SOME SUBSPACES OF ANALYTIC FUNCTIONS

By Donald E. Myers

Let S be a strip in C of the form $S = \{z \mid \tau < R(z) < \eta, z \varepsilon C\}$. The main results of this paper are: an L_2 topology on square integrable analytic functions on S is stronger than uniform convergence on compact subsets of S and the imbedding of analytic functions of polynomial growth into the space of square integrable functions as a complete inductive limit space. The first result is a generalization of that in Reference [2].

In previous papers [1], [2], this author has been interested in representation theorems for distributions as analytic functions or functionals and in particular, appropriate topologies for these spaces of analytic functions. For functions analytic in the strip S, two topologies in particular are considered, uniform convergence on compact subsets of S and that generated by an L_2 norm with respect to the variable y, $z = x + iy$. The first topology can be used for arbitrary collections of analytic functions and the ring of all functions analytic in S is known to be complete in this topology. The L_2 topology is only applicable to the subspace of functions that are bounded and go to zero sufficiently fast as $|y| \to \infty$.

Notation. $S = \{z \mid \tau < R(z) < \eta\}$.

0_S denotes the algebra of all functions analytic in S with the topology of uniform convergence on compact subsets of S. Of course, 0_S is complete.

$$0_S^2 = \{f \mid f \varepsilon 0_S , ||f|| < \infty\}$$

$$||f|| = \left[\sup \Upsilon < x < \eta \int_{-\infty}^{\infty} |f(x + iy)|^2 \, dy\right]^{\frac{1}{2}}.$$

Lemma 1. *If $\{f_n\}$ is a Cauchy sequence in 0_S^2, then it is a Cauchy sequence in 0_S.*

Proof. Suppose there exists a compact subset K of S such that $f_n - f_m$ does not converge uniformly to zero on K. Since $||f_n - f_m|| \to 0$, $|f_n(z) - f_m(z)|$ is bounded on K for each n, m. Let $g_{nm}(K) = \sup_{z \varepsilon K} |f_n(z) - f_m(z)|$. With K compact and f_n, f_m analytic, there exists for each n, m, $z_{nm} \varepsilon K$ such that $g_{nm}(K) = |f_n(z_{nm}) - f_m(z_{nm})|$. If $|f_n - f_m| \not\to 0$ uniformly on K, then for some $\epsilon > 0$, there exist two unbounded sequences $\{n_k\}$, $\{m_k\}$ such that $g_{n_k m_k}(K) > \epsilon$ for all k. Denote by K_ϵ the closure of $\{z_k\}$, $z_{m_k n_k} = z_k$. By the continuity of $|f_n(z) - f_m(z)|$, for each k there is a neighborhood $N_{z_k}(\delta_k) = N_k$ of z_k such that $|f_{n_k}(z) - f_{m_k}(z)| > \epsilon/2$ for all $z \varepsilon N_{z_k}(\delta_k)$. K is covered by the union of those neighborhoods and, since as a closed subset of a compact set, K_ϵ is compact, there is a finite subcover

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$N_{k_1}, \cdots, N_{k_J}$. Let $\delta = \min(\delta_{k_1}, \cdots, \delta_{k_J})$. We note now that

$$\begin{aligned} ||f_n - f_m|| &\geq \int_{-\infty}^{\infty} |f_n(x_k + iy) - f_m(x_k + iy)|^2 \, dy \\ &\geq \int_{y_k-\delta}^{y_k+\delta} |f_n(x_k + iy) - f_m(x_k + iy)|^2 \, dy \\ &\geq \frac{\epsilon}{2} \delta \quad \text{if} \quad n = n_k, \qquad m = m_k . \end{aligned}$$

Since $||f_n - f_m|| \to 0$ as $n, m \to \infty$, there exists an integer I such that for $n, m > I$

$$||f_n - f_m|| < \frac{\epsilon}{4} \delta$$

which provides a contradiction to the above for all n_k, $m_k > I$; and since $\{n_k\}$ $\{m_k\}$ are unbounded, there are an infinite number of pairs n_k, $m_k > I$. We conclude then that $f_n - f_m \to 0$ uniformly on all compact subsets of S.

LEMMA 2. *If $\{f_n\}$ is a Cauchy sequence in 0_S^2, then $\{f_n\}$ is convergent in 0_S.*

Proof. From [3; 139], 0_S is complete, and the lemma follows from Lemma 1.

We note that neither Lemma 1 nor Lemma 2 puts any condition on Υ, η, except $\Upsilon \infty \eta$. Let $x = R(z)$ for some $z \in S$ and denote by $F^x(y) = f(x + iy)$. Then since L_2 spaces are complete there is for each x and each Cauchy sequence $\{f_n\}$ in 0_S^2 an L_2 function $F_0^x(y)$ such that

$$\int_{-\infty}^{\infty} |F_0^x(y) - F_n^x(y)|^2 \, dy \to 0 \quad \text{as} \quad n \to \infty .$$

We define $f_0(x + iy) = F^x(y)$ point-wise as a function of x. Furthermore

$$\int_{-\infty}^{\infty} |f_0(x + iy)|^2 \, dy < \infty, \qquad \Upsilon < x < \eta$$

since

$$\begin{aligned} &\int_{-\infty}^{\infty} |f_0(x + iy)|^2 \, dy \\ &\qquad \leq \int_{-\infty}^{\infty} |f_0(x + iy) - f_n(x + iy)|^2 \, dy + \int_{-\infty}^{\infty} |f_n(x + iy)|^2 \, dy < \infty, \end{aligned}$$

so that $f_0 \in 0_S^2$, and this completes the proof that 0_S^2 is complete since by the uniqueness of the limit function, f_0 must be in 0_S. The theorem then is the following

THEOREM 3. *For each S, $-\infty < \Upsilon < \eta < \infty$, 0_S^2 is complete.*

We observe that Lemma 2 is a generalization of the result in [2] since the condition that the analytic functions be the bilateral Laplace transforms of functions satisfying

$$\int_0^\infty |e^{-\tau t} F(t)|^2 \, dt < \infty$$

$$\int_{-\infty}^0 |e^{-\eta t} F(t)|^2 \, dt < \infty$$

has been replaced by the finite norm condition.

DEFINITION 4. If f is in 0_S, then f is said to be of polynomial growth in S if there exists a polynomial P_j (degree j) such that

$$|f(z)| < P_j(|z|), \qquad z \in S.$$

Let

$P_k = \{f \mid f \in 0_S, f \text{ of poly growth of degree } j, j \leq k\}$ then map P_k into 0_S^2 by

$$f(z) \xrightarrow{\mu_k} \frac{f(z)}{(\alpha + z)^{(k+2)/2}}$$

where either $\tau + \alpha > 0$ or $\eta + \alpha < 0$.

We note that if $i < j$, then $P_i \subset P_j$ and if I_{ij} is the identity map P_i into P_j, then

$$\mu_j \cdot I_{ij} \neq \mu_i ,$$

however, the identity map is continuous from

$$\mu_i P_i \quad \text{into} \quad \mu_j I_{ij} P_i .$$

LEMMA 5. *Let $\{f_n\}$ be a sequence in P_k converging uniformly on compact subsets of S; then the limit, f, is in P_k.*

Proof. It is clear from the completeness of 0_S that f is an analytic function so it only remains to be shown that f is of polynomial growth in S and of degree $\leq k$.

Consider

$$|f(z)| \leq |f(z) - f_n(z)| + |f_n(z)|;$$

but $|f_n(z)| < P(|z|)$ for all $z \in S$ and all n, therefore

$$|f(z)| < |f(z) - f_n(z)| + P(|z|)$$

for all n and all $z \in S$. Now let $\epsilon > 0$ and $\{K_i\}$ an increasing sequence of compact subsets of S whose union is S. Then for each i, there is an $N(i)$ such that for $n > N(i)$ and $z \in K_i$

$$|f(z) - f_n(z)| < \epsilon;$$

and hence

$$|f(z)| < \epsilon + P(|z|).$$

Since ϵ is arbitrary, it follows that

$$|f(z)| < P(|z|)$$

and $f \in P_k$.

THEOREM 6. *For each j, $\mu_j P_j$ is a complete linear subspace of 0_S^2, and hence $\bigcup_{j=1}^{\infty} \mu_j P_j$ is a complete inductive limit subspace of 0_S^2.*

Proof. Since $\bigcup_{j=1}^{\infty} \mu_j P_j$ is complete only if $\mu_j P_j$ is complete for each j, it is sufficient to show the latter. Let $\{g_n\}$ be a Cauchy sequence in $\mu_j P_j$, i.e.

$$\sup_{x \varepsilon S} \int_{-\infty}^{\infty} |g_n(x+iy) - g_m(x+iy)|^2 \, dy \to 0 \quad \text{as} \quad n, m \to \infty$$

However, by Lemma 1

$$g_n(z) - g_m(z) \to 0$$

uniformly on compact subsets, but

$$g_n(z) = \frac{f_n(z)}{(\alpha + z)^{(j+2)/2}}, \qquad f_n \varepsilon P_j$$

and

$$g_m(z) = \frac{f_m}{(\alpha + z)^{(j+2)/2}} \qquad f_m \varepsilon P_j$$

so that

$$f_n(z) - f_m(z) \to 0$$

uniformly on compact subsets of S.

By Lemma 5 then

$$f_n \to f, \qquad f \varepsilon P_j$$

uniformly on compact subsets of S, and hence

$$\frac{f_n(z)}{(\alpha + z)^{(j+2)/2}} \to \frac{f(z)}{(\alpha + z)^{(j+2)/2}}$$

uniformly on compact subsets of S. By the uniqueness of limits and the completness of 0_S^2

$$g_n \to \mu_j f$$

in 0_S^2 which shows that $\mu_j P_j$ is complete.

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