*Irrationals, area, and probability**

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An interpretation of the area of a set D as the probability that a point chosen at random from R *will be in D*

CONSIDER THE FOLLOWING probability problem. Let R be a unit square, that is, with sides of length one and denoting both the curve or perimeter and the area enclosed. Divide R into two parts, R_1 and R_2, by constructing a line parallel to one pair of sides and bisecting each of the other sides. Suppose a point p is chosen at random from R, a reasonable intuitive notion of the probability that p is in R_1 is $\frac{1}{2}$. Alternately construct the diagonal of R and obtain two "equal" parts, R_1' and R_2'. Again intuitively the probability that p is in R_1' is $\frac{1}{2}$. For an elementary area R_1 (for example, a circle, triangle, rectangle) enclosed in R the probability that the point p is in the elemental area R_1 would be the value of the area $A(R_1)$. In the usual introduction to probability in an algebra course the probability, $P(E)$, of an event, E, is defined to be the ratio of the number of ways E can occur to the number of ways "something" can happen. This definition yields only rational numbers for probabilities. In the preceding example, however, irrational $P(E)$ will result easily. If a circle with a rational radius is used the area is irrational and hence so is the probability of the point randomly chosen occurring in the circle.

The word area has been used in two ways, (1) to denote a set of points and (2) to denote a number associated with the set of points. In this paper only the latter usage will occur. To relieve the necessity for the ambiguity a coordinate system will be utilized. This enables us to distinguish easily between a "square," meaning the curve, and "square," meaning the points "enclosed" by the square. The former is denoted by

$$\begin{aligned}\mathrm{bd}(R) = &\{(x, y) \mid a \le x \le b,\ y = c\} \cup\\ &\{(x, y) \mid a \le x \le b,\ y = d\} \cup\\ &\{(x, y) \mid x = a,\ c \le y \le d\} \cup\\ &\{(x, y) \mid x = b,\ c \le y \le d\},\end{aligned}$$

and the latter by

$$\mathrm{int}(R) = \{(x, y) \mid a < x < b,\ c < y < d\};$$

in each case $b-a=d-c$. bd(R) is read "boundary of R" and int(R) is read "interior of R."

The coordinate system also allows us to avoid the word "enclosed" when referring to a set of points. The pitfalls inherent in the intuitive usage of "enclosed" are illustrated by Figure 1. The point p is not "enclosed" by the closed curve, although

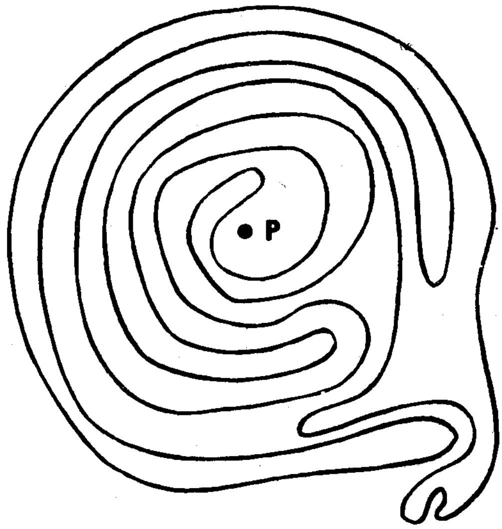

Figure 1

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it appears to be. With the distance formula from analytic geometry, the distance between $P_1:(x_1, y_1)$ and $P_2:(x_2, y_2)$ is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = |(x_1, y_1)-(x_2, y_2)| = |P_1-P_2|.$$

An ϵ-neighborhood of a point $p:(x_1, y_1)$ is the set of points interior to a circle center at p radius ϵ or analytically $N_p(\epsilon)=\{(x, y)\,|\,|(x, y)-p|<\epsilon\}$. Using neighborhoods, interior and boundary points can be defined. Let R be an arbitrary set of points in the plane, then p is an interior point of R if there exists an ϵ-neighborhood of p contained in R. p is a boundary point of R if every ϵ-neighborhood of p contains points in R and points not in R. The sets referred above as bd(R) and int(R) are seen to be sets of boundary points and interior points respectively.

To construct a theory of area for planar sets area must be an undefined concept for some sets. In particular if S is a rectangular set, i.e., $S=\{(x, y)\,|\,a\leq x\leq b, c\leq y\leq d\}$, then $A(S)=(b-a)(d-c)$. The usual process of subdividing S into unit squares to compute the area of S simply implies that area should be additive for sets which pairwise have no interior points. Since we are interested in area as a probability measure, all sets will be assumed to be contained in the unit square. That is, all points will have coordinates (x, y) which are subject to the general restriction $0<x\leq 1$, $0<y\leq 1$. If the coordinates are not subject to additional restrictions then the set of such points is the unit square. Let R denote the unit square and D an arbitrary subset of R. Let R be partitioned into subrectangles by the construction of lines parallel to one or the other of the pairs of opposite sides of R. This is determined analytically by letting $0<a_1<a_2<\cdots<a_n=1$ and $0<b_1<\cdots<b_m=1$ and then for each $j=1, 2, \cdots, n$ connect the points $(0, a_j)$ and $(1, a_j)$; for each $j=1, 2, \cdots, m$ connect $(b_j, 0)$ and $(b_j, 1)$. The rectangles are numbered from the lower left-hand corner across the bottom to the right, then left to right in the next to bottom row, etc., as in Figure 2.

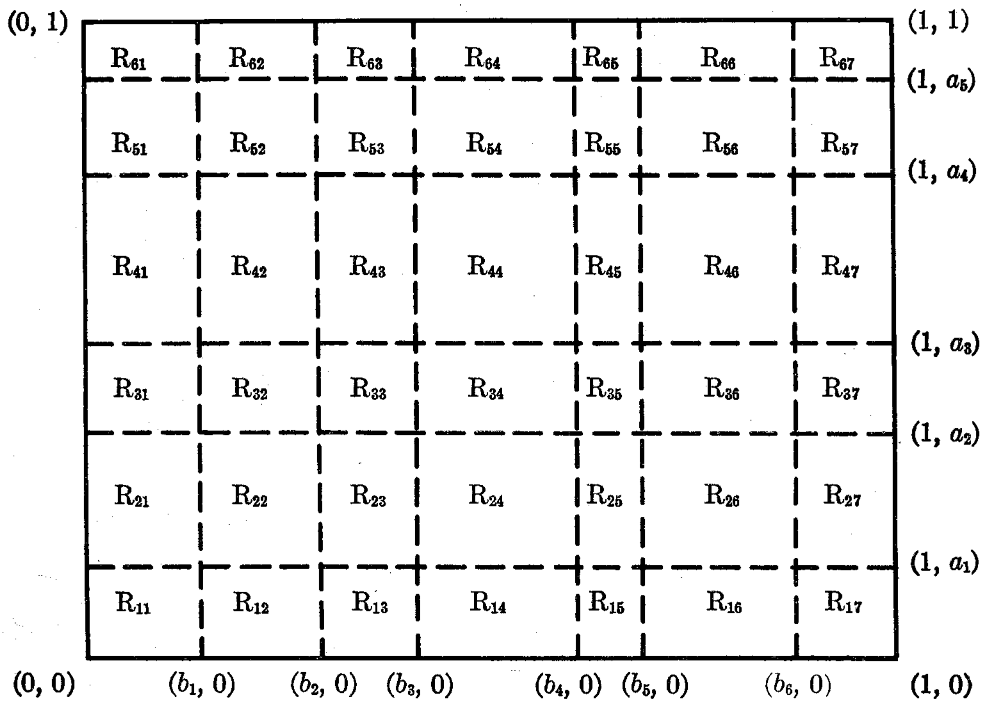

Figure 2

Call this partition N_1, then N_2 is another partition obtained from N_1 by the construction of additional lines. Continue this to obtain a sequence of partitions $\{N_n\}$ such that the length of the maximal diagonal of the subrectangles $\to 0$ as $n \to \infty$. Each succeeding partition is said to refine the previous ones. The requiring of the maximal diagonal to go to zero assures two things, the maximal area of the subrectangles will go to zero and the lengths of both sides of all subrectangles will go to zero. Increasing the number of lines does not necessarily insure either of these. Denote by $\{R_{ij}^n\}$ the subrectangles in the partition N_n where $R_{ij}^n = \{(x, y) \mid a_{i_n-1} < x \leq a_{i_n}$ and $b_{j_n-1} < y \leq b_{j_n}\}$. With respect to the set D, each is in one and only one of the following categories:

i Those R_{ij}^n not containing any points of D or its boundary.
ii Those R_{ij}^n containing only interior points of D.
iii Those R_{ij}^n containing at least one boundary point of D.

For each partition, then, there is an outer sum $\overline{A}_n(\mathrm{D})$, namely the sum of the areas of the rectangles in categories (ii) and (iii). The inner sum $\underline{A}_n(\mathrm{D})$, is the sum of the areas of the rectangles in category (ii). Obviously, $\underline{A}_n(\mathrm{D}) \leq \overline{A}_n(\mathrm{D})$; less obvious is the relation $\underline{A}_n(\mathrm{D}) \leq \underline{A}_m(\mathrm{D}) \leq \overline{A}_m(\mathrm{D}) \leq \overline{A}_n(\mathrm{D})$ for $m \geq n$. Even if one partition N' is not obtained from another N'' by constructing additional lines, there is a partition N''' such that its inner and outer sums bear the preceding relation to the inner and outer sums of the partitions N', N'', separately. Utilizing the completeness property of the real numbers we know there is a number $\overline{A}(\mathrm{D})$, such that $\overline{A}(\mathrm{D}) \leq \overline{A}_n(\mathrm{D})$ for all n and is the greatest number satisfying that inequality. $\overline{A}(\mathrm{D})$ then is the greatest lower bound for the outer sums of D and similarly $\underline{A}(\mathrm{D})$ is the least upper bound of all the inner sums; if $\underline{A}(\mathrm{D}) = \overline{A}(\mathrm{D})$ this common value is the area and is denoted $A(\mathrm{D})$. Consider the following examples:

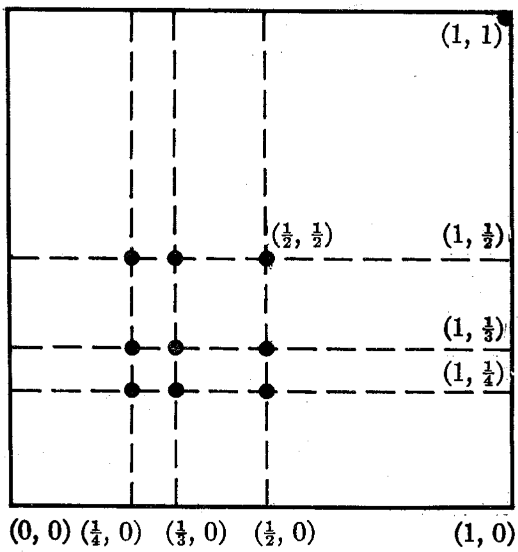

Figure 3

Example 1. Let (x_1, y_1) be any point in R and N a partition of R as previously described but including lines through the point (x_1, y_1). Considering (x_1, y_1) as a set consisting of one point it is seen that there are no members of category (ii), exactly one in category (iii) and all the rest are in (i). $\underline{A}(\mathrm{D}) = 0$ and $\overline{A}(\mathrm{D})$ also is zero because the outer sums can be made arbitrarily small. Hence the area is zero and this extends easily to any finite set of points.

Example 2. Let Q be the set of points in R with rational coordinates. That is, $Q = \{(x, y) \mid x = \frac{m}{n}, \ y = \frac{p}{q}, \ n \neq 0, \ q \neq 0, \ m \leq n, \ p \leq q$ positive integers$\}$. The inner sums are always zero and outer sums one, hence area is not defined for Q.

Example 3. Let W be the subset of Q obtained by setting $m = p = 1$ or $W = \{(x, y) \mid x = \frac{1}{n}, \ y = \frac{1}{q}, \ n, q$ positive integers$\}$. The inner sums are all zero and hence the inner area is zero. We will show now that $\overline{A}(W) = 0$ also and hence $A(W) = 0$. Let N_1 be obtained by constructing the bisecting lines, $x = \frac{1}{2}$, $y = \frac{1}{2}$ and the lines $x = 1 - \epsilon$, $y = 1 - \epsilon$, $\frac{1}{4} > \epsilon > 0$. There are nine rectangles, three of which are in category (i); the area of these three is $\frac{1}{4} - \epsilon^2$, hence $\overline{A}_1(W) \leq \frac{3}{4} + \epsilon^2$. Now construct the lines $x=\frac{1}{4}$ and $y=\frac{1}{4}$. If $\frac{1}{16}>\epsilon>0$, then there are five rectangles in category (i) with a total area of $\frac{1}{2}-\epsilon^2-\frac{\epsilon}{2}$; another three rectangles contain only five points of W, $(1, 1)$, $(1, \frac{1}{2})$, $(1, \frac{1}{3})$, $(\frac{1}{2}, 1)$ and $(\frac{1}{3}, 1)$, the area of these three is $\epsilon^2+\frac{\epsilon}{2}$, hence $\overline{A}_2(W) \leq \frac{1}{2}+\epsilon^2+\frac{\epsilon}{2}$. In general, construct the additional lines

$$x=\frac{1}{2^k}, \quad x=\frac{1}{2^k}-\epsilon_k, \quad y=\frac{1}{2^k}, \quad y=\frac{1}{2^k}-\epsilon_k$$

where

$$\frac{1}{2^{k+1}} > \epsilon_k > 0;$$

in this way it is seen that $\overline{A}(W)=0$ and therefore $A(W)=0$.

In more intuitive terms we have constructed circumscribing and inscribed sets of rectangles. Several properties are obtained from the method of construction:

a. If D_1 and D_2 are sets in R, the union of the circumscribing regions is a circumscribing region for the union of D_1 and D_2, similarly for the inscribed regions.

b. Further, if $A(D_1)$ and $A(D_2)$ both exist then $A(D_1 \cup D_2)$ exists and is less than or equal to $A(D_1)+A(D_2)$. Equality is obtained if D_1 and D_2 have no common interior points.

c. Denoting by $C_R(D)$ the set of points in R but not in D, read the complement of D with respect to R, then we have shown that a circumscribing region for D is an "inscribed" region for $C_R(D)$ and vice versa. Using the relations $1-\overline{A}_n(D) \leq 1-\underline{A}_n(D)$, $1-\overline{A}_n(D) = \underline{A}_n[C_R(D)]$, $1-\underline{A}_n(D)=\overline{A}_n[C_R(D)]$ the existence of $A(D)$ implies the existence of $A[C_R(D)]$ or conversely.

d. Finally, if $A(D_1)$ and $A(D_2)$ both exist, $A(D \cap_1 D_2)$ exists, $D_1 \cap D_2$ being the intersection of D_1 and D_2 or set of points common to both.

When the "completeness" property is used to obtain the existence of $\overline{A}(D)$ and $\underline{A}(D)$, the possibility of irrational numbers occurs even if all the lengths of the sides of the subrectangles are rational. In fact, all the irrationals between zero and one can be obtained in this way. This allows us to make a correspondence between irrational numbers and planar sets of points. The number $\frac{\pi}{4}$ is represented by the set $C=\{(x, y) \mid |(x, y)-(\frac{1}{2}, \frac{1}{2})| \leq \frac{1}{2}\}$. Some ambiguity is present, since for C, above, the equality may or may not be left off without changing the area. Contracting or expanding sequences of sums of rectangles illustrate rational approximation to irrational numbers. Furthermore, an intuitive proof of what is known in the advanced courses as the Nested Set Theorem is immediate.

As mentioned previously, the usual introduction to probability implies the existence of only rational values for probabilities. Some of the other principal results are independent of this, but sometimes have only trivial illustrations. Recall our interpretation of the area of a set D as the probability that a point chosen from R will be in D. In a probability space there must be a collection of admissible events. The definition of area given provides a criteria for admissibility, such that if E_1, E_2 have area $E_1 \cup E_2$, $E_1 \cap E_2$, $C_R(E_1)$, $C_R(E_2)$ all have area. Further, the empty set has zero area and R has area one, the latter two being the "impossible" and "certain" events, respectively. Examples 1 and 3 illustrate that an event can have zero probability and still not be the "impossible" event. By considering the complements of such sets we have examples where the probability is 1, but neither is the "certain" event. Many of the problems of constructing a satisfactory theory of area are seen to be the same or related to the problems of a theory of probability and the extension of the rational number system to all of the reals.

Many of the important results of modern probability theory are due to the introduction of random variables. Briefly, a random variable is a function whose values are determined by probabilities rather than values of the independent variable. Using these we construct a final example relating area, probabilities, and irrationals. Denote by X_n a random variable associated with each positive integer n, as follows, let D be a subset of R such that $A(\mathrm{D})$ is assumed to exist but is unknown, a point p is selected from R and X_1 is 0 or 1 according as p is not in D or p is in D. Repeating X_n is 0 or 1 according as the n^{th} point selected is not in D or is in D. For each X_n, the probability that $X_n=1$ is $A(\mathrm{D})$. Let

$$S_n=\frac{X_1+X_2+\cdots+X_n}{n};$$

this is the relative frequency of the number of successes to the number of trials. S_n is an approximation for $A(\mathrm{D})$, and in fact a theorem of probability theory known as "Strong Law of Large Numbers" asserts that with probability 1, $S_n-A(\mathrm{D})\rightarrow 0$ as $n\rightarrow\infty$. If $A(\mathrm{D})$ is irrational, we again have rational approximations to an irrational. Mechanically two things are needed, a method for choosing the points "randomly" and a method for determining whether or not a selected point is in D or not. The former problem is essentially solved: tables of random numbers to use for coordinates are available and programs have been written to generate random numbers on digital computers. Two examples will be given to illustrate how the second of these problems can sometimes be solved.

Example 4. Let $T=\{(x,\ y)\,|\,0\leq x\leq y,\ 0\leq y\leq 1\}$. As a subset of the unit square R, T is the set of points of R on the diagonal connecting the lower left-hand corner to the upper right corner and those lying above and to the left of it. For any point p in R, p is not in T if $x>y$. So that if p is selected by a pair of random numbers a simple comparison of the two determines the value of X_n.

Example 5. $C=\{(x,\ y)\,|\,|(x,\ y)-(\frac{1}{2},\ \frac{1}{2})|\leq\frac{1}{2}\}$. Again determining whether p is in D or not reduces to an arithmetic comparison; either $(x,\ y)$ satisfies the distance relation above or it does not. With respect to Example 4, it might be remarked that the formula for the area of a triangle does not result from that for a rectangle by piecing together two such triangles. That would be assuming that any subset of a set having area itself has area defined. Example 2 refutes this conjecture; however, the difficulty is easily surmounted by complementation.

Nearly every statement in the preceding can be extended to volume instead of area. Whether or not this interpretation of area as a probability is correct or not cannot be answered, but for some applications it seems to work quite well.

Letter to the editor

Dear Editor:

My thanks for printing the fine article "Some thoughts about curriculum revision" by Irving Adler in the November, 1963, issue. The thoughts expressed have significance in the education of the handicapped.

I teach mathematics to deaf students who are preparing for college. The ability of these students, most of them deaf from birth or infancy, to master not only mathematics but also English and all other subjects required for college entrance is in itself an amazing accomplishment, but on top of that we are continually finding untapped resources and unreached limits. In switching from traditional to modern math (UICSM) I find I am gaining two to three years in this subject. Mr. Adler's penetrating statements on the "possible" and the "impossible" and his remarks on the disastrous effects of low IQ ratings illuminate some of our successes and failures. In the education of the deaf it is so easy for poor initial performance and low mental ratings to lead to lower expectations, lower performance, and the vicious circle Mr. Adler deplores. Yet these "special" students leave school when they have "reached the limit of their potential" only to find good jobs, sometimes to make more than their teachers, and often to return to school years later with a greatly improved command of English gained in a different environment than the schoolroom in which they could learn no more. The "impossible" often becomes "possible" with a change in techniques and motivation in any subject, not just mathematics.

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