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Multivariate Correlation in the Framework of Support and Spatial Scales of Variability[1]

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This paper extends the concept of dispersion variance to the multivariate case where the change of support affects dispersion covariances and the matrix of correlation between attributes. This leads to a concept of correlation between attributes as a function of sample supports and size of the physical domain. Decomposition of dispersion covariances into the spatial scales of variability provides a tool for computing the contribution to variability from different spatial components. Coregionalized dispersion covariances and elementary dispersion variances are defined for each multivariate spatial scale of variability. This allows the computation of dispersion covariances and correlation between attributes without integrating the cross-variograms. A correlation matrix, for a second-order stationary field with point support and infinite domain, converges toward constant correlation coefficients. The regionalized correlation coefficients for each spatial scale of variability, and the cases where the intrinsic correlation hypothesis holds are found independent of support and size of domain. This approach opens possibilities for multivariate geostatistics with data taken at different support. Two numerical examples from soil textural data demonstrate the change of correlation matrix with the size of the domain. In general, correlation between attributes is extended from the classic Pearson correlation coefficient based on independent samples to a most general approach for dependent samples taken with different support in a limited domain.

KEY WORDS: dispersion covariances, Pearson correlation, multivariate geostatistics.

INTRODUCTION

Regionalized random functions describing the behavior of spatial dependent attributes may be multivariate. The collected data will be in $\Re^d$ physical space represented by n sampling locations with p attributes (e.g., soil variables). The space of the attributes has a dimension less than p because of their correlation. A $p \times p$ correlation matrix of the Pearson correlation coefficients is classically computed as an estimate of the population correlation between attributes. Such

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an approach assumes independent samples or a pure nugget variogram. From geostatistics, each of the n sampling locations or any other point in the studied field is considered as a multivariate random variable. Then random variables are governed by a random function which following a linear decomposition into spatial scales of variability can be considered as the sum of q independent random functions each governing one scale of variability (Journel and Huijbregts, 1978; Wackernagel, 1985, 1988). In this paper, we are concerned with the role of spatial scales of variability, sample supports, and size of the domain when computing classic correlation matrices for attributes. We also clarify the difference between the size of the domain and spatial scale of variability.

Univariate autocorrelation for second-order stationary random functions can be computed from the variogram model. The normalized autocovariance is the autocorrelation function which is equivalent to the correlogram that quantifies the similarities between point locations separated by a vector **h**. The univariate correlogram can be used to find an integral scale in $\Re^1$ for a normalized $\rho(h)$ (Russo and Jury, 1987):

$$J_1 = \int_0^\infty \rho(h)dh \tag{1}$$

Such an approach provides information from the average correlogram between pairs of samples from zero to infinity.

In the multivariate case the correlation between attributes is classically estimated in statistics with the Pearson correlation coefficient:

$$r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}s_{jj}}} \tag{2}$$

where i and j are two attributes with sample variances s_{ii}, s_{jj}, and with sample covariance s_{ij}. The Pearson correlation coefficients consider independent samples taken with point support.

A regionalized correlation coefficient is given by Wackernagel (1985) and (1988) as a spatial approach for the correlation between two attributes

$$r_{ij}^u = \frac{b_{ij}^u}{\sqrt{b_{ii}^u b_{jj}^u}} \tag{3}$$

where u is a particular spatial scale of variability and b_{ij}^u the elements of the matrix of coregionalization $\mathbf{B}^u$

$$\mathbf{B}^u = \begin{bmatrix} b_{11}^u & & b_{1p}^u \\ b_{21}^u & \cdots & b_{2p}^u \\ & \vdots & \\ b_{p1}^u & \cdots & b_{pp}^u \end{bmatrix} \tag{4}$$

In the multivariate geostatistics approach, coregionalization matrices from the linear model of coregionalization are the sills for bounded components of a nested multivariate variogram which can be computed by the iterative algorithm of Goulard and Voltz (1992) or simultaneous diagonalization explained by Myers (1994) and Xie and Myers (1995). For cross-covariograms, coregionalization entries can be interpreted as covariances between two attributes (e.g., soil variables). As in the case before, point support and infinite domains are assumed.

As can be observed from the explanation above, regionalized correlation coefficients depend on computed scales of variability (Myers, 1997). They provide information that can be computed from the model matrix variogram but they cannot be measured directly from unfiltered data from the field. Filtering of the spatial scales is sometimes applied to get data in the required spatial scales of variability (Wackernagel, 1985). Such filtered data has spatial components $\mathbf{z}^u$:

$$\mathbf{z} = \sum_{u=1}^{q} \mathbf{z}^u \tag{5}$$

Components $\mathbf{z}^u$ should show a correlation for the attributes which is equal to the regionalized correlation coefficients r_{ij}^u.

To understand spatial scales of variability as a different but related concept to size of a domain, dispersion variances should be analyzed. It is common practice to think that changing the sample support and the size of the domain is directly equivalent to changing spatial scales of variability. This is not true. In the present paper, we will show how sample support, size of the domain, and spatial scales of variability are related. Also we show how these considerations are directly related to correlation matrices between attributes. For such a goal, dispersion variance is invoked as an approach to the problem. Finally, the paper addresses the effect of change of support in the estimation of correlation between attributes.

Linear Coregionalization

The multivariate linear model of coregionalization allows a better understanding of multivariate spatial variability by decomposing it in some independent spatial scales of variability (Journel and Huijbregts, 1978; Wackernagel, 1985, 1988). The multivariate variogram of the sum of independent processes is equal to the sum of the variograms. Then

$$\overline{\Gamma}_z = \sum_{u=1}^{q} \overline{\Gamma}_z^u \tag{6}$$

or

$$\overline{\Gamma}_z = \begin{bmatrix} \gamma_{11} & & \gamma_{1p} \\ \gamma_{21} & \cdots & \gamma_{2p} \\ & \vdots & \\ \gamma_{p1} & \cdots & \gamma_{pp} \end{bmatrix} = \begin{bmatrix} \gamma_{11}^j & & \gamma_{1p}^j \\ \gamma_{21}^j & \cdots & \gamma_{2p}^j \\ & \vdots & \\ \gamma_{p1}^j & \cdots & \gamma_{pp}^j \end{bmatrix} + \cdots + \begin{bmatrix} \gamma_{11}^q & & \gamma_{1p}^q \\ \gamma_{21}^q & \cdots & \gamma_{2p}^q \\ & \vdots & \\ \gamma_{p1}^q & \cdots & \gamma_{pp}^q \end{bmatrix} \tag{7}$$

Applying the linear model of coregionalization

$$\overline{\Gamma}_z = \sum_{u=1}^{q} \mathbf{B}^u g^u(h) \tag{8}$$

$$\overline{\Gamma}_z = \begin{bmatrix} b_{11}^j & & b_{1p}^j \\ b_{21}^j & \cdots & b_{2p}^j \\ & \vdots & \\ b_{p1}^j & \cdots & b_{pp}^j \end{bmatrix} g^j(h) + \cdots + \begin{bmatrix} b_{11}^q & & b_{1p}^q \\ b_{21}^q & \cdots & b_{2p}^q \\ & \vdots & \\ b_{p1}^q & \cdots & b_{pp}^q \end{bmatrix} g^q(h) \tag{9}$$

Dispersion Variances

Dispersion variance is a central topic in geostatistics. It rises from the basic fact that variance between dependent samples is reduced when a larger physical size of each sample (sample support) is taken. Fundamental explanations based on the works of Krige and Matheron can be found in Journel and Huijbregts (1978) and Rendu (1978). A discussion is provided by Zhang, Warrick, and Myers (1987) and Myers (1993). Analytical procedures and numerical methods for computing dispersion variances are also provided by Journel and Huijbregts. An early work by Smith (1938) demonstrates practically that variance is lost through the mechanical averaging produced by increasing the physical size of the samples v called the size of nonpoint support. An example of the effect of size of domain is found in Miesch (1975), where Krige's formula is experimentally demonstrated. All these studies have restricted the analysis of dispersion variances to the univariate case. Thus, total variance in a large domain or block V can be considered fixed and invariant for constant size V. Dispersion variance is the difference between total variance in a domain and variance lost by taking nonpoint samples. However, when sample support v is held constant and the domain V becomes larger, dispersion variance is expected to increase. Depending on the stationarity of the field, such an increase might lead to a limit. Variance inside a volume is computed by the mean variogram inside a volume. From the work of Krige and Matheron, it is common knowledge that dispersion variance for the univariate case is:

$$D^2(v|V) = \bar{\gamma}(V,V) - \bar{\gamma}(v,v) \tag{10}$$

where

$$\bar{\gamma}(V,V) = \frac{1}{V^2}\int_V dx \int_{V'} \gamma(x - x')dx' \tag{11}$$

The support of block or domain V and the support of the pixel elements or samples v belong to a physical $\Re^d$ space of dimension d.

THEORY

Multivariate Dispersion Covariances

Now we extend dispersion variance to the multivariate case. To compute the dispersion covariance matrix for the p attributes from the multivariate variogram $\overline{\Gamma}_z$ we define a multivariate dispersion covariance matrix $\mathbf{D}^2(v|V)$ by

$$\mathbf{D}^2(v|V) = \overline{\overline{\Gamma}}_z(V,V) - \overline{\overline{\Gamma}}_z(v,v) \tag{12}$$

where $\overline{\overline{\Gamma}}_z(v,v')$ is the mean value of the function multivariate variogram $\overline{\Gamma}_z(\mathbf{h})$ when one extremity of $\mathbf{h}$ moves in a volume v and the other extremity moves in v'. Then, $\mathbf{D}^2(v|V)$ can also be written:

$$\mathbf{D}^2(v|V) = \frac{1}{V^2}\int_V dx \int_{V'} \overline{\Gamma}_z(x - x')dx' - \frac{1}{v^2}\int_v dx \int_{v'} \overline{\Gamma}_z(x - x')dx' \tag{13}$$

The diagonal entries of matrix $\mathbf{D}^2(v|V)$ correspond to the univariate dispersion variances computed for each attribute. However, the off-diagonal entries correspond to a new definition that we call dispersion covariances between the attributes. In this way, the concept of regularization of variograms can be extended to regularization of cross-variograms.

Dispersion Covariances and Spatial Scales

To examine what happens when decomposing the dispersion variance matrix into spatial scales of variability, we invoke the linear model of coregionalization and substitute (8) into Equation (13). The result is:

$$\mathbf{D}^2(v|V) = \frac{1}{V^2}\int_V dx \int_{V'} \sum_{u=1}^{q} \mathbf{B}^u g^u(x-x')dx' - \frac{1}{v^2}\int_v dx \int_{v'} \sum_{u=1}^{q} \mathbf{B}^u g^u(x-x')dx' \tag{14}$$

For q spatial scales of variability, this is:

$$\mathbf{D}^2(v|V) = \frac{1}{V^2}\int_V dx\left[\mathbf{B}^j \int_{V'} g^j(x-x')dx' + \cdots + \mathbf{B}^q \int_{V'} g^q(x-x')dx'\right] - \frac{1}{v^2}\int_v dx\left[\mathbf{B}^j \int_{v'} g^j(x-x')dx' + \cdots + \mathbf{B}^q \int_{v'} g^q(x-x')dx'\right] \tag{15}$$

Coregionalization matrices are constant so they can be moved out of the integrals to give:

$$\mathbf{D}^2(v|V) = \left[\frac{\mathbf{B}^j}{V^2}\int_V dx \int_{V'} g^j(x-x')dx' + \cdots + \frac{\mathbf{B}^q}{V^2}\int_V dx \int_{V'} g^q(x-x')dx'\right] - \left[\frac{\mathbf{B}^j}{v^2}\int_v dx \int_v g^j(x-x')dx' + \cdots + \frac{\mathbf{B}^q}{v^2}\int_v dx \int_{v'} g^q(x-x')dx'\right] \tag{16}$$

or

$$\mathbf{D}^2(v|V) = \sum_{u=1}^{q} \left(\overline{\overline{\Gamma}}^u(V,V) - \overline{\overline{\Gamma}}^u(v,v)\right) \tag{17}$$

We define coregionalized dispersion covariance matrices for each spatial scale by $\mathbf{D}^{u2}(v|V)$ with

$$\mathbf{D}^{u2}(v|V) = \overline{\overline{\Gamma}}^u(V,V) - \overline{\overline{\Gamma}}^u(v,v) \tag{18}$$

From Equation (17), the multivariate matrix of dispersion covariance for a domain or block V is the sum of the dispersion covariances for the spatial scales of variability, i.e.,

$$\mathbf{D}^2(v|V) = \sum_{u=1}^{q} \mathbf{D}^{u2}(v|V) \tag{19}$$

Elementary variograms $g^u(h)$ from the coregionalization model can be used to compute dispersion covariances with $d^{u2}(v|V)$ as the difference between mean elementary variogram functions, to give from Equations (16)–(19) the result

$$\mathbf{D}^{u2}(v|V) = \mathbf{B}^u(\bar{g}^u(V,V) - \bar{g}^u(v,v)) = \mathbf{B}^u d^{u2}(v|V) \tag{20}$$

Thus, the ratio between the coregionalized dispersion covariance and the coregionalization matrix for scale u is given by $d^{u2}(v|V)$. It will be called elementary dispersion variance because it is computed from the elementary variogram.

Correlation Coefficients Under Varying Support

Dispersion covariances are the difference between average cross-variograms computed for supports v and V. Therefore, $\mathbf{D}^2(v|V)$ can be normalized:

$$\mathbf{R}(v|V) = ((\mathbf{D}^2(v|V))(\mathbf{S}^2)^{-1/2})^T(\mathbf{S}^2)^{-1/2} \tag{21}$$

where $(\mathbf{S}^2)^{1/2}$ is a dispersion standard deviations diagonal matrix constructed with the root square of the diagonal terms of the dispersion covariance matrix $\mathbf{D}^2(v|V)$. When computed for the Z global random function, $\mathbf{R}(v|V)$ is a matrix estimator of correlation that accounts for sample support and size of domain. Therefore, Equation (21) is a general form of the classic Pearson correlation coefficient. Also $\mathbf{R}(v|V)$ is an estimator closer to reality than the traditional estimator which is a particular case where samples are point support and independent. For second order stationarity, $\mathbf{R}(v|V)$ converges towards a constant correlation coefficient as the domain gets much larger than the largest range scale. If cross-variograms do not reach a sill or the drift is not proportional for the attributes, the correlation does not reach a constant. If an element of matrix $\mathbf{R}(v|V)$ is plotted as a function of V for a constant v, we get a correlation graph as a function of size of domain V. As will be shown later, the correlation structure changes as V increases except for the case when the intrinsic hypothesis holds for correlation between attributes.

Now we can compute the dispersion covariances and a restricted regionalized correlation $\mathbf{R}^u(v|V)$ for each individual scale. At first glance, the regionalized correlation coefficients r^u_{ij} defined by Wackernagel (1985) look like $\mathbf{R}^u(v|V)$ when V is a domain much larger than the correlation length and v is point support $\mathbf{R}^u(0|\infty)$. The result is

$$\mathbf{R}^u(v|V) = ((\mathbf{D}^{u2}(v|V))(\mathbf{S}^{u2})^{-1/2})^T(\mathbf{S}^{u2})^{-1/2} \tag{22}$$

From Equation (20), $\mathbf{R}^u(v|V)$ is

$$\mathbf{R}^u(v|V) = ((\mathbf{B}^u d^{u2}(v|V))(d^{u2}(v|V)\mathbf{b}^u)^{-1/2})^T(d^{u2}(v|V)\mathbf{b}^u)^{-1/2} \tag{23}$$

where $(\mathbf{b}^u)^{-1/2}$ is the diagonal matrix of dispersion standard deviations for each scale of variability. As the $d^{u2}(v|V)$ terms cancel out, $\mathbf{R}^u(v|V)$ is reduced to the constant matrix of coregionalized correlation independent of support:

$$\mathbf{R}^u(v|V) = (\mathbf{B}^u(\mathbf{b}^u)^{-1/2})^T(\mathbf{b}^u)^{-1/2} \tag{24}$$

This leads to:

$$\mathbf{R}^u(v|V) = \mathbf{R}^u(0|\infty) = \mathbf{R}^u \tag{25}$$

for any $v < V$. Thus, the interesting result is that correlation at each spatial scale of variability is independent of the size of domain and support.

The next step is of course to find the relationship between the regionalized correlation coefficients $\mathbf{R}^u$ and the general support dependent correlation $\mathbf{R}(v|V)$ for Z. From Equations (19) and (21)

$$\mathbf{R}(v|V) = \left(\sum_{u=1}^{q} \mathbf{D}^{u2}(\mathbf{S}^2)^{-1/2}\right)^T (\mathbf{S}^2)^{-1/2} \tag{26}$$

Computing $\mathbf{D}^{u2}(v|V)$ from Equation (22) leads to

$$\mathbf{D}^{u2}(v|V) = [\mathbf{R}^u(v|V)(\mathbf{S}^{u2})^{1/2}]^T(\mathbf{S}^{u2})^{1/2} \tag{27}$$

$$\mathbf{R}(v|V) = \sum_{u=1}^{q} [[[\mathbf{R}^u(v|V)(\mathbf{S}^{u2})^{1/2}]^T(\mathbf{S}^{u2})^{1/2}](\mathbf{S}^2)^{-1/2}]^T(\mathbf{S}^2)^{-1/2} \tag{28}$$

Computing the standard deviations from the coregionalized dispersion matrices, Equation (20) gives

$$\mathbf{R}(v|V) = \sum_{u=1}^{q}\left[d^{u2}(v|V)[[\mathbf{R}^u\mathbf{b}^{u(1/2)}]^T\mathbf{b}^{u(1/2)}]\left[\sum_{u=1}^{q} d^{u2}(v|V)\mathbf{b}^u\right]^{-1/2}\right]^T \cdot \left[\sum_{u=1}^{q} d^{u2}(v|V)\mathbf{b}^u\right]^{-1/2} \tag{29}$$

Note that this time the dispersion variances do not cancel out. Introducing the last two terms into the summation and defining "weight" matrices $\mathbf{W}^u$ gives

$$\mathbf{W}^u = (d^{u2}(v|V)\mathbf{b}^u)^{1/2}\left[\sum_{u=1}^{q}(d^{u2}(v|V)\mathbf{b}^u)\right]^{-1/2} = \mathbf{S}^u\mathbf{S}^{-1} \tag{30}$$

and

$$\mathbf{R}(v|V) = \sum_{u=1}^{q}[(\mathbf{R}^u\mathbf{W}^u)^T\mathbf{W}^u] \tag{31}$$

From these equations, we observe that $\mathbf{R}(v|V)$ depends on dispersion variances but not on total dispersion covariances. So sample support does not affect the correlation structure inside each spatial scale u but it does affect the correlation for the combined random function Z. Note this result may allow computation of the correlation after mixing independent populations u. So, the attributes could be measured in separated physical domains as well.

$\mathbf{R}(v|V)$ does not depend on dispersion variances when the random function follows the intrinsic correlation hypothesis. From Sandjivy (1984) the linear model of coregionalization becomes

$$\overline{\Gamma}_z = \mathbf{B}\sum_{u=1}^{q} c_z^u g^u(h) \tag{32}$$

This means all the coregionalization matrices are proportional and a single correlation structure between attributes occurs independently of the spatial correlation. From Equation (20), $\mathbf{D}^{u2}(v|V)$ is

$$\mathbf{D}^{u2}(v|V) = \mathbf{B}^u(\overline{g}^u(V,V) - \overline{g}^u(v,v)) = \mathbf{B}c_z^u d^{u2}(v|V) \tag{33}$$

$\mathbf{R}^u(v|V)$ is found equal to the intrinsic correlation $\mathbf{R}^{\text{int}}$ by applying Equation (24)

$$\mathbf{R}^u(v|V) = (\mathbf{B}(\mathbf{b})^{-1/2})^T(\mathbf{b})^{-1/2} = \mathbf{R}^{\text{int}} \tag{34}$$

Note that c_z^u cancels out; the correlation matrix would be the same for any scale at any support v and size of domain V. Substitution of this result into Equation (29) and factoring out $\mathbf{R}^{\text{int}}$ leads to

$$\mathbf{R}(v|V) = \mathbf{R}^{\text{int}}\sum_{u=1}^{q} d^{u2}(v|V)c^u\left[\sum_{u=1}^{q} d^{u2}(v|V)c^u\right]^{-1} = \mathbf{R}^{\text{int}} \tag{35}$$

The cross-variogram of an attribute with itself is the variogram. So highly correlated variables should show similarities in shape between their variograms at each lag distance. This occurs if variograms are proportional. If the correlation vs. support diagrams are not horizontal lines, the intrinsic correlation hypothesis does not hold.

It is important to observe that support considerations for computing correlations are not necessary if the intrinsic correlation holds. Equation (34) can be demonstrated easily if measured attributes come from samples at different support. Under the intrinsic correlation hypothesis, samples may be taken at larger or smaller support without altering the correlation between attributes. It is obvious that in all cases spatial information about sample correlation is lost because of large sample support.

An interesting situation occurs when the structure of variables does not follow the intrinsic correlation hypothesis. In the case of nonintrinsic correlation, support and size of domain are critical for mixing the spatial scales. If domains are too small or too large, support of sample affects correlation between attributes on the total random function. In such a case, Equation (30) can be generalized to predict the effect of different support. Elementary dispersion variances in Equation (30) are scalar because they are the same for all attributes at spatial scale of variability u. In the case of different support, Equation (20) is modified by changing the scalar elementary dispersion variance by a diagonal matrix of elementary dispersion variances.

$$\begin{aligned}\mathbf{D}^{u2}(v_i \mid V_i) &= \mathbf{B}^u(\overline{g}^u(V_i, V_i) - \overline{g}^u(v_i, v_i)) \\ &= \mathbf{B}^u \mathbf{d}^{u2}(v_i \mid V_i)\end{aligned} \tag{36}$$

where an element of $\mathbf{d}^{u2}(v_i \mid V_i)$ is the elementary dispersion variance computed for specific $v_i \mid V_i$. So Equation (30) becomes

$$\mathbf{W}^u = (\mathbf{d}^{u2}(v_i \mid V_i)\mathbf{b}^u)^{1/2}\left[\sum_{u=1}^{q}(\mathbf{d}^{u2}(v_i \mid V_i)\mathbf{b}^u)\right]^{-1/2} = \mathbf{S}^u\mathbf{S}^{-1} \tag{37}$$

Then, Equation (31) can be applied to compute such correlations. The computed values would be the correlations for attributes from samples at different support and even at different sizes of domain. However, in most cases, we might like to avoid such computations and proceed with samples at the same support. A useful tool is the traditional regularization of variograms. The methods for regularization of variograms are rather simple. Each time the sample support is increased, the dispersion variance is reduced according to Krige's formula. Subtracting such reduction from the variograms is a good approximation (e.g., Journel and Huijbregts, 1978, p. 78). However, for our purpose, we need regularized cross-variograms as a consequence of our multivariate extension of dispersion variances. After some analysis, it has been found that a regularized cross-variogram is found by computing a reduction of the dispersion covariance as defined by the matrix in Equations (18), (19), and (20). Suppose attribute A has been measured on samples support v_i and domain V_i. On the other hand, attribute B has been measured on samples support v_j and size of domain V_j. The regularized variograms $\gamma_{Av_i}(h)$, $\gamma_{Bv_j}(h)$, and cross-variogram $\gamma_{Av_iBv_j}(h)$ are approximated by:

$$\gamma_{Av_i}(h) = \gamma_{A0}(h) - \bar{\gamma}_{A0}(v_i, v_i) = \sum_{u=1}^{q} b_A^u g^u(h) - \sum_{u=1}^{q} b_A^u \bar{g}^u(v_i, v_i)$$

$$\gamma_{Bv_j}(h) = \gamma_{B0}(h) - \bar{\gamma}_{B0}(v_j, v_j) = \sum_{u=1}^{q} b_B^u g^u(h) - \sum_{u=1}^{q} b_B^u \bar{g}^u(v_j, v_j)$$

$$\gamma_{Av_iBv_j}(h) = \gamma_{AB0}(h) - \bar{\gamma}_{AB0}(v_i, v_j) = \sum_{u=1}^{q} b_{AB}^u g^u(h)$$

$$- \sum_{u=1}^{q} b_{AB}^u \sqrt{\bar{g}^u(v_i, v_i)}\sqrt{\bar{g}^u(v_j, v_j)} \tag{38}$$

In practice, we might like to do the opposite procedure, i.e., compute all variograms at point support $\gamma_{A0}(h)$, $\gamma_{B0}(h)$, and $\gamma_{AB0}(h)$.

Krige's Formula and Spatial Scales of Variability

In practice, we can split the field (G) into domains (V), and the domain into blocks (v), and the block into samples (0). In such a case, Krige's formula can be applied. Krige's formula becomes multivariate as dispersion covariance matrices are defined:

$$\mathbf{D}^2(0|G) = \mathbf{D}^2(0|v) + \mathbf{D}^2(v|V) + \mathbf{D}^2(V|G) \tag{39}$$

This formula can be expressed as a function of the spatial scales of variability in the same way as explained with Equation (20).

$$\mathbf{D}^2(0|G) = \sum_{u=1}^{q} \mathbf{B}^u d^{u2}(0|v) + \sum_{u=1}^{q} \mathbf{B}^u d^{u2}(v|V) + \sum_{u=1}^{q} \mathbf{B}^u d^{u2}(V|G) \tag{40}$$

Dispersion (cross)-covariance and correlation between attributes classically computed within each domain carries the contribution of all the spatial scales of variability and the effect of sample support and size of the block and domain. However, for stationary fields, correlation approaches a constant when the domain is much larger than the larger range scale. If such a larger spatial scale of variability is not bounded within the field of study, weaker stationarity and even drift may occur. If the nonstationary attributes do not follow the intrinsic correlation hypothesis, correlation between paris of soil attributes or the studied variables continues to change with the size of the field.

Finally, the effect of anisotropy on univariate dispersion variance is known. Shape of the samples, blocks and domains affects the computations, (Zhang, Warrick, and Myers, 1987). In the multivariate approach, the problem becomes more complicated because anisotropy can show different behavior for different attributes. This topic is, for the most part, case specific.

FIELD CASE EXAMPLES

Two multivariate variograms for clay, sand, and silt have been modeled with the linear model of coregionalization. The two datasets come from studies in soils of the Maricopa Agricultural Center of the University of Arizona (MAC). The size of the domains studied in both datasets are very different. In both cases, the sample support may be considered close to point because of the small size relative to the size of the domain. However, results for small domains have been avoided. The first data set called MAC Fields 28-31 has an area of 1000 m by 1400 m (Warrick and others, 1990). The second dataset is from a 50 m long trench sampled at a depth around 1.5 m at horizontal intervals of 0.5 m. This dataset is called MAC EMS (Environmental Monitoring Site). From previous exploratory data analysis, it is known that each dataset corresponds to an isotropic single multivariate population. Additionally, MAC Fields 28-31 is known to approximate second-order stationarity.

The sample variogram for MAC Fields 28-31 was computed from 182 samples in the upper 0.25 m of soil. Such samples were taken is such a way that they report information at almost all lag distances (Warrick and Myers, 1987). The model for clay, sand, and silt obtained is

$$\overline{\Gamma}_z(\mathbf{h}) = \begin{bmatrix} 7.098 & -7.820 & 0.721 \\ -7.820 & 8.616 & -0.794 \\ 0.721 & -0.794 & 0.073 \end{bmatrix} g^1(\mathbf{h})$$

$$+ \begin{bmatrix} 15.336 & -23.182 & 8.303 \\ -23.182 & 36.096 & -12.928 \\ 8.303 & -12.928 & 4.631 \end{bmatrix} g^2(\mathbf{h})$$

$$+ \begin{bmatrix} 1.918 & -4.086 & 3.281 \\ -4.086 & 20.918 & -16.797 \\ 3.281 & -16.797 & 13.489 \end{bmatrix} g^3(\mathbf{h})$$

where the elementary variograms $g^1(\mathbf{h}) = (1 - \delta(\mathbf{h}))$ is nugget with ($\delta(\mathbf{h}) = 1$ if $\mathbf{h}$

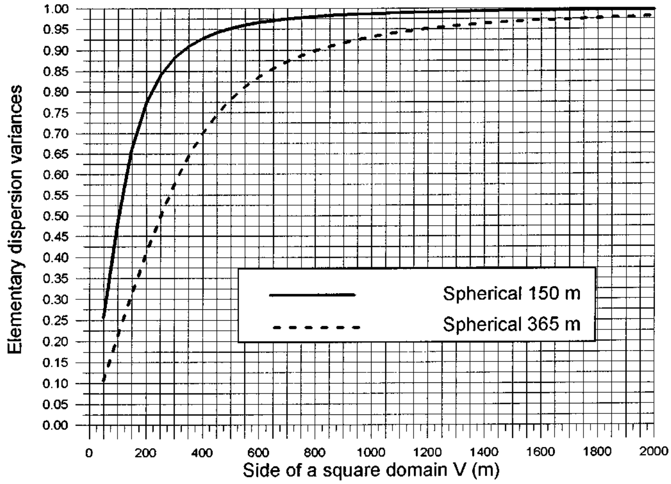

Figure 1. Elementary dispersion variances $d^{u2}(0|V)$ MAC Fields 28-31.

$= 0$ and $\delta(\mathbf{h}) = 0$ otherwise), $g^2(\mathbf{h})$ = spherical range 150 m, and $g^3(\mathbf{h})$ = spherical range 365 m have unit sills. The regionalized correlation matrices for clay, sand, and silt are calculated by normalizing the coregionalization matrices. Equation (31) is applied to compute the correlation for different sizes of domain.

$$\mathbf{R}(0|V) = \left[\begin{bmatrix} 1 & -0.9999 & 0.9994 \\ -0.9999 & 1 & -0.999 \\ 0.9994 & -0.999 & 1 \end{bmatrix} \mathbf{W}^0\right]^T \mathbf{W}^0$$

$$+ \left[\begin{bmatrix} 1 & -0.9853 & 0.9853 \\ -0.9853 & 1 & -1 \\ 0.9853 & -1 & 1 \end{bmatrix} \mathbf{W}^1\right]^T \mathbf{W}^1$$

$$+ \left[\begin{bmatrix} 1 & -0.645 & 0.6451 \\ -0.645 & 1 & -1 \\ 0.6451 & -1 & 1 \end{bmatrix} \mathbf{W}^2\right]^T \mathbf{W}^2$$

The “weight” diagonal matrices $\mathbf{W}$ for each V are computed with Equation (30). The required elementary dispersion variances $d^{(1)2}(v|V)$ and $d^{(2)2}(v|V)$ (Fig. 1) for square domains have been computed analytically with the method of auxiliary functions described in Journel and Huijbregts (1978). From Equation (31), Figure 2 shows correlations between two attributes as a function of size of domain V. The support of samples has been held to a constant point support.

Figures 3 and 4 for MAC EMS are obtained by applying the same procedure for clay, sand, and silt on the next multivariate variogram:

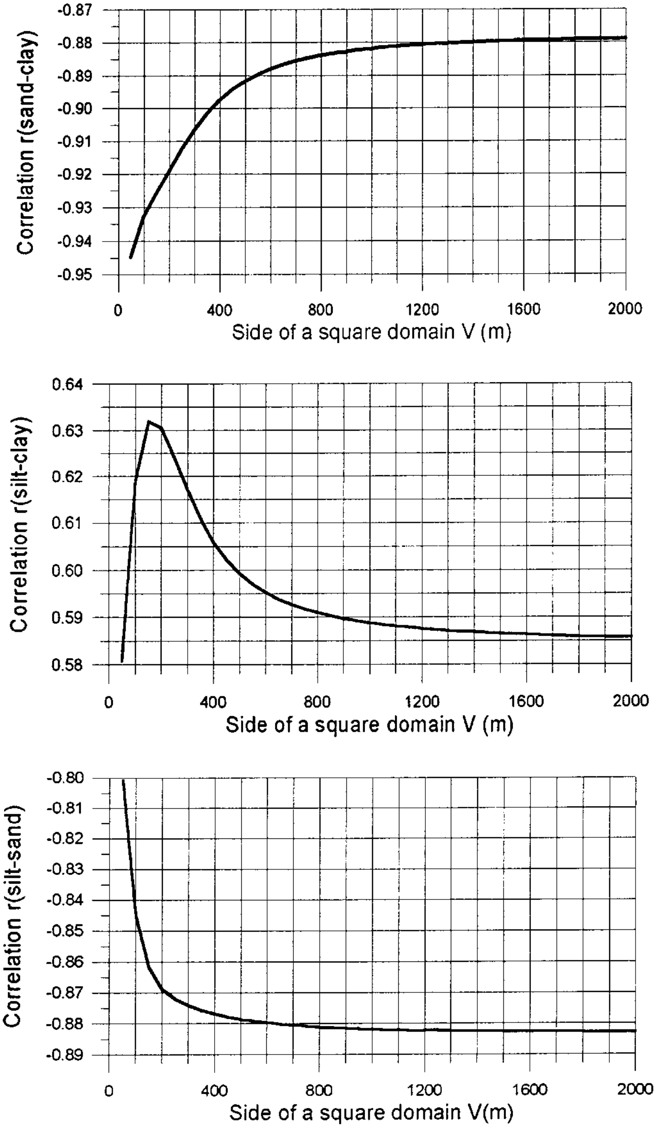

Figure 2. Correlation between attributes MAC Fields 28-31.

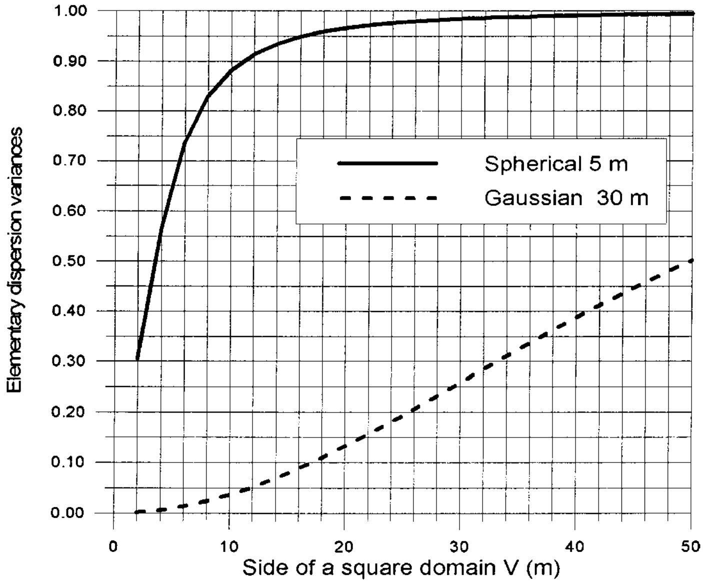

Figure 3. Elementary dispersion variances $d^2(0|V)$ MAC EMS.

$$\overline{\Gamma}_Z(\mathbf{h}) = \begin{bmatrix} 4.085 & -6.003 & 1.925 \\ -6.003 & 22.724 & -16.754 \\ 1.925 & -16.754 & 14.857 \end{bmatrix} g^1(\mathbf{h})$$

$$+ \begin{bmatrix} 3.724 & -3.789 & 0.096 \\ -3.789 & 10.4359 & -6.679 \\ 0.096 & -6.679 & 6.586 \end{bmatrix} g^2(\mathbf{h})$$

$$+ \begin{bmatrix} 2.312 & -0.394 & -1.931 \\ -0.394 & 0.067 & 0.329 \\ -1.931 & 0.329 & 1.613 \end{bmatrix} g^3(\mathbf{h})$$

where $g^1(\mathbf{h}) = (1-\delta(\mathbf{h}))$ is nugget with ($\delta(\mathbf{h}) = 1$ if $\mathbf{h} = 0$ and $\delta(\mathbf{h}) = 0$ otherwise), $g^2(\mathbf{h})$ = spherical range 5 m, and $g^3(\mathbf{h})$ = gaussian practical range 30 m are the elementary variograms. Correlation between clay, sand, and silt attributes as a function of size of domain is given by the following relationship:

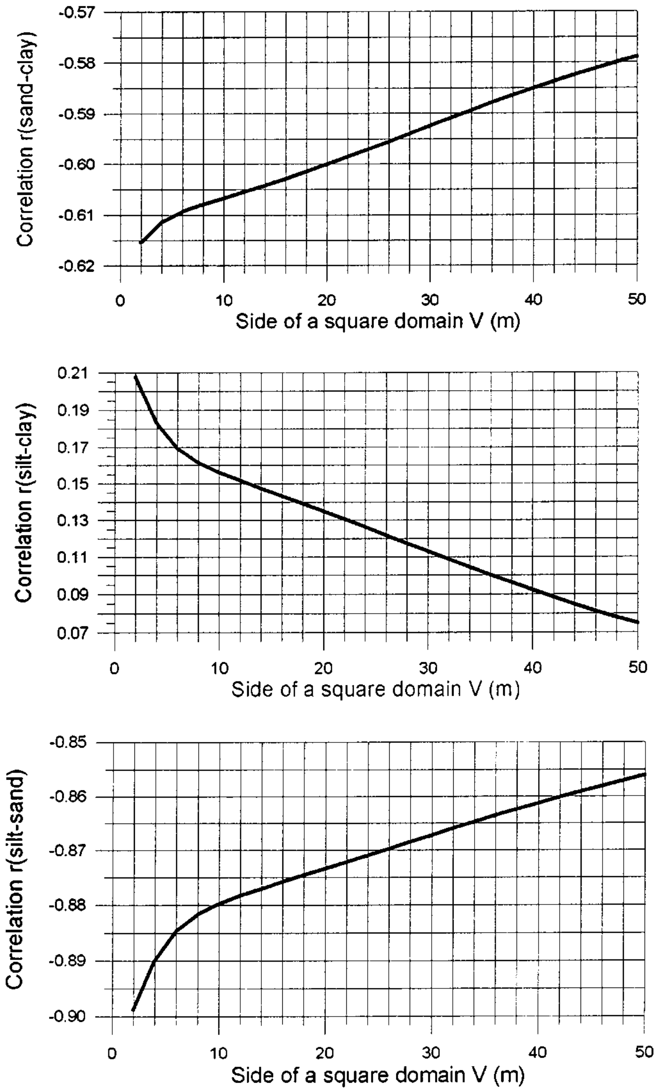

Figure 4. Correlation between attributes MAC EMS.

$$\mathbf{R}(0|V) = \left[\left[\begin{array}{ccc} 1 & -0.623 & 0.247 \\ -0.623 & 1 & -0.9118 \\ 0.247 & -0.9118 & 1 \end{array}\right]\mathbf{W}^0\right]^T \mathbf{W}^0$$

$$+\left[\left[\begin{array}{ccc} 1 & -0.6079 & 0.0193 \\ -0.6079 & 1 & -0.8056 \\ 0.0193 & -0.8056 & 1 \end{array}\right]\mathbf{W}^1\right]^T \mathbf{W}^1$$

$$+\left[\left[\begin{array}{ccc} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{array}\right]\mathbf{W}^2\right]^T \mathbf{W}^2$$

Figure 3 shows the values of the spatial mean function of the elementary variogram or elementary dispersion variances $d^2(0|V)$ for square 2-D domains and point support. The rate of increase of dispersion variances depends on the range of the variograms and model of the elementary variogram; $d^2(0|V)$ may or not reach a constant inside the boundaries of the domain studied. Elementary dispersion variances for nonpoint support can be computed from Figures 1 and 3 by applying Krige's formula.

From Figures 2 and 4, the values of correlation between attributes are shown to be dependent on the size of the domain and may approach constant values for stationary fields (Fig. 2). Correlation between pairs of attributes can increase or decrease depending on the contribution of total dispersion variances. The shape of the curves depends on Equation (30), where the "weights" depend on the diagonal of coregionalization matrices and elementary dispersion variances but not on dispersion (cross)-covariances. This is an important observation, because regionalized correlations $\mathbf{R}^u(0|\infty)$ are independent of the size of domain but allow the computation of total correlation $\mathbf{R}(v|V)$ dependent on the size of domain.

In general, for both datasets, MAC Fields 28-31 and MAC EMS, the correlation values computed from data with the classic Pearson correlation formula give similar results as $\mathbf{R}(0|\infty)$ predicted with Equation (31).

CONCLUSIONS

The approach presented here extends dispersion variance to the multivariate case. Regionalized dispersion covariance matrices were defined for each spatial scale of variability. Depending on sample support and size of the domain, such covariances have been found to represent the contribution of each scale of variability and each attribute to the total dispersion. This approach allows generalization of the Pearson correlation coefficient to a relationship that accounts for support and size of blocks and domains. The intrinsic correlation hypothesis and the coregionalized correlation coefficient have been demonstrated analytically to be independent of support. Krige's formula and the linear model of coregionalization have been related in terms of spatial scales of variability. This approach offers practical applications for utilizing data obtained at different support. It also provides the reader with more tools to work with spatial scales of variability at different sizes of domains.

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REFERENCES

Goulard, M., and Voltz, M., 1992, Linear coregionalization: Tools for estimation and choice of cross-variogram matrix: Math. Geology, v. 24, no. 3, p. 269–286.

Journel, A. G., and Huijbregts, Ch. J., 1978, Mining geostatistics: Academic Press, New York, 600 p.

Miesch, A. T., 1975, Variograms and variance components in geochemistry and ore evaluation: Geol. Soc. America, Mem. 142, p. 333–340.

Myers, D. E., 1993, Change of support and transformations, *in* Dimitrakopoulos, R., ed., Geostatistics for the next century: Kluwer Academic Publ., Dordrecht, pp. 253–258.

Myers, D. E., 1994, The linear model and simultaneous diagonalization of the variogram matrix function, *in* Fabbri, A. G., and Royer, J. J., eds., 3rd CODATA Conference on Geomathematics and Geostatistics: Sci. de la terre, Sér, Inf., Nancy, v. 32, p. 125–139.

Myers, D. E., 1997, Statistical models for multiple scaled analysis, *in* Quattrochi, D. A., and Goodchild, M., eds., Scale in remote sensing and GIS: Lewis Publishers, p. 273–294.

Rendu, J. M., 1978, An introduction to geostatistical methods of mineral evaluation: South African Institute of Mining and Metallurgy, Johannesburg, 84 p.

Russo, D., and Jury, W., 1987, A theoretical study of the estimation of the correlation scale in spatially variable fields: Water Resources Res., v. 23, no. 7, p. 1257–1268.

Sandjivy, L., 1984, The factorial kriging analysis of regionalized data. Its application to geochemical prospecting, *in* Verly, G., and others, eds., Geostatistics for natural resources characterization, NATO-ASI Series C: v. 122, Reidel Publ. Co., Dordrecht, p. 559–572.

Smith, H. F., 1938, An empirical law describing heterogeneity in the yields of agricultural crops: Jour. Agr. Science, v. 28, p. 1–23.

Wackernagel, H., 1985, The inference of the linear model of coregionalization in the case of a geochemical data set: Ecole des Mines de Paris, Centre de Geostatistique et de Morphologie Mathematique, Fontainebleau, p. 14.

Wackernagel, H., 1988, Geostatistical techniques for interpreting multivariate spatial information, *in* Chung, C. F., and others, eds., Quantitative analysis of mineral and energy resources, NATO ASI series C 223: Reidel Publ. Co., Dordrecht, p. 393-409.

Warrick, A. W., and Myers, D. E., 1987, Optimization of sampling locations for variogram calculations: Water Resources Res., v. 23, no. 3, p. 496–500.

Warrick, A. W., Musil, S. A., Artiola, J. F., Hendricks, D. E., and Myers, D. E., 1990, Sampling strategies for hydrological properties and chemical constituents in the upper vadose zone, final technical report: University of Arizona, Tucson, Arizona, 117 p.

Xie, T., and Myers, D. E., 1995, Fitting matrix valued variogram models by simultaneous diagonalization: I Theory: Math. Geology, v. 27, no. 7, p. 867–876.

Zhang, R., Warrick, A. W., and Myers, D. E., 1987, Variance as a function of sample support size: Math. Geology, v. 22, no. 1, p. 107–121.