Hence, if f is continuous, the theorem identifies f as the derivative of F directly.

Of course even if it is conceded that bypassing the definite integral is possible it might be maintained that such a step is undesirable. The writer's considered opinion on this question is: (a) the material in this topic is so difficult that the student is almost always presented with a frankly incomplete treatment, an incorrect treatment or an incomprehensible treatment; (b) at this level all the functions are piece-wise analytic and require neither the generality nor the subtlety of the usual approach.

The theorem can also be applied meaningfully if F is given and f is unknown. For instance, to identify speed as ds/dt we usually tell our students that this is how the physicists do or should define it. While it is true that many physicists are adept in the use of limits, it is not reasonable to presuppose this skill in practitioners of other disciplines in which calculus can play a role. An alternative approach, using speed as an example, would be to note that we could plausibly expect $\Delta t \cdot \min$ speed to underestimate distance and $\Delta t \cdot \max$ speed to overestimate it. The theorem then asserts that, however speed is defined, ds/dt is a correct formula for it in the usual situations.

Reference

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DEFINITION OF RADIUS OF CONVERGENCE

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In most calculus texts, the standard definition of the radius of convergence of a power series

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{1}$$

is

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

With this definition the problems are carefully selected so that the limit always exists. In addition to obtaining a certain skill in computing this limit, the student is asked to grasp two more subtle ideas. On the one hand, r is represented intuitively as maximal such that the series converges for all x with $|x| < r$ and then, somewhat in contrast, the student is cautioned to check independently for convergence at the end points of the interval. Often, to enforce the student's retention of this latter concept, the examples and problems are so chosen that divergence occurs in at least one of the end points. As a consequence, many students expect that this will always occur.

At the next level, perhaps in an advanced calculus or introductory analysis course, the radius of convergence is generally defined by the root-test whereby the problems of existence of the limit are avoided. The computation of r still seems to rely, however, on the ratio test with "Lim" replaced by "Lim Sup"; consequently, the student still thinks of radius of convergence in terms of maximal region of convergence or the value of

$$\operatorname*{Lim\,Sup}_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

The student next encounters the same problem in the context of complex variable theory complicated by the study of singularities. At this point, the circle of convergence becomes identified with the location of the singularities of the function represented by the series.

We come now to a very interesting example, intended to justify the preceding discussion. Let $f(z)=\sum_{n=1}^{\infty} z^n/n^2$. It is easily seen that the series converges not only for $|z|<1$ but for $|z|=1$. The student wonders where the singularities of f are. If r represents the maximum such that convergence occurs for $|z|<r$, why is the radius for this example 1 and not $1+\epsilon$ $(\epsilon>0)$? The answer is that in this case the singularity is manifested not at a point of divergence, but rather in the nondifferentiability of f. The singularity is easily found by testing the convergence of the differentiated series. It occurs at $z=1$. Of course, if one differentiates the series twice then the resulting series diverges at every point on the circle $|z|=1$.

A FOURIER SERIES DERIVATION OF THE EULER-LAGRANGE EQUATION

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THEOREM. *Suppose that* $y=\bar{y}(x)$ *extremizes* $\int_a^b f(x, y(x), y'(x))dx$ *with respect to all functions which are continuously differentiable on* $[a, b]$ *and which satisfy* $y(a)=y_1$, $y(b)=y_2$. *Then there is a constant c such that*

$$f_{y'}(x, \bar{y}(x), \bar{y}'(x)) - \int_a^x f_y(x, \bar{y}(x), \bar{y}'(x))dx = c \qquad \textit{for all } x \textit{ in } [a, b].$$

Proof. (We assume that $f(x, y, y')$, $f_y(x, y, y')$ and $f_{y'}(x, y, y')$ are continuous for all x in $[a, b]$ and for all y and y'.) We may work over the interval $[-\pi, \pi]$ rather than over $[a, b]$ without loss of generality. Let $I_n(\alpha)=\int_{-\pi}^{\pi} f(x, \bar{y}+\alpha \sin nx, \bar{y}'+\alpha n \cos nx)dx$ for $n=1, 2, 3, \cdots$. Since $\bar{y}(x)$ extremizes $\int_{-\pi}^{\pi} f(x, y(x), y'(x))dx$ with respect to all continuously differentiable functions having the appropriate end-point conditions, it extremizes $I_n(\alpha)$ with respect to functions of the form $\bar{y}+\alpha \sin nx$ for $n=1, 2, 3, \cdots$. Hence

$$I_n'(0) = 0 = \int_{-\pi}^{\pi} [f_y(x, \bar{y}, \bar{y}') \sin nx + f_{y'}(x, \bar{y}, \bar{y}')n \cos nx]dx$$