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Coregionalization by Linear Combination of Nonorthogonal Components[1]

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This paper applies the relationship between the matrix multivariate covariance and the covariance of a linear combination of a single attribute to analyze modeling with nested structures. This analysis for modeling of covariances is introduced to the multivariate case for nonorthogonal vector spatial components. Results validate the classic linear model of coregionalization for a more general case of nonorthogonality, that produces additional terms including cross-covariance in the nested structures. Linear combinations of nested structures have been applied in the frequency domain to a more general case where the coefficients are nonconstant but valid transfer functions. This allows for a tool for the production of cross-covariance and covariance models that are convolutions of valid models. An example for modeling of the hole effect is illustrated.

KEY WORDS: covariance modeling, cross-covariance, multivariate variogram, hole effect.

INTRODUCTION

For covariance modeling purposes, the stationary and ergodic random function $Z(x)$ may be assumed to be the sum of linearly independent or spatially orthogonal components $Z_u(x)$ (e.g., Journel and Huijbregts, 1978; Sandjivy, 1984). This is

$$Z(x) = \sum_{u=1}^{m} a Z_u(x) \tag{1}$$

Modeling of covariances is usually performed with nested structures, each one corresponding to a spatial component $Z_u(x)$ as

$$C(h) = \sum_{u=1}^{m} a_u^2 c_u(h) \tag{2}$$

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where a_u^2 are regionalization constant coefficients and $c_u(h)$ are valid elementary covariance models that have zero lag distance covariance equal to one and must be positive definite. Nested structures in the model of Equation (2) hold for independent spatial components $Z_u(x)$ in Equation (1). Choosing a combination of valid elementary models is somewhat arbitrary and subjective and may or may not have a relationship with the studied phenomena.

An extension to the case of several attributes modeled with a vector random function $\mathbf{Z}(x)$ is the linear model of coregionalization (LMC) (Journel and Huijbregts, 1978, p. 171). This is

$$\mathbf{C}(h) = \sum_{u=1}^{m} \mathbf{B}_u c_u(h) \tag{3}$$

where $\mathbf{B}_u$ are positive-definite coregionalization matrices. Then, the nested structures in Equation (3) are multivariate and assume the vector random function is

$$\mathbf{Z}(x) = \sum_{u=1}^{m} \mathbf{Z}_u(x) \tag{4}$$

where the spatial components or vectors $\mathbf{Z}_u(x)$ are independent to each other. Notice that each $\mathbf{Z}_u(x)$ is made of intrinsically correlated attributes (Wackernagel, 1995).

From a practical point of view, the spatial components may be thought as a way for modeling realities that physically exist and add to the total observed phenomena even though frequently they might not be identified in a unique way. Moreover, it may be reasonable to assume that in a natural process the unknown spatial components may be cross-correlated to each other. For example, cumulative spatial events on time are correlated to each other. More important is that negative cross-correlations may appear and produce more complicated shapes in the sample covariance function of the resulting random function. In this case, it may be reasonable that the sample covariance could be modeled with the inclusion of cross-covariances in the nested structures.

This paper derives the model of regionalization starting from positive-definite multivariate matrix of covariance functions for *nonorthogonal spatial components* and the corresponding nested covariance for their linear combination. In this way, the cross-covariances between linearly combined random functions are included and the method is extended to the multivariate case of a vector random function. The approach is analyzed for linear combinations in the frequency domain, leading to a generalization using linear filters. This frequency domain version of the LMC opens new possibilities for modeling cross-correlated spatial components and allows for studying the conditions for valid covariances of linear combinations. The purpose is to show the limitations of LMC and introduce more advanced modeling possibilities.

COVARIANCES OF LINEAR COMBINATIONS FOR A SINGLE ATTRIBUTE

Linear Combinations

From Myers (1983), the covariance $C_W(h)$ of a linear combination of random functions is related to the corresponding matrix covariance $\mathbf{C}(h)$ as

$$C_W(h) = \mathbf{A}^{\mathrm{T}}\mathbf{C}(h)\mathbf{A} \tag{5}$$

where $\mathbf{A}$ is a column vector of coefficients in the linear combination

$$W(x) = \mathbf{A}^{\mathrm{T}}\mathbf{Z}(x) \tag{6}$$

From Myers (1982) the covariance of a sum of two random functions is

$$C_W(h) = C_1(h) + C_2(h) + 2C_{12}(h) \tag{7}$$

A related concept is the multivariate covariance (Bourgault and Marcotte, 1991). This is

$$G(h) = \mathrm{Trace}[\mathbf{V}^{\mathrm{T}}\mathbf{C}(h)\mathbf{V}] \tag{8}$$

where $\mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{M}$ is an Euclidean metric. They present a method for identification of orthogonal spatial components. A question to be answered is about the validity of the covariances of linear combinations.

Nested Structures and the Frequency Domain

Bochner's Theorem for Nested Structures

The following analysis in the frequency domain validates modeling with nested structures and shows the possibilities of using it. Recall that the inverse Fourier transform of the spectrum or spectral density function $s(\omega)$ is the covariance $C(h)$, and ω is the frequency. Then

$$C(h) = \int e^{i\omega h} s(\omega)\, d\omega \tag{9}$$

This is Bochner's theorem expressed as a Riemann integral, and for positive-definite covariance, only $s(\omega) > 0$ shall be allowed (e.g., Bochner, 1949; Cressie, 1993). Following the Stieltjes representation Equation (9) may be written as

$$C(h) = \int e^{i\omega h}\, dS(\omega) \tag{10}$$

where $S(\omega)$ is the spectral distribution function. Because of the condition above, $S(\omega)$ has the property that $dS(\omega)/d\omega > 0$ for every ω. An additional condition is

$$\int dS(\omega) < \infty \tag{11}$$

and

$$s(\omega) = \frac{dS(\omega)}{d\omega} \tag{12}$$

The Fourier–Stieltjes representation of a random function $Z(x)$ is

$$Z(x) = \int e^{iwx}\, dZ(\omega) \tag{13}$$

(e.g., Priestley, 1981).

Consider m random functions $Z_u(x)$ each one having a Fourier–Stieltjes representation as

$$Z_u(x) = \int e^{iwx}\, dZ_u(\omega) \tag{14}$$

Let $s_u(\omega)$ be the spectral density function or spectrum of a u spatial component $Z_u(x)$. Equation (9) is still consistent if the total spectrum is split into m spectra for independent nested spatial components as in Equation (2). Then

$$C(h) = \sum_{u=1}^{m} \left[\int e^{i\omega h} s_u(\omega)\, d\omega \right] \tag{15}$$

Equation (15) assures that the nested structures provide positive-definite covariances if cross-covariances between nested random functions are zero. Moreover, it also provides methodology for generating covariances as linear combinations or nested structures.

The Role of Transfer Functions

Let $s_{uZ}(\omega)$ be the complex cross-spectrum between one spatial component $Z_u(x)$ and the random function $Z(x)$ respectively. Then, the transfer function is the linear relationship between two random functions computed as the ratio of the cross-spectrum and the spectrum. This is

$$T_u(\omega) = \frac{s_{uZ}(\omega)}{s_u(\omega)} \tag{16}$$

In general, the linear combination of spatial components can be generalized in the frequency domain introducing transfer functions $T_u(\omega)$ instead of constant coefficients. Then

$$dZ(\omega) = \sum_{u=1}^{m} T_u(\omega)\, dZ_u(\omega) \tag{17}$$

where $dZ(\omega)$ is the Fourier–Stieltjes representation of $Z(x)$. We introduce the case where the spatial components $Z_u(x)$ are not independent, and the transfer functions need to be computed in an analogous way to a linear regression in the frequency domain. This projection is valid since frequencies are independent of each other. Let $\mathbf{s}_{uZ}(\omega)$ be the complex column vector of spectra of all m spatial components $dZ_u(\omega)$ versus $dZ(\omega)$. That is each term comes from

$$s_{uZ}(\omega) = \frac{E\,[dZ(\omega)\, dZ_u(\omega)]}{d\omega} \tag{18}$$

and let $\mathbf{s}_{uv}(\omega)$ be the square matrix of spectra between the m nested spatial components. Each term is as

$$s_{uv}(\omega) = \frac{E[dZ_u(\omega)\, dZ_v^*(\omega)]}{d\omega} \tag{19}$$

Next for the sake of simplicity we consider Hermitian matrices and a regression in the frequency domain gives a vector of smoothing transfer functions.

Proposition 1. *If the orthogonal projection of spatial component random functions on a linear combination is applied in the frequency domain we have a direct linear relationship, and the transfer function is*

$$\mathbf{T}_u(\omega) = [\mathbf{s}_{uv}(\omega)]^{-1}\, \mathbf{s}_{uZ}(\omega) \tag{20}$$

Conditions for Valid Correlated Nested Structures

The above result is valid under two conditions that relate random variables at each frequency.

Proposition 2. *The cross-spectrum between the linear combination and the nested component must assure smoothing in Equation (16); then it is bounded because of Schwarz inequality, which is* $s_{uZ}(\omega) \leq \sqrt{s_u(\omega)s(\omega)}$. *And the cross-spectrum between a pair of nested components that assures the solution of Equation (20) is bounded as* $s_{uv}(\omega) \leq \sqrt{s_u(\omega)s_v(\omega)}$. *The above allows for a result as given in the following proposition.*

Proposition 3. *Equation (15) can be generalized as*

$$C(h) = \sum_{v=1}^{m} \sum_{u=1}^{m} \left[\int e^{i\omega h} T_u(\omega) T_v(\omega) s_{uv}(\omega)\, d\omega \right] \tag{21}$$

Since the spatial components are nonorthogonal and have cross-covariances, Equation (21) is an extension of Bochner's theorem for a linear combination of random functions which forms a multivariate space where spectral structures follow Schwarz inequality in the frequency domain.

As reported in Chiles and Delfiner (1999), the use of Schwarz inequality may encounter limitations in the spatial domain. Notice that Schwarz inequality in the spatial domain may not always guarantee positive definite multivariate covariance matrices. However, it does in the frequency domain because the frequencies are mutually independent.

As has been shown above, the Fourier–Stieltjes representation provides a more complete view of the problem of nested structures. This has been shown to be the same as linear combinations of random functions in the frequency domain. A simplification of Equation (21) for independent spatial components is

$$C(h) = \sum_{u=1}^{m} \left[\int e^{i\omega h} T_u^2(\omega) s_u(\omega)\, d\omega \right] \tag{22}$$

To check for the conditions in the propositions above some tools are suggested. The coherency should be bounded and play the role of linear correlation for each frequency. For symmetric matrices of nested structures, a verification tool is that all eigenvalues must be positive and larger than zero. Proposition 3 may lead to use of a fast Fourier transform to obtain valid covariances.

From Bochner (1949), the inverse Fourier transform of a Hermitian matrix of joint spectra is a positive definite matrix of covariance. The numerical application of Bochner's theorem for numerical modeling of covariances has been introduced by Yao and Journel (1998). As a fundamental difference, the approach proposed here is based on standard valid models for covariance functions.

The Spectrum of the Linear Combination With Constant Coefficients

For the scalar linear model of regionalization, or commonly used nested structures, the transfer functions are constant for all frequencies. As an example,

consider the case of constant transfer functions for just two nested structures. This implies the relationship

$$s(\omega) = a^2 s_u(\omega) + b^2 s_v(\omega) + ab\, s_{uv}(\omega) + ba\, s_{vu}(\omega) \tag{23}$$

which after inverse Fourier transform yields a linear model of regionalization that includes cross-covariances. Exclusion of the cross-spectra reduces Equation (23) to the classic model with orthogonal nested structures. Notice that such a limitation of using constants is overcome if modeling is done numerically, if not analytically, in the frequency domain.

Covariances for Nonorthogonal Nested Structures

Covariance of a Single Attribute

In the spatial domain the above model of regionalization involves valid convolutions. Thus, covariances made of nesting linear filter transformations in the frequency domain are transformed into the physical space. The simplest case as explained before is using constant transfer functions to avoid convolutions in the spatial domain.

First consider the classic case of m spatially nonorthogonal scalar random functions or spatial components $Z_u(x)$, where $u = \{1, 2, \ldots, m\}$. The random functions are second-order stationary and they have the following covariance matrix

$$\mathbf{C}_s(h) = \begin{pmatrix} C_{11}(h) & C_{12}(h) & \cdots & C_{1m}(h) \\ \vdots & & \ddots & \vdots \\ C_{m1}(h) & C_{m2}(h) & \cdots & C_{mm}(h) \end{pmatrix} \tag{24}$$

For example, $C_{11}(h)$ is the covariance for the first spatial component. The off diagonal terms are cross-covariances between the spatial components. The covariance for a single attribute modeled with nested structures is

$$C(h) = \mathbf{A}^{\mathrm{T}} \mathbf{C_s}(h) \mathbf{A} \tag{25}$$

where $\mathbf{A}$ is a column vector of coefficients. This is an application of Equation (5) to nested structures. A simple example for just two nested structures is

$$C(h) = [a\ b]^{\mathrm{T}} \begin{bmatrix} c_1(h) & c_{12}(h) \\ c_{21}(h) & c_2(h) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{26}$$

This yields,

$$C(h) = ac_1(h) + bc_2(h) + 2abc_{12}(h) \tag{27}$$

This result looks the same as the classic covariance for the sum of two random functions but in this case is expressed as a model including cross-covariance in the nested structures. This last formula shows that two nonorthogonal random functions can be modeled adding the cross-covariance. Notice Equation (27) is valid only under the propositions above. The asymmetry of the cross-covariance matrix may be considered, and the covariance is

$$C(h) = ac_1(h) + bc_2(h) + abc_{12}(h) + bac_{21}(h) \tag{28}$$

The multivariate covariance $G(h)$ as Equation (8) is equivalent to the classic use of nested structures. The particular case where $\mathbf{C_s}(h)$ is a diagonal matrix reduces Equation (27) to a model of independent nested structures.

COVARIANCES OF LINEAR COMBINATIONS FOR MULTIPLE ATTRIBUTES

The Multivariate Case and the Frequency Domain

The Complete Matrix of Spectra for the Spatial Components

A second-order stationary vector random function $\mathbf{Z}(x)$ can be represented by the Fourier–Stieltjes integral $d\mathbf{Z}(\omega)$ extending Equation (13). A complete matrix of multivariate spectra $\mathbf{s}_s(h)$ can be introduced made of matrices of spectra for each of the p attributes, and composed by the spatial components. This is

$$\mathbf{s}_s(h) = \begin{pmatrix} \begin{pmatrix} s^1_{11}(\omega) & s^1_{12}(\omega)\cdots s^1_{1m}(\omega) \\ \vdots & \ddots \quad \vdots \\ s^1_{m1}(\omega) & s^1_{m2}(\omega)\cdots s^1_{mm}(\omega) \end{pmatrix} & \cdots & \begin{pmatrix} s^{1p}_{11}(\omega) & s^{1p}_{12}(\omega)\cdots s^{1p}_{1p}(\omega) \\ \vdots & \ddots \quad \vdots \\ s^{1m}_{m1}(\omega) & s^{1m}_{m2}(\omega)\cdots s^{1m}_{mp}(\omega) \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \begin{pmatrix} s^{p1}_{11}(\omega) & s^{p1}_{12}(\omega)\cdots s^{p1}_{1p}(\omega) \\ \vdots & \ddots \quad \vdots \\ s^{p1}_{m1}(\omega) & s^{p1}_{m2}(\omega)\cdots s^{p1}_{mm}(\omega) \end{pmatrix} & \cdots & \begin{pmatrix} s^{p}_{11}(\omega) & s^{p}_{12}(\omega)\cdots s^{p}_{1p}(\omega) \\ \vdots & \ddots \quad \vdots \\ s^{p}_{m1}(\omega) & s^{p}_{m2}(\omega)\cdots s^{p}_{mm}(\omega) \end{pmatrix} \end{pmatrix} \tag{29}$$

Another matrix of cross-spectra between all spatial components for all attributes versus the global vector random function is

$$\mathbf{s}_{sZ}(h) = \begin{pmatrix} \begin{pmatrix} s_{1Z}^{1}(h) \\ \vdots \\ s_{mZ}^{1}(h) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} s_{1Z}^{p1}(h) \\ \vdots \\ s_{mZ}^{p1}(h) \end{pmatrix} \end{pmatrix} \tag{30}$$

The Role of Transfer Functions

Considering the case where the real $\mathbf{s}_s(h)$ is Hermitian, a matrix of valid transfer functions is

$$\mathbf{T}(\omega) = [\mathbf{s}_s(\omega)]^{-1}\, \mathbf{s}_{sZ}(\omega) \tag{31}$$

Notice that this is an estimation to obtain a single scalar random function in the frequency domain. The complete linear combination spectrum in the frequency domain is

$$s(\omega) = \mathbf{T}^{\mathrm{T}} \mathbf{s}_s(\omega) \mathbf{T} \tag{32}$$

The corresponding covariance of the inverse Fourier transform of this linear combination is a scalar measure that combines all attributes and all correlated nested structures. See the analogy with the multivariate covariance in the spatial domain (Bourgault and Marcotte, 1991).

Nested Structures for a Vector Linear Combination in the Frequency Domain

From the result obtained above it is obvious that combining the spectra by blocks for the attributes including all spatial components may produce the matrix of spectra for the vector random function. Equation (29) is split into blocks or the transfer functions arrayed in an adequate way. This means each block is treated separately for all nested structures. This is similar to the univariate case. Once Equation (31) has been solved the matrix of transfer functions is split in vectors for each attribute $\mathbf{T}_i(\omega)$ and then

$$s_{ij}(\omega) = \mathbf{T}_i^{\mathrm{T}} \mathbf{s}_{ij}(\omega) \mathbf{T}_j \tag{33}$$

where $\mathbf{s}_{ij}(\omega)$ is one of the block matrices in Equation (29), and the resulting cross-spectra is for the specific attributes $s_{ij}(\omega)$. If $\mathbf{s}_{ij}(\omega)$ is in the diagonal, $\mathbf{T}_i = \mathbf{T}_j$ and the result is the spectrum for a given attribute. This result shows that modeling of multivariate linear combinations requires a complete knowledge of cross-covariances. Thus, the problem of computing valid matrices of covariances of a multivariate linear combination is conditioned to the solution of Equation (31).

This is similar to the model in the physical spatial domain. In fact, when the transfer functions are constant for all frequencies and the spectra for each spatial scale proportional among all attributes, the problem is reduced to a linear model of coregionalization for correlated spatial components.

General Form of the LMC

The linear combination, of m nonorthogonal spatial component random functions, is applied to a second-order stationary vector random function of p attributes. A covariance matrix $\mathbf{C}_Z(h)$ of functions characterizes a second-order stationary vector random function. Each vector spatial component is also characterized by a matrix covariance $\mathbf{C}_u(h)$. The spatial components also have a covariance matrix $\mathbf{C}_s(h)$ that is an array for blocks of $\mathbf{C}_u(h)$. This is

$$\mathbf{C}_s(h) = \begin{pmatrix} \mathbf{C}_{11}(h) & \mathbf{C}_{12}(h) \cdots \mathbf{C}_{1m}(h) \\ \vdots & \ddots & \vdots \\ \mathbf{C}_{m1}(h) & \mathbf{C}_{m2}(h) \cdots \mathbf{C}_{mm}(h) \end{pmatrix} \tag{34}$$

The diagonal terms of $\mathbf{C}_s(h)$ are the multivariate matrices of covariances for one spatial component, and the off diagonal terms the cross-covariance matrices for two spatial components. The off diagonal terms are included to show the general matrix of covariance for spatial components of linear combinations in space. This is

$$\mathbf{C}_s(h) = \begin{pmatrix} \begin{pmatrix} C_{11}^{1}(h) & C_{12}^{1}(h) \cdots C_{1p}^{1}(h) \\ \vdots & \ddots & \vdots \\ C_{p1}^{1}(h) & C_{p2}^{1}(h) \cdots C_{pp}^{1}(h) \end{pmatrix} & \cdots & \begin{pmatrix} C_{11}^{1m}(h) & C_{12}^{1m}(h) \cdots C_{1p}^{1m}(h) \\ \vdots & \ddots & \vdots \\ C_{p1}^{1m}(h) & C_{p2}^{1m}(h) \cdots C_{pp}^{1m}(h) \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \begin{pmatrix} C_{11}^{m1}(h) & C_{12}^{m1}(h) \cdots C_{1p}^{m1}(h) \\ \vdots & \ddots & \vdots \\ C_{p1}^{m1}(h) & C_{p2}^{m1}(h) \cdots C_{pp}^{m1}(h) \end{pmatrix} & \cdots & \begin{pmatrix} C_{11}^{m}(h) & C_{12}^{m}(h) \cdots & C_{1p}^{m}(h) \\ \vdots & \ddots & \vdots \\ C_{p1}^{m}(h) & C_{p2}^{m}(h) \cdots & C_{pp}^{m}(h) \end{pmatrix} \end{pmatrix} \tag{35}$$

Define a matrix of coefficients for each spatial component $\mathbf{A}_u$. In general, $\mathbf{A}_u$ matrices do not need to be diagonal and the linear combination is equivalent to rotation. Recall here that the $\mathbf{Z}_u(x)$ random functions spatial components are cross-correlated at all lag distances.

A matrix (vector of matrices) $\mathbf{A}$ with the m coefficient matrices $\mathbf{A}_u$ can be constructed in the following way

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{pmatrix} \tag{36}$$

Now, extend the linear combinations relationship, given for a scalar attribute, to the multivariate attribute case. Then, a matrix covariance for the vector linear combination is obtained as

$$\mathbf{C}(h) = \mathbf{A}^{\mathrm{T}}\mathbf{C}_s(h)\mathbf{A} \tag{37}$$

or

$$\mathbf{C}(h) = (\mathbf{A}_1\ \mathbf{A}_2\ \cdots \mathbf{A}_m) \begin{pmatrix} \mathbf{C}_{11}(h) & \mathbf{C}_{12}(h) \cdots \mathbf{C}_{1m}(h) \\ \vdots & \ddots \qquad \vdots \\ \mathbf{C}_{m1}(h) & \mathbf{C}_{m2}(h) \cdots \mathbf{C}_{mm}(h) \end{pmatrix} \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{pmatrix} \tag{38}$$

If an additional simplification is introduced by making the structures within each vector spatial component proportional, the result is the matrix model introduced as a "linear model of coregionalization for nonorthogonal spatial components." It is a more general case than the classic linear model of coregionalization LMC. To see this, consider an example with just two spatial components and three attributes. Assuming intrinsic correlation between attributes at a single nested structure. This is

$$\mathbf{C}(h) = [\mathbf{A}_1]^{\mathrm{T}}\mathbf{A}_1 c_1(h) + [\mathbf{A}_2]^{\mathrm{T}}\mathbf{A}_2 c_2(h) + [\mathbf{A}_1]^{\mathrm{T}}\mathbf{A}_2 c_{12}(h) + [\mathbf{A}_2]^{\mathrm{T}}\mathbf{A}_1 c_{21}(h) \tag{39}$$

The classic coregionalization matrices $\mathbf{B}^u = [\mathbf{A}^u]^{\mathrm{T}}\,\mathbf{A}^u$ are identified. Now, we introduce cross-coregionalization matrices for the cross terms and in general this is

$$\mathbf{B}^{uv} = \left[\mathbf{A}^u\right]^{-1}\mathbf{A}^v \tag{40}$$

The functions $c_{ij}(h)$ are elementary covariances for $i = j$, and elementary cross-covariances for $i \neq j$. Note that the classic linear model of coregionalization assumes that the cross-covariances between the nested components are zero. This is a condition that the spatial components need be directly and cross-attribute orthogonal to make the classic LMC valid.

EXAMPLE

The following example illustrates a simple case of hole effect for one attribute modeled with correlated nested structures I and II with multivariate matrix of cross-covariance

$$C(h) = \begin{bmatrix} c_{\mathrm{I}}(h) & c_{\mathrm{I-II}}(h) \\ c_{\mathrm{II-I}}(h) & c_{\mathrm{II}}(h) \end{bmatrix}$$

The covariance functions are $c_{\mathrm{I}}(h) = \exp(-h/150)$, $c_{\mathrm{II}}(h) = \exp(-h/30)$ and the cross-covariance $c_{\mathrm{I-II}}(h) = -0.65\exp(-h/100)$ between the nested structures is symmetric. Notice the covariances and cross-covariances are not proportional. To check the matrix we proceed computing the real part of the spectra and cross-spectra by Fast Fourier Transform for positive h. Dividing the cross-spectra by the product of the square root of the two spectra the computed coherency satisfies Schwarz inequality for frequencies (see Fig. 1). The nested model in this case is the covariance of the sum. This is

$$c_s(h) = \exp\left(\frac{-h}{150}\right) + \exp\left(\frac{-h}{30}\right) - 0.65 \times 2\ \exp\left(\frac{-h}{100}\right)$$

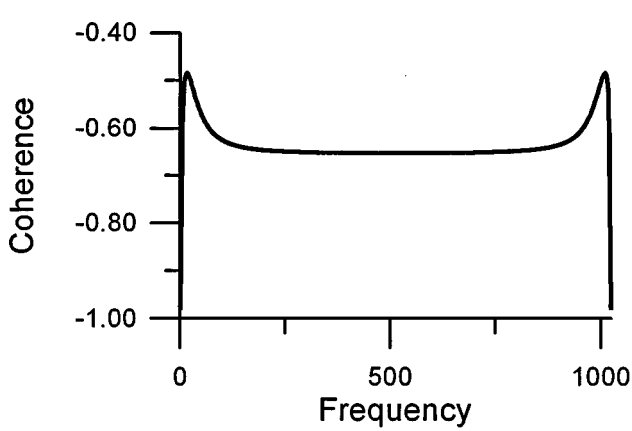

Figure 1. Coherency for the two nested components.

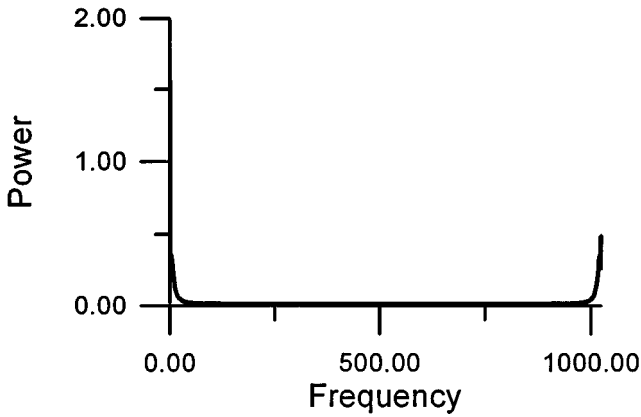

Figure 2. Power spectrum of the linear combination.

Following Bochner's theorem, the spectrum of this covariance exists and is positive for all frequencies. Notice that values close to zero are not exactly zeroes (see Fig. 2). The corresponding variogram is $\gamma_s(h) = 0.7 - [c_{\mathrm{I}}(h) + c_{\mathrm{II}}(h) + 2c_{\mathrm{I-II}}(h)]$ and is plotted in Figure 3. Notice that in general the hole effect may be valid up to some negative linear correlation between the components. If the scalar unit variances and the corresponding cross-covariance are replaced by matrices of

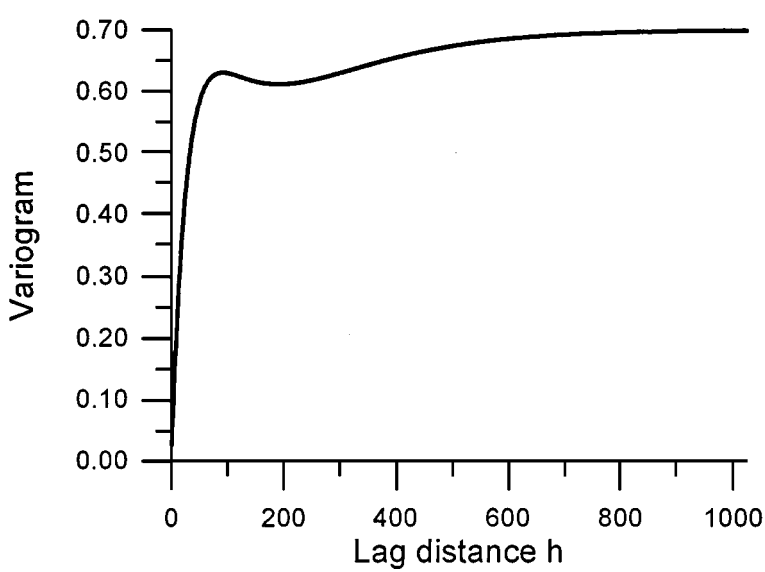

Figure 3. Variogram of the linear combination (hole effect).

coregionalization the extension for a multivariate case of a vector random function is straightforward, provided that the coherency as in Figure 1 is between -1 and 1 for all combinations of nested components and attributes.

DISCUSSION

This paper has developed a theory for the LMC which has been analyzed including correlated spatial components. It provides additive hints about the strong conditions of orthogonality that apply to the classic LMC. By considering linear combinations of correlated random functions, the model allows inclusion of nonorthogonality between the spatial components. This has the importance that negative cross-covariance may be included in the LMC.

The approach also uses the possibilities for more exact modeling by treating spectra instead of covariances. This allows for transfer functions that constrain models to valid positive definite matrices. Bochner's theorem is also combined with Schwarz's inequality in the frequency domain providing guidelines for analysis of spectra. The use of the frequency domain shows possibilities for more sophisticated numerical modeling and it also supports the classic LMC. Linear combinations in the frequency domain and nonconstant transfer functions lead to model covariances with nested convolutions. This comes from the linear filter with transfer functions, and seems applicable to the generation of models of covariances and cross-covariances. The case of constant transfer functions validates the classic LMC, and in the case of correlated spatial components the LMC remains valid with the addition of cross-covariance terms. New cross-regionalization matrices are introduced generalizing the LMC.

In some real cases, a linear combination may be required to couple several correlated attributes in a single model. For example, in mining two or more commodities may be linearly combined to provide an equivalent attribute that is used in the economic evaluation. In hydrology, water content in the vadose zone changes due to correlated events in time domain. In the oil industry data integration is frequently done using linear combinations. In general, geological phenomena may be due to the overlapping of some nonorthogonal events.

As a simple example the hole effect is modeled with two negatively correlated nested components with different ranges. It shows that the validity of the covariance of the sum or a linear combination is in general restricted to the formulated propositions that integrate Bochner's theorem with Schwarz inequality.

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