t}\sum_{k=1}^{n_s}\sum_{l=1}^{n_t} a_{ij}a_{kl}\gamma_{s,t}(\mathbf{s}_i - \mathbf{s}_k, t_j - t_l) \geq 0, \tag{7}$$

with the constraint:

$$\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} a_{ij} = 0,$$

for any $(\mathbf{s}_i, t_j) \in D \times T$, $i = 1, \dots, n_s$, $j = 1, \dots, n_t$, and any positive integers n_s and n_t. It is easy to show that the above conditions ensure the variance of any linear combination of spatial-temporal random variables:

$$Y = \sum_{i=1}^{n_s}\sum_{j=1}^{n_t} a_{ij} Z(\mathbf{s}_i, t_j)$$

is non-negative.

As in the purely spatial context, using (6) or (7) to determine whether a function is valid as a covariance or variogram is quite difficult, hence it is more practical to look for the best model among the parametric families whose members are known to be non-negative or better strictly positive in order to assure a unique solution in the kriging system.
Moreover, the users can utilize some useful results, herein see next section 2.1, to derive classes of valid space-time covariance or variogram models.

2.1 Theoretical aspects

Covariances or variograms in $\Re^n$ can be derived from other valid functions in different ways. The main results used to obtain valid covariance functions follow.

1. From the *convexity property* of the family of covariances, if $C_1(\mathbf{h})$ and $C_2(\mathbf{h})$ are covariances in $\Re^n$ and $b > 0$, then both $C_1(\mathbf{h}) + C_2(\mathbf{h})$ and $b \cdot C_1(\mathbf{h})$ are covariances in $\Re^n$. The same results hold for the family of variograms.

2. From the 1^{st} *stability property* (Chilès and Delfiner, 1999, p. 60), if $C_1(\mathbf{h})$ and $C_2(\mathbf{h})$ are covariances in $\Re^n$, then their product, $C_1(\mathbf{h}) \cdot C_2(\mathbf{h})$, is still a covariance in $\Re^n$ (Matern, 1960, p. 10).

3. From the 2^{nd} *stability property* (Chilès and Delfiner, 1999, p.60), if $\mu(a)$ is a positive measure in $U \subseteq \Re$ (see Appendix 1) and C($\mathbf{x}$,$\mathbf{y}$;a) is a covariance function in $\Re^n$ for each $a \in V \subseteq U$, which is integrable over the subset V of U for every pair $(\mathbf{x}, \mathbf{y})$, then $C(\mathbf{x}, \mathbf{y})$, defined as follows:

$$C(\mathbf{x}, \mathbf{y}) = \int_V C(\mathbf{x}, \mathbf{y}; a) d\mu(a),$$

is a covariance in $\Re^n$ (Matern, 1960, p. 10).

4. From the generalization of the turning band method to n-dimensional space (Matheron, 1973, p. 461) and the general principle of the method appeared in Matern (1960, p. 16), it is proved that if $C_1(h)$ is the covariance of a random function Z_1 definited in $\Re$, then an isotropic covariance function in $\Re^n$ can be obtained as follows:

$$C_n(\|\mathbf{h}\|) = \frac{2}{\sqrt{\pi}} \Gamma\Big(\frac{n}{2}\Big) \Big[\Gamma\Big(\frac{n-1}{2}\Big)\Big] \int_0^1 C_1(u\|\mathbf{h}\|)(1-u^2)^{\frac{(n-3)}{2}} du,$$

where $\Gamma(\cdot)$ is the gamma function (or Euler's integral).

5. From *Bochner's theorem*, a continuous real function $C(\mathbf{h})$, defined in $\Re^n$, is a covariance if and only if it is the Fourier transform of a positive bounded symmetric measure $F(d\mathbf{u})$, the *spectral measure* (where $F(d\mathbf{u}) = f(\mathbf{u})d\mathbf{u}$ if $C(\mathbf{h})$ falls off fast enough to ensure that $C(\mathbf{h})$ is integrable in $\Re^n$, that is $\int |C(\mathbf{h})| d\mathbf{h} < \infty$). Thus, it is possible to generate covariance (or bounded variogram) functions by choosing proper spectral functions.

6. Difference or differential equations (Christakos, 1984, p. 253; Christakos, 1992, p. 138) can lead to several classes of isotropic models. For example, the following difference equation in $\Re^2$, useful for soil patterns:

$$Z(x_i, x_j) = b[Z(x_{i+1}, x_j) + Z(x_{i-1}, x_j) + Z(x_i, x_{j+1}) + Z(x_i, x_{j-1})] + a(x_i, x_j),$$

where $a(x_i, x_j)$ is usually white noise and b is a deterministic coefficient, leads to variograms of the form:

$$\gamma(r) = A\Big(1 - \frac{r}{a} K_1(r/a)\Big),$$

where K_1 is the modified Bessel function of the second kind.

7. Representing the random field in $\Re^n$ as a combination of independent components in the separate domains, one can utilize the following separable models (Chilès and Delfiner, 1999, p. 95):

a) the *factorized or separable covariance*, such as:

$$C(\mathbf{h}) = \prod_{i=1}^{n} C_i(h_i) \tag{8}$$

if $Z(\mathbf{u}) = \prod_{i=1}^{n} Z_i(u_i)$ and Z_i are independent random fields in $\Re$;

b) the *nested structure variogram*, such as:

$$\gamma(\mathbf{h}) = \sum_{i=1}^{n} \gamma_i(h_i) \tag{9}$$

if $Z(\mathbf{u}) = \sum_{i=1}^{n} Z_i(u_i)$ and Z_i are independent random fields in $\Re$ (see Appendix 2).

In both cases h_i are the components of the vector $\mathbf{h}$ and $C_i(h_i)$ and $\gamma_i(h_i)$ are, respectively, covariances and variograms in $\Re, i = 1, \ldots, n$.

3 Some parametric families of space-time stationary covariances

As already pointed out, most space-time covariance or variogram models in literature, have been derived by utilizing the above results.

Under the convenient assumption of treating space and time separately, the factorized covariance (8) and the nested variogram (9) represent one of the first attempts to generate parametric families of space-time covariances and variograms. The well known product model (Rodriguez-Iturbe and Mejia, 1974; Posa, 1993; De Cesare et al., 1997) and the linear model (Bilonick, 1987; Rouhani and Hall, 1989) belong to this category. Note that the linear model will in general not be strictly positive definite but only semi-definite, (Myers and Journel, 1990).

3.1 The product-sum covariance model

An extension of the two simple models (product model and linear model)is given by the class of valid product-sum covariance models, introduced in De Cesare et al., 2000:

$$C_{s,t}(\mathbf{h}_s, h_t; \theta) = k_1 C_s(\mathbf{h}_s; \theta_1) C_t(h_t; \theta_2) + k_2 C_s(\mathbf{h}_s; \theta_1) + k_3 C_t(h_t; \theta_2), \tag{10}$$

where $\theta = (\theta_1, \theta_2, k_1, k_2, k_3)$, C_t and C_s are valid temporal and spatial covariance models, respectively. Note that the above nonseparable parametric family of space-time covariance functions has been derived by applying the convexity property of the family of covariances in $\Re^n$ (see section 2.1).
Thus, any linear combination of the previous product and sum covariance models with coefficients $k_1 > 0, k_2 \geq 0$ and $k_3 \geq 0$ is positive definite.

3.2 Cressie-Huang models

Cressie and Huang (1999) have recently shown how to construct Recently some nonseparable classes of integrable space-time covariances. Using the representation :

$$C_{s,t}(\mathbf{h}_s, h_t) = \int_{\Re^n} e^{i\mathbf{h}_s'\omega} \rho(\omega; h_t) k(\omega) d\omega,$$

where $\rho(\omega; \cdot)$ is a continuous autocorrelation function for each $\omega \in \Re^n$,

$$\int_{\Re_+} \rho(\omega; h_t) dh_t < \infty,$$

$$k(\omega) > 0 \text{ and } \int_{\Re^n} k(\omega) d\omega < \infty.$$

These spatial-temporal covariances are generated by using Bochner's theorem (see section 2.1) and by assuming a proper spectral density.

4 New parametric families of space-time covariance models

One objective of this paper is to apply a new approach for generating nonseparable parametric families of space-time covariance models.

Theorem 1 *Let $\mu(a)$ be a positive measure over $U \subseteq \Re$, let $C_s(\mathbf{h}_s; a)$ and $C_t(h_t; a)$ be covariances, respectively in $D \subset \Re^n$ and $T \subset \Re_+$, for each $a \in V \subseteq U$.*
a) If $C_s(\mathbf{h}_s; a) \cdot C_t(h_t; a)$ is integrable with respect to the measure μ over V for each $\mathbf{h}_s$ and h_t, given $k > 0$, then:

$$C_{s,t}(\mathbf{h}_s, h_t) = \int_V k C_s(\mathbf{h}_s; a) C_t(h_t; a) d\mu(a) \tag{11}$$

is a covariance in $D \times T$.

b) Likewise, if $k_1 C_s(\mathbf{h}_s; a) C_t(h_t; a) + k_2 C_s(\mathbf{h}_s; a) + k_3 C_t(h_t; a)$ *is integrable with respect to the measure* μ *over* V *for each* $\mathbf{h}_s$ *and* h_t*, given* $k_1 > 0$, $k_2 \geq 0$ *and,* $k_3 \geq 0$*, then:*

$$C_{s,t}(\mathbf{h}_s, h_t) = \int_V [k_1 C_s(\mathbf{h}_s; a) C_t(h_t; a) + k_2 C_s(\mathbf{h}_s; a) + k_3 C_t(h_t; a)] d\mu(a) \qquad (12)$$

is a covariance in $D \times T$*.*

This result follows from the 2^{nd} stability property (see section 2.1) and the previous models, such as the product and product-sum covariance models.
Since the product and the product-sum covariance models can be written in terms of the variograms (De Cesare et al. 1997; De Cesare et al., 2000), from theorem 1 it follows that:

$$\gamma_{s,t}(\mathbf{h}_s, h_t) =$$

$$= \int_V k[C_t(0; a)\gamma_s(\mathbf{h}_s; a) + C_s(\mathbf{0}; a)\gamma_t(h_t; a) - \gamma_s(\mathbf{h}_s; a)\gamma_t(h_t; a)] d\mu(a) \qquad (13)$$

and

$$\gamma_{s,t}(\mathbf{h}_s, h_t) =$$

$$= \int_V [(k_2 + k_1 C_t(0; a))\gamma_s(\mathbf{h}_s; a) + (k_3 + k_1 C_s(\mathbf{0}; a))\gamma_t(h_t; a) - k_1 \gamma_s(\mathbf{h}_s; a)\gamma_t(h_t; a)] d\mu(a) \quad (14)$$

where $\gamma_s(\mathbf{h}_s; a)$ and $\gamma_t(h_t; a)$ are valid spatial and temporal variogram models for each choice of $a \in V$, while $C_s(\mathbf{0}; a)$ and $C_t(0; a)$ are the corresponding sill values.

Remarks.

- In general product-sum covariance models are not integrable over $\mathbf{h}_s$ and h_t, are not separable and do not correspond to the use of a space-time metric. Models obtained by an integrated product-sum representation will have the same characteristics.

- Although the product covariance models are separable and integrable, the integrated product representation (11) can also produce non-separable and non-integrable models, as it will be shown in the following examples. Hence this new class of models cannot be obtained, in general, from Cressie-Huang's representation (Cressie and Huang, 1999).

- Since the complex exponential can be written as:

  $$e^{i\mathbf{h}'\omega} = cos(\mathbf{h}'\omega) + isin(\mathbf{h}'\omega),$$

  if $\rho(\omega; h_t)k(\omega)$ is symmetric about the origin in $\Re^n$, then Cressie-Huang's representation can be viewed as a special case of (11). Hence,

  $$C_{s,t}(\mathbf{h}_s, h_t) = \int_{\Re^n} e^{i\mathbf{h}_s'\omega} \rho(\omega; h_t) k(\omega) d\omega = \int_{\Re^n_+} C_s(\mathbf{h}_s; \omega) C_t(h_t; \omega) k(\omega) d\omega$$

where $k(\omega)$ is defined, positive and integrable over $\Re_+^n = \Re_+ \times \ldots \times \Re_+$ n times, $C_s(\mathbf{h}_s;\omega)$ is only positive semi-definite spatial covariance function for each $\omega \in \Re_+^n$ and $C_t(h_t;\omega)$ is a temporal covariance. In this special case only, the non-separable Cressie-Huang covariance models could be re-written in terms of variograms as in (13) and take advantage that variograms at zero lags are zero. Note that all the covariance models obtained by Cressie and Huang (1999) satisfy the symmetry property. Moreover, from Cressie-Huang space-time covariance models, $\gamma_{s,t}(\mathbf{h}_s, h_t)$, $\gamma_{s,t}(\mathbf{h}_s, 0)$ and $\gamma_{s,t}(\mathbf{0}, h_t)$ have the same sill values.

- By using non-separable Cressie-Huang models, one could take into account of separate space and time components only by adding them to the model considered (Cressie and Huang, 1999) , while in the above product-sum and the integrated product-sum models, separate spatial and temporal structures are a part of the models by construction.

Obviously there are practical problems in choosing among the parametric families of variograms, that can be generated from (13) and (14), one that is closest to the empirical space-time variogram. These problems are considered in section 5, while in the following section, Theorem 1 is used to generate examples of parametric families of space-time covariance functions.

4.1 Some examples

By applying Theorem 1, a wide class of parametric families of space-time covariance models are obtained. Suppose that the measure μ is generated by an absolutely continuous function Φ, then there exists a function ϕ such that $d\Phi(a) = \phi(a)da$ almost everywhere. The following examples, based on particular choices of isotropic covariances and functions ϕ, show that one can obtain other families by using the same hypothesis and criteria. The integrals considered in the following examples are easily evaluated (Gradshteyn and Ryzhik, 1965).

Example 1. Given the following functions:

$$C_s(\mathbf{h}_s; a, b, \alpha) = e^{-\frac{a||\mathbf{h}_s||^\alpha}{b}}, \quad 1 \leq \alpha \leq 2, \quad a > 0, \quad b > 0,$$

$$C_t(h_t; a, c, \delta) = e^{-\frac{a h_t^\delta}{c}}, \quad 1 \leq \delta \leq 2, \quad c > 0,$$

$$\phi(a, n, \beta) = \frac{\beta^{n+1}}{\Gamma(n+1)}\, a^n e^{-\beta a}, \quad n \geq 0, \quad \beta > 0,$$

since $C_s(\mathbf{h}_s; a, b, \alpha)$ and $C_t(h_t; a, c, \delta)$ are, respectively, valid spatial and temporal covariance models for each choice of a over the interval $V = [0; +\infty[$, the integrability conditions of

Theorem 1 are satisfied, two new classes of nonseparable space-time covariances can be obtained:

$$C_{s,t}(\mathbf{h}_s, h_t; \theta_1) = \int_V k e^{-\frac{a||\mathbf{h}_s||^\alpha}{b}} \cdot e^{-\frac{a h_t^\delta}{c}} \cdot \frac{\beta^{n+1}}{\Gamma(n+1)} \, a^n e^{-\beta a} da =$$

$$= \frac{k\beta^{n+1}}{\Gamma(n+1)} \int_V a^n e^{-a(\frac{||\mathbf{h}_s||^\alpha}{b} + \frac{\mathbf{h}_t^\delta}{c} + \beta)} da = \frac{k\beta^{n+1}}{(\frac{||\mathbf{h}_s||^\alpha}{b} + \frac{h_t^\delta}{c} + \beta)^{n+1}}, \tag{15}$$

where $\theta_1 = (b, c, n, k, \alpha, \beta, \delta)$;

$$C_{s,t}(\mathbf{h}_s, h_t; \theta_2) = \int_V \left[k_1 e^{-\frac{a||\mathbf{h}_s||^\alpha}{b}} \cdot e^{-\frac{a h_t^\delta}{c}} + k_2 e^{-\frac{a||\mathbf{h}_s||^\alpha}{b}} + k_3 e^{-\frac{a h_t^\delta}{c}} \right] \frac{\beta^{n+1}}{\Gamma(n+1)} \, a^n e^{-\beta a} da =$$

$$= k_1 \frac{\beta^{n+1}}{(\frac{||\mathbf{h}_s||^\alpha}{b} + \frac{h_t^\delta}{c} + \beta)^{n+1}} + k_2 \frac{\beta^{n+1}}{(\frac{||\mathbf{h}_s||^\alpha}{b} + \beta)^{n+1}} + k_3 \frac{\beta^{n+1}}{(\frac{h_t^\delta}{c} + \beta)^{n+1}}, \tag{16}$$

where $\theta_2 = (b, c, n, k_1, k_2, k_3, \alpha, \beta, \delta)$.
Note that, when $\alpha = \delta = 1$ and $n = 0$, $\left(\frac{||\mathbf{h}_s||}{b} + \frac{h_t}{c}\right)$ in (15) and (16) might correspond to a space-time metric and $\frac{\beta}{(\frac{||\mathbf{h}_s||^\alpha}{b} + \frac{h_t^\delta}{c} + \beta)}$ would belong to a well known family of covariance models, namely:

$$C(\mathbf{h}; w_1, w_2) = \frac{w_1}{w_2 + ||\mathbf{h}||}.$$

Example 2. Given the following functions:

$$C_s(\mathbf{h}_s; a, b, \alpha) = e^{-a^2||\mathbf{h}_s||^\alpha}, \quad 1 \le \alpha \le 2, \quad a > 0, \quad b > 0,$$

$$C_t(h_t; a, c, \delta) = e^{-\frac{a^2 h_t^\delta}{c}}, \quad 1 \le \delta \le 2, \quad c > 0,$$

$$\phi(a, n, \beta) = \frac{2\beta^{(n+1)/2}}{\Gamma((n+1)/2)} \, a^n e^{-\beta a^2}, \quad n \ge 0, \quad \beta > 0,$$

since $C_s(\mathbf{h}_s; a, b, \alpha)$ and $C_t(h_t; a, c, \delta)$ are, respectively, valid spatial and temporal covariance models for each choice of a over the interval $V = [0; +\infty[$, the integrability conditions of Theorem 1 are satisfied, two new classes of nonseparable space-time covariances can be obtained:

$$C_{s,t}(\mathbf{h}_s, h_t; \theta_\mathbf{1}) = \int_V k e^{-\frac{a^2||\mathbf{h}_s||^\alpha}{b}} \cdot e^{-\frac{a^2 h_t^\delta}{c}} \cdot \frac{2\beta^{(n+1)/2}}{\Gamma((n+1)/2)} \, a^n e^{-\beta a^2} da =$$

$$= \frac{2k\beta^{(n+1)/2}}{\Gamma((n+1)/2)} \int_V a^n e^{-a^2(\frac{||\mathbf{h}_s||^\alpha}{b}+\frac{\mathbf{h}_t^\delta}{c}+\beta)} da = \frac{k\beta^{(n+1)/2}}{(\frac{||\mathbf{h}_s||^\alpha}{b}+\frac{h_t^\delta}{c}+\beta)^{(n+1)/2}}, \tag{17}$$

where $\theta_1 = (b, c, n, k, \alpha, \beta, \delta)$;

$$C_{s,t}(\mathbf{h}_s, h_t; \theta_2) = \int_V \left[k_1 e^{-\frac{a||\mathbf{h}_s||^\alpha}{b}} \cdot e^{-\frac{ah_t^\delta}{c}} + k_2 e^{-\frac{a||\mathbf{h}_s||^\alpha}{b}} + k_3 e^{-\frac{ah_t^\delta}{c}} \right] \frac{2\beta^{(n+1)/2}}{\Gamma((n+1)/2)} a^n e^{-\beta a^2} da =$$

$$= k_1 \frac{\beta^{(n+1)/2}}{(\frac{||\mathbf{h}_s||^\alpha}{b}+\frac{h_t^\delta}{c}+\beta)^{(n+1)/2}} + k_2 \frac{\beta^{(n+1)/2}}{(\frac{||\mathbf{h}_s||^\alpha}{b}+\beta)^{(n+1)/2}} + k_3 \frac{\beta^{(n+1)/2}}{(\frac{h_t^\delta}{c}+\beta)^{(n+1)/2}}, \tag{18}$$

where $\theta_2 = (b, c, n, k_1, k_2, k_3, \alpha, \beta, \delta)$.

Example 3. Given the following functions:

$$C_s(\mathbf{h}_s; a, \omega) = \cos[a(\omega||\mathbf{h}_s||)], \quad a > 0, \quad \omega \in \Re,$$

$$C_t(h_t; a, c, \delta) = e^{-\frac{ah_t^\delta}{c}}, \quad 1 \le \delta \le 2, \quad c > 0,$$

$$\phi(a, \beta) = \beta e^{-a\beta}, \quad \beta > 0,$$

since the hypotheses of Theorem 1 are satisfied, two new classes of nonseparable space-time covariances are obtained:

$$C_{s,t}(\mathbf{h}_s, h_t; \theta_1) = \int_V k e^{-\frac{ah_t^\delta}{c}} \cos[a(\omega||\mathbf{h}_s||)] \cdot \beta e^{-a\beta} da =$$

$$= k\beta \int_V e^{-a(\frac{h_t^\delta}{c}+\beta)} \cos[a(\omega||\mathbf{h}_s||)] da = \frac{k\beta(\frac{h_t^\delta}{c}+\beta)}{(\frac{h_t^\delta}{c}+\beta)^2 + (\omega||\mathbf{h}_s||)^2}, \tag{19}$$

where $\theta_1 = (c, k, \omega, \beta, \delta)$;

$$C_{s,t}(\mathbf{h}_s, h_t; \theta_2) = \int_V \{k_1 e^{-\frac{ah_t^\delta}{c}} \cos[a(\omega||\mathbf{h}_s||)] + k_2 \cos[a(\omega||\mathbf{h}_s||)] + k_3 e^{-\frac{ah_t^\delta}{c}}\} \cdot \beta e^{-a\beta} da =$$

$$= k_1 \frac{\beta(\frac{h_t^\delta}{c}+\beta)}{(\frac{h_t^\delta}{c}+\beta)^2 + (\omega||\mathbf{h}_s||)^2} + k_2 \frac{\beta^2}{\beta^2 + (\omega||\mathbf{h}_s||)^2} + k_3 \frac{\beta}{\frac{h_t^\delta}{c}+\beta}, \tag{20}$$

where $\theta_2 = (c, k_1, k_2, k_3, \omega, \beta, \delta)$.

Example 4. Given the following functions:

$$C_s(\mathbf{h}_s; a, \omega) = \cos[a(2\omega||\mathbf{h}_s||)], \quad a > 0, \quad \omega \in \Re,$$

$$C_t(h_t;a,c,\delta)=e^{-\frac{a^2h_t^\delta}{c}},\quad 1\le\delta\le 2,\quad c>0,$$

$$\phi(a,\beta)=(2\sqrt{\frac{\beta}{\pi}})e^{-a^2\beta},\quad \beta>0,$$

since the hypotheses of Theorem 1 are satisfied, two new classes of nonseparable space-time covariances are obtained:

$$C_{s,t}(\mathbf{h}_s,h_t;\theta_1)=\int_V k\cos[a(2\omega||\mathbf{h}_s||)]e^{-\frac{a^2h_t^\delta}{c}}\left(2\sqrt{\frac{\beta}{\pi}}\right)e^{-a^2\beta}da=$$

$$=2k\sqrt{\frac{\beta}{\pi}}\int_V e^{-a^2(\frac{h_t^\delta}{c}+\beta)}\cos[a(2\omega||\mathbf{h}_s||)]da=k\sqrt{\frac{\beta}{\frac{h_t^\delta}{c}+\beta}}\,e^{-\frac{\omega^2||\mathbf{h}_s||^2}{\frac{h_t^\delta}{c}+\beta}},\tag{21}$$

where $\theta_1=(c,k,\omega,\beta,\delta)$;

$$C_{s,t}(\mathbf{h}_s,h_t;\theta_2)=$$

$$=\int_V\{k_1\cos[a(2\omega||\mathbf{h}_s||)]e^{-\frac{a^2h_t^\delta}{c}}+k_2\cos[a(2\omega||\mathbf{h}_s||)]+k_3e^{-\frac{a^2h_t^\delta}{c}}\}\left(2\sqrt{\frac{\beta}{\pi}}\right)e^{-a^2\beta}da=$$

$$=k_1\sqrt{\frac{\beta}{\frac{h_t^\delta}{c}+\beta}}\,e^{-\frac{\omega^2||\mathbf{h}_s||^2}{\frac{h_t^\delta}{c}+\beta}}+k_2e^{-\frac{\omega^2||\mathbf{h}_s||^2}{\beta}}+k_3\sqrt{\frac{\beta}{\frac{h_t^\delta}{c}+\beta}},\tag{22}$$

where $\theta_2=(c,k_1,k_2,k_3,\omega,\beta,\delta)$.

Example 5. Given the following functions:

$$C_s(\mathbf{h}_s;a,b,\alpha)=e^{-\frac{a^2||\mathbf{h}_s||^\alpha}{b}}e^{-\frac{||\mathbf{h}_s||}{a^2}},\quad 1\le\alpha\le 2,\quad b>0,$$

$$C_t(h_t;a,c,\delta)=e^{-\frac{a^2h_t^\delta}{c}}e^{-\frac{h_t}{a^2}},\quad 1\le\delta\le 2,\quad c>0,$$

$$\phi(a,\beta)=2\sqrt{\frac{\beta}{\pi}}e^{-a^2\beta},\quad \beta>0,$$

since the hypotheses of theorem 1 are satisfied, two new classes of nonseparable space-time covariances are obtained:

$$C_{s,t}(\mathbf{h}_s,h_t;\theta_1)=\int_V ke^{-(\frac{a^2||\mathbf{h}_s||^\alpha}{b}+\frac{||\mathbf{h}_s||}{a^2})}e^{-(\frac{a^2h_t^\delta}{c}+\frac{h_t}{a^2})}2\sqrt{\frac{\beta}{\pi}}e^{-a^2\beta}da=$$

$$=2\sqrt{\frac{\beta}{\pi}}\int_V ke^{-\left[a^2(\beta+\frac{||\mathbf{h}_s||^\alpha}{b}+\frac{h_t^\delta}{c})+\frac{||\mathbf{h}_s||+h_t}{a^2}\right]}da=$$

$$= k\sqrt{\frac{\beta}{\beta+\frac{||\mathbf{h}_s||^\alpha}{b}+\frac{h_t^\delta}{c}}}\; e^{-2\sqrt{(\beta+\frac{||\mathbf{h}_s||^\alpha}{b}+\frac{h_t^\delta}{c})(||\mathbf{h}_s||+h_t)}}, \tag{23}$$

where $\theta_1 = (b, c, k, \alpha, \beta, \delta)$;

$$C_{s,t}(\mathbf{h}_s, h_t; \theta_2) =$$

$$= \int_V [k_1 e^{-(\frac{a^2||\mathbf{h}_s||^\alpha}{b}+\frac{||\mathbf{h}_s||}{a^2})} e^{-(\frac{a^2 h_t^\delta}{c}+\frac{h_t}{a^2})} + k_2 e^{-(\frac{a^2||\mathbf{h}_s||^\alpha}{b}+\frac{||\mathbf{h}_s||}{a^2})} + k_3 e^{-(\frac{a^2 h_t^\delta}{c}+\frac{h_t}{a^2})}]2\sqrt{\frac{\beta}{\pi}} e^{-a^2\beta} da =$$

$$= k_1\sqrt{\frac{\beta}{\beta+\frac{||\mathbf{h}_s||^\alpha}{b}+\frac{h_t^\delta}{c}}}\; e^{-2\sqrt{(\beta+\frac{||\mathbf{h}_s||^\alpha}{b}+\frac{h_t^\delta}{c})(||\mathbf{h}_s||+h_t)}} +$$

$$+ k_2\sqrt{\frac{\beta}{\beta+\frac{||\mathbf{h}_s||^\alpha}{b}}}\; e^{-2\sqrt{(\beta+\frac{||\mathbf{h}_s||^\alpha}{b})||\mathbf{h}_s||}} + k_3\sqrt{\frac{\beta}{\beta+\frac{h_t^\delta}{c}}}\; e^{-2\sqrt{(\beta+\frac{h_t^\delta}{c})h_t}}, \tag{24}$$

where $\theta_2 = (b, c, k_1, k_2, k_3, \alpha, \beta, \delta)$.

Remarks.

- Even though the space-time covariance or variogram models (since any of the above covariance models can be written in terms of variograms) look different from the spatial and temporal structures which are used in the integrals (11), (12), (13) and (14), they retain the main features of the separate components in space and time;
- the integrated product-sum variogram models corresponding to (16) and (18), derived from examples 1 and 2, with

  $$\theta_2 = (4000; 8; 2; 180; 220; 70; 1; 3; 1),$$

  are shown in Fig. 1 and 2. Note that the space-time variogram surface is convex both in $\|\mathbf{h}_s\|$ and in h_t, starting from spatial and temporal exponential variogram models;
- when the separate spatial and temporal structures are Gaussian models, the space-time variogram models corresponding to (16) and (18), with

  $$\theta_2 = (4000^2; 8^2; 2; 180; 220; 70; 2; 3; 2),$$

  turn out to be concave both in $\|\mathbf{h}_s\|$ and in h_t, especially for small spatial and temporal lags, as it is shown in Fig. 3 and 4;

- models from (20) and (22) present parabolic behaviour in $\|\mathbf{h}_s\|$ for small spatial lags; they are also either convex in h_t (Fig. 5 and 7), where

$$\theta_2 = (8; 180; 220; 70; 10^{-4}; 3; 1),$$

if the temporal variogram is an exponential model, or concave in h_t (Fig. 6 and 8), with

$$\theta_2 = (8; 180; 220; 70; 10^{-4}; 3; 2),$$

especially for small lags, if the temporal variogram is a Gaussian model;

- as it can be seen from Fig. 9, model (24), with

$$\theta_2 = (4000^2; 8^2; 180; 220; 70; 1; 3; 1),$$

can be used to describe an almost complete lack of space-time correlation, that is, it is close to a pure nugget effect model in space-time domain.

5 Some practical aspects

Given a spatial-temporal data set, it is necessary to know how to use the data to generate a model of the form (13) or (14), that is, how to choose the function ϕ, the spatial and the temporal variograms as well as the coefficients using the data.
Note that in the following only the practical aspects of using (14) are considered, since users can easily extend the following technique in order to utilize (13).
The first step is to take advantage of one of the defining properties of the variogram function, that is $\gamma(0) = 0$. Hence, from (14) it follows that :

$$\gamma_{s,t}(\mathbf{h}_s, 0) = \int_V (k_2 + k_1 C_t(0; a))\gamma_s(\mathbf{h}_s; a)\phi(a)da \tag{25}$$

and

$$\gamma_{s,t}(\mathbf{0}, h_t) = \int_V (k_3 + k_1 C_s(\mathbf{0}; a))\gamma_t(h_t; a)\phi(a)da. \tag{26}$$

Secondly, if it is assumed that:

1. ϕ is a density function;

2. $\gamma_s(\mathbf{h}_s; a)$ and $\gamma_t(h_t; a)$ are standardized variograms with sill values equal to 1, that is $C_s(\mathbf{0}; a) = 1$ and $C_t(0; a) = 1$;

3. $(k_2 + k_1 C_t(0; a)) = (k_2 + k_1)$ and $(k_3 + k_1 C_s(\mathbf{0}; a)) = (k_3 + k_1)$ are the sill values, respectively, of $\gamma_{s,t}(\mathbf{h}_s, 0)$ and $\gamma_{s,t}(\mathbf{0}, h_t)$;

then the relations between $\gamma_{s,t}(\mathbf{h}_s, 0)$ and $\gamma_s(\mathbf{h}_s; a)$, $\gamma_{s,t}(\mathbf{0}, h_t)$ and $\gamma_t(h_t; a)$ are strictly linked to the function ϕ. Note that the above assumption is satisfied, for example, if a is a parameter which is related only to the ranges of the spatial and temporal variogram models, $\gamma_s(\mathbf{h}_s; a)$ and $\gamma_t(h_t; a)$ (see examples in section 4.1).
The function ϕ, which is needed to derive the spatial-temporal variogram model described in (14), can be obtained the following procedure:

- Models for $\gamma_s(\cdot; a)$ and $\gamma_t(\cdot; a)$, dependent on a, can be chosen by looking at the behaviour of the sample spatial and temporal variograms (denoted by $\widehat{\gamma}_{s,t}(\mathbf{r}_s, 0)$ and $\widehat{\gamma}_{s,t}(\mathbf{0}, r_t)$, respectively);

- For a discrete number of values $a_1, \ldots, a_m$ for the parameter a, one can obtain multiple spatial and temporal theoretical curves

$$\gamma_s(\cdot; a_j), \gamma_t(\cdot; a_j), j = 1, \ldots, m,$$

  which provide different fits to the corresponding sample variograms $\widehat{\gamma}_{s,t}(\mathbf{r}_s, 0)$ and $\widehat{\gamma}_{s,t}(\mathbf{0}, r_t)$. In practice, the minimum and maximum values of the sequence $(a_i, i = 1, \ldots, m)$ are chosen in such a way that the corresponding theoretical variograms are not too "far" from the sample variograms;

- Evaluate how well each $\gamma_s(\cdot; a_j)$ and $\gamma_t(\cdot; a_j)$ fits the data. This measure of the goodness of fit can be used to define a likelihood of fit and hence a probability density :

$$w_j^s = \frac{1}{\sum_{N_s} \left(\frac{\widehat{\gamma}_{s,t}(\mathbf{r}_s, 0) - (k_2 + k_1)\gamma_s(\mathbf{r}_s; a_j)}{(k_2 + k_1)\gamma_s(\mathbf{r}_s; a_j)} \right)^2}, \quad j = 1, \ldots, m \tag{27}$$

$$w_j^t = \frac{1}{\sum_{N_t} \left(\frac{\widehat{\gamma}_{s,t}(\mathbf{0}, r_t) - (k_3 + k_1)\gamma_t(r_t; a_j)}{(k_3 + k_1)\gamma_t(r_t; a_j)} \right)^2}, \quad j = 1, \ldots, m \tag{28}$$

  where N_s and N_t are, respectively, the number of spatial and temporal lags for the sample variograms;

- by plotting a_j versus w_j^s and $w_j^t, j = 1, \ldots, m$, one can easily define the measure ϕ;

- finally, a space-time variogram model can be obtained by solving the integral (14).

It is evident that the above model still has an unknown parameter k_1, it can be estimated in either of two ways:

- by minimizing $W(k_1)$, the weighted least-squares value (Cressie, 1993), given by:

$$W(k_1) = \sum_{s}^{N_s} \sum_{t}^{N_t} |L(\mathbf{r}_s, r_t)| \left(\frac{\widehat{\gamma}_{s,t}(\mathbf{r}_s, r_t) - \gamma_{s,t}(\mathbf{r}_s, r_t; k_1)}{\gamma_{s,t}(\mathbf{r}_s, r_t; k_1)} \right)^2$$

  where $\mathbf{r}_s$, r_t and $|L(\mathbf{r}_s, r_t)|$ have been defined in section 2, N_s and N_t are, respectively, the number of spatial vector lags and the number of temporal lags, while $\widehat{\gamma}_{s,t}$ is the sample space-time variogram;

- by computing the sill value of $\gamma_{s,t}$ from the sample space-time variogram $\widehat{\gamma}_{s,t}$ and solving the linear system of the following equations:

$$\begin{aligned} k_2 + k_1 = &\text{ estimated sill value of } \gamma_{s,t}(\mathbf{r}_s, 0), \\ k_3 + k_1 = &\text{ estimated sill value of } \gamma_{s,t}(\mathbf{0}, r_t), \\ k_1 + k_2 + k_3 = &\text{ estimated sill value of } \gamma_{s,t}(\mathbf{r}_s, r_t). \end{aligned}$$

The method described above for generating the measure μ provides a very useful result for approximation and optimization.

Theorem 2 *Let $\phi(a)$ be a density function, that vanishes for negative values, and let $\gamma_s(\mathbf{r}_s; a)$ and $\gamma_t(r_t; a)$ be standardized variograms. The spatial and temporal variograms, defined, respectively, in (25) and (26), satisfy the following inequalities:*

$$\sum_{N_s} [\gamma_{s,t}(\mathbf{r}_s, 0) - \widehat{\gamma}_{s,t}(\mathbf{r}_s, 0)]^2 \leq E_a \left(\sum_{N_s} [\gamma_s(\mathbf{r}_s; a) - \widehat{\gamma}_{s,t}(\mathbf{r}_s, 0)]^2 \right)$$

$$\sum_{N_t} [\gamma_{s,t}(\mathbf{0}, r_t) - \widehat{\gamma}_{s,t}(\mathbf{0}, r_t)]^2 \leq E_a \left(\sum_{N_t} [\gamma_t(r_t; a) - \widehat{\gamma}_{s,t}(\mathbf{0}, r_t)]^2 \right)$$

where $\mathbf{r}_s$ and r_t have been defined in section 2, N_s and N_t are, respectively, the number of spatial vector lags and the number of temporal lags, $\widehat{\gamma}_{s,t}(\mathbf{r}_s, 0)$ and $\widehat{\gamma}_{s,t}(\mathbf{0}, r_t)$ the sample spatial and temporal variograms.

Proof. Since ϕ is supposed to be a density function, which vanishes for negative values, defining:

$$\gamma_{s,t}(\mathbf{r}_s, 0) = \int_0^\infty \gamma_s(\mathbf{r}_s, a)\phi(a)da,$$

$$\gamma_{s,t}(\mathbf{0}, r_t) = \int_0^\infty \gamma_t(r_t, a)\phi(a)da,$$

it follows that:

$$\gamma_{s,t}(\mathbf{r}_s, 0) = E_a(\gamma_s(\mathbf{r}_s, a)),$$
$$\gamma_{s,t}(\mathbf{0}, r_t) = E_a(\gamma_t(r_t, a)).$$

By recalling the variance definition, it comes out:

$$\mathrm{Var}_a(\gamma_s(\mathbf{r}_s, a)) = E_a(\gamma_s(\mathbf{r}_s, a) - \gamma_{s,t}(\mathbf{r}_s, 0))^2 =$$

$$= E_a(\gamma_s(\mathbf{r}_s, a) - \widehat{\gamma}_{s,t}(\mathbf{r}_s, 0))^2 - (\gamma_{s,t}(\mathbf{r}_s, 0) - \widehat{\gamma}_{s,t}(\mathbf{r}_s, 0))^2,$$

$$Var_a(\gamma_t(r_t, a)) = E_a(\gamma_t(r_t, a) - \gamma_{s,t}(\mathbf{0}, r_t))^2 =$$
$$= E_a(\gamma_t(r_t, a) - \widehat{\gamma}_{s,t}(\mathbf{0}, r_t))^2 - (\gamma_{s,t}(\mathbf{0}, r_t) - \widehat{\gamma}_{s,t}(\mathbf{0}, r_t))^2.$$

Since the variance is non-negative, the following inequalities hold for any lag $\mathbf{r}_s$ and r_t:

$$(\gamma_{s,t}(\mathbf{r}_s, 0) - \widehat{\gamma}_{s,t}(\mathbf{r}_s, 0))^2 \leq E_a(\gamma_s(\mathbf{r}_s, a) - \widehat{\gamma}_{s,t}(\mathbf{r}_s, 0))^2,$$

$$(\gamma_{s,t}(\mathbf{0}, r_t) - \widehat{\gamma}_{s,t}(\mathbf{0}, r_t))^2 \leq E_a(\gamma_t(r_t, a) - \widehat{\gamma}_{s,t}(\mathbf{0}, r_t))^2,$$

hence, the theorem follows.

This result ensures that the spatial and temporal variogram models, $\gamma_{s,t}(\mathbf{r}_s, 0)$ and $\gamma_{s,t}(\mathbf{0}, r_t)$, obtained by the above procedure, provide a better average fitting to the spatial and temporal sample variograms than the fitting obtained by using $\gamma_s(\mathbf{r}_s, \cdot)$ and $\gamma_t(r_t, \cdot)$.

6 A case study

The methods described above have been applied to hourly average concentrations of NO_2 measured during August 1997 in 18 survey stations in Milan district. After removing the seasonal component by the standard technique of moving averages (Brockwell and Davis, 1987), residuals, available for all stations, were used for the structural analysis.

The steps for generating the space-time variogram model are listed below.

- Examining the shapes of the sample spatial and temporal variograms, $\widehat{\gamma}_{s,t}(\cdot, 0)$ and $\widehat{\gamma}_{s,t}(\mathbf{0}, \cdot)$, exponential models have been chosen for $\gamma_s(\cdot, a)$ and $\gamma_t(\cdot, a)$ whose analytical expressions are, respectively:

$$\gamma_s(\mathbf{h}_s, a) = 1 - exp\{-a\frac{\|\mathbf{h}_s\|}{4414}\}, \qquad (29)$$

$$\gamma_t(h_t, a) = 1 - exp\{-a\frac{h_t}{8.22}\}, \tag{30}$$

and the sill values have been set to 400 and 250, respectively, for the spatial and temporal structures, i.e. $k_2 + k_1 = 400$, and $k_3 + k_1 = 250$. Note that, for $a = 1$, $400\gamma_s(\mathbf{h}_s, a)$ and $250\gamma_t(h_t, a)$ are considered to be a good fitting to the estimated spatial and temporal variograms.

- After choosing 10 values of a, from $a_1 = 0.25$ to $a_{10} = 2.75$, as many spatial and temporal theoretical curves from (29) and (30) have been obtained.
- By computing (27) and (28) for the 10 values of a and by plotting a_j versus w_j^s and $w_j^t, j = 1, \ldots, 10$, the following density function ϕ (Fig. 10) has been derived:

$$\phi(a) = 9.84a^2 exp\{-2.7a\}.$$

- By using (25) and (26), the following spatial and temporal variogram models have been obtained:

$$\gamma_{s,t}(\mathbf{h}_s, 0) = \int_0^\infty [400(1 - exp\{-a\frac{\|\mathbf{h}_s\|}{4414}\})][9.84a^2 exp\{-2.7a\}]da =$$

$$= 400\Big(1 - \frac{2.7^3}{(2.7 + \frac{\|\mathbf{h}_s\|}{4414})^3}\Big);$$

$$\gamma_{s,t}(\mathbf{0}, h_t) = \int_0^\infty [250(1 - exp\{-a\frac{h_t}{8.22}\})][9.84a^2 exp\{-2.7a\}]da =$$

$$= 250\Big(1 - \frac{2.7^3}{(2.7 + \frac{h_t}{8.22})^3}\Big).$$

Fig. 11 and 12 show, respectively for spatial and temporal aspect, the sample variograms of the residuals, the exponential models for different values of a and the integrated models. Note that Fig. 12 also shows the sample temporal variogram of the original data.

- Finally, the sill value of $\gamma_{s,t}$, evaluated from the sample space-time variogram $\widehat{\gamma}_{s,t}$ (Fig. 13), has been set to 470 ($k_1 + k_2 + k_3 = 470$). Hence, $k_1 = 180$, $k_2 = 220$, $k_3 = 70$ and the space-time variogram model (Fig. 14) has been obtained by solving the integral (14):

$$\gamma_{s,t}(\mathbf{h}_s, h_t) = 470 - 220\Big(\frac{2,7}{2.7 + \frac{\|\mathbf{h}_s\|}{4414}}\Big)^3 - 70\Big(\frac{2.7}{2.7 + \frac{h_t}{8.2}}\Big)^3 - 180\Big(\frac{2.7}{2.7 + \frac{\|\mathbf{h}_s\|}{4414} + \frac{h_t}{8.22}}\Big)^3$$

7 Conclusions

Estimating and modeling the correlation of a space-time process is a very important problem . In this paper, beginning with the product and the product-sum covariance models, non-integrable space-time covariance models have been generated. These parametric families cannot be obtained, in general, from Cressie-Huang representation. To use these models, the user still must fit a model to the data. Possible choices can be determined after computing the sample spatial-temporal variogram and inspecting its main features (such as behaviour near the origin for small spatial and temporal lags, the sill values along spatial and temporal directions). Using these features, compare with the examples given in section 4.1. It is evident, for example, that one can choose between the integrated product and the integrated product-sum just by looking at the spatial and temporal sill values, since only the product-sum model can be used when the sill values are equal.

Other practical aspects, linked with the problems of fitting the covariance or variogram model to the data available, were discussed and a case study has been presented.

Appendix

1. **Measure theory.** Given a space $(\Omega,\ A)$, a measure μ on a σ-algebra A of Ω is a set function which assigns to each set of A a number of the interval $]-\infty, +\infty]$ such that:

$$\mu(\{\emptyset\}) = 0,$$

$$\mu\Big(\bigcup_{i=1}^{n} \mathbf{A}_i\Big) = \sum_{i=1}^{n} \mu(\mathbf{A}_i)$$

   where $\mathbf{A}_i$ are pairwise disjoint sets of A. If $\mu(\mathbf{A}) > 0$ for any $\mathbf{A}$, then the measure is called *positive*; moreover, if:

$$\sup\{\mu(\mathbf{A}) : \mathbf{A} \in A\} < \infty$$

   then the measure is *bounded*. A positive measure μ is called *finite* if $\mu(\Omega) < \infty$.

2. As a particular case of the 2^{nd} stability property, the nested structure variogram can be generalized by integrating one-dimensional anisotropic components, (Myers,1987) $\gamma(\mathbf{u}'\mathbf{h})$ with respect to a bounded positive measure $\mu(\mathbf{u})$ on the unit half-sphere of $\Re^n$, then it follows:

$$\gamma(\mathbf{h}) = \int \ldots \int \gamma(\mathbf{u}'\mathbf{h})\mu(d\mathbf{u}).$$

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- Fig. 5: Integrated product-sum variogram model (20) obtained from a temporal exponential variogram model with $\theta_2 = (8; 180; 220; 70; 10^{-4}; 3; 1)$.
- Fig. 6: Integrated product-sum variogram model (22) obtained from a temporal exponential variogram model with $\theta_2 = (8; 180; 220; 70; 10^{-4}; 3; 2)$.
- Fig. 7: Integrated product-sum variogram model (20) obtained from a temporal Gaussian variogram model with $\theta_2 = (8; 180; 220; 70; 10^{-4}; 3; 1)$.
- Fig. 8: Integrated product-sum variogram model (22) obtained from a temporal Gaussian variogram model with $\theta_2 = (8; 180; 220; 70; 10^{-4}; 3; 2)$.
- Fig. 9: Integrated product-sum variogram model (24) with θ_2=(4000^2; 8^2; 180; 220; 70; 1; 3; 1).
- Fig. 10: Probability density used to check the goodness of fit.
- Fig. 11: Sample spatial variogram of No_2, different models for different values of the parameter a and the integrated model.
- Fig. 12: Sample temporal variogram of residuals of No_2, different models for different values of the parameter a and the integrated model.
- Fig. 13: Sample space-time variogram surface of No_2.
- Fig. 14: Integrated space-time variogram model of No_2.