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Space-Time Radial Basis Functions

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Abstract—Radial basis functions are "isotropic"; i.e., under a rotation, the basis function is left unchanged and is obtained as a function of a distance on the space. For Euclidean space this is not a problem since there is a natural metric. To extend radial basis functions to space-time, i.e., $R^m \times T$, either a zonal anisotropy has to be incorporated or a metric must be defined on space-time. While the sum of two valid radial basis functions defined on different dimensional spaces is generally only semidefinite on the product space, the product of two positive definite functions on lower dimensional spaces is positive definite on the product space. This construction can be extended in several ways including a product-sum, integrated product, and the integrated product-sum. Examples are given for each construction and an application is given. The constructions are equally applicable to extending from space to space-time or for splitting higher-dimensional Euclidean spaces into the product of several lower-dimensional spaces.

Keywords—Space-time covariances, Integrated product-sum covariance models, Radial basis functions, Positive definite functions.

1. INTRODUCTION

The basis functions in a radial basis function interpolator are isotropic; i.e., the values of these kernel functions depend only on the length of the separation vector and not on its direction. To interpret the values of the basis functions, one must distinguish between the positive definite case and the conditionally positive definite cases (of various orders). In the case of a positive definite function which will have its maximum value at zero, the values might be interpreted as measures of similarity—that is, as measures of the similarity of the values of the unknown function at pairs of points separated by a vector increment. In the case of conditionally positive definite

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functions, the values of the basis functions might be interpreted as measures of dissimilarity. In either case, it is plausible that the similarity or dissimilarity is dependent not only on the length of the separation vector, but also on its direction. There are at least two different schemes for incorporating direction: one corresponds to the use of a modified metric, and the other corresponds to splitting the space into several lower-dimensional subspaces and separating the dependence of the radial basis function on these lower-dimensional subspaces. In the sense that the splitting emphasizes the orthogonality of the subspaces, this might be considered as a limiting case of the directionally dependent metric, but the radial basis function on the split space cannot be constructed from the one obtained from a modified metric.

Let u be a point, i.e., a vector in m-dimensional Euclidean space, R^m, and $\|u\|$ the usual metric or norm. Let Q be an $m \times m$ positive definite matrix. Then $u^{\mathrm{T}} Q u$ is another metric or distance and can be used in a radial basis function interpolator in lieu of the usual Euclidean metric. Q can be thought of as corresponding to an affine transformation, e.g., a rotation and a stretching/shrinking. Thus, a given length vector in one direction will correspond to an increased or decreased length vector after rotation to a new direction—that is, the measure of similarity or dissimilarity changes with direction even for a fixed length vector. One difficulty of using such a transformation is in choosing Q.

The change in the measure of similarity or dissimilarity might also be thought of as a separation of the dependence in one direction from that of another direction. An extreme case of this separation is appropriate when the interpolating functions are defined on $R^m \times T$ where T is the time axis. While one might define a metric or distance function on this space to be of the form $[\|u\|^2 + c^2|t|^2]^{0.5}$, there is the critical question of how to choose the constant c. The choice of such a constant should reflect a realistic relationship between space and time. Such a metric corresponds to considering the time axis as being "orthogonal" to all of the space axes but with a change of units.

Instead of relying on a space-time metric or on a positive definite matrix to define a quadratic form, a more general form is used as the starting point, and then several methods are given that will result in valid radial basis functions. That is, rather than considering radial basis functions in the form $g(u) = \phi(\|u\|)$, they are considered to be of the form $g(u,t)$. In this form there is no assumed geometrical relationship between the space axes and the time axis. The various constructions described subsequently can easily lead to models that are differentiable with respect to time but not space and vice-versa. Most of the functions described in the following sections are not "radial", but are still to be used in an interpolator of the same form as "radial" basis function interpolator. For simplicity the terminology will be retained even for "nonradial" functions.

2. OVERVIEW

2.1. Preliminaries and Notation

If g satisfies the appropriate positive definiteness condition, then the general form of the radial basis function interpolator is

$$Z^*(u) = \sum_{i=1}^{n} b_i g(u - u_i) + \sum_{k=0}^{p} a_k f_k(u), \tag{1}$$

where $Z(u)$ is the function to be interpolated and the u_i, $i = 1, \dots, n$, are the data locations. The $f_k(u)$, $k = 0, \dots, p$, are linearly independent functions, usually taken to be monomials in the coordinates of u. Micchelli [1] has shown that coefficients are determined, given the condition on g and on the f_ks. Neither this form nor the system of equations to determine the coefficients are dimension dependent. Nor are they dependent on the assumption that $g(u) = \phi(\|u\|)$. The definition of positive definiteness (conditional positive definiteness) is not dependent on the use of a metric. Moreover, the interpolator is easily extended to vector-valued functions as shown by Myers [2]. The construction of Madych and Nelson [3] can be extended to the space-time context.

Hence, the radial basis function interpolator will be considered to be of the form

$$Z^*(u,t) = \sum_{i=1}^{n} b_i g(u - u_i, t - t_i) + \sum_{k=0}^{p} a_k f_k(u,t), \tag{2}$$

where $Z(u,t)$ is the function to be interpolated and the (u_i, t_i), $i = 1, \dots, n$, are the data locations. The $f_k(u,t)$, $k = 0, \dots, p$, are linearly independent functions. Although it would be possible to simply choose these to be monomials in the coordinates of (u,t), it is more realistic to also incorporate sin, cos functions of time as well; see [4].

The problem is how to construct a function of the form $g(u,t)$ with the appropriate positive definiteness condition. The standard models are all isotropic, i.e., are functions of length only. The obvious choice would be to construct a function $g(u,t) = F(g_s(u), g_t(t))$ where $g_s(u)$, $g_t(t)$ are valid radial basis functions on R^m and T, respectively. The two simplest choices for F are the sum and the product. While the sum of two valid radial basis functions defined on R^m is again a radial basis function on R^m, that property does not extend to $g_s(u) + g_t(t)$. Such a function will be only semidefinite as shown in [5,6]. If $g_s(u)$, $g_t(t)$ are positive definite on R^m and T, respectively, then $g_s(u) \times g_t(t)$ is positive definite on $R^m \times T$. If one or the other of the two factors is only conditionally positive definite, then the product may not be conditionally positive definite. However, the sum and product constructions can be combined for a more general form. Kyriakidis and Journel [7] have given a general overview of the construction of space-time models.

It is well known that if the $g_j(u)$ (in space or time) are positive definite (or conditionally positive definite) and c_j are positive numbers, then the linear combination is again of the same form. This can be extended to integrals. Let $\theta(a)$ be a probability density function on R and suppose that $g(u;a)$ is positive definite (or conditionally positive definite) for each a in R. Then

$$\int_R g(u;a)\theta(a)\, da \tag{3}$$

is of the same form. In some instances, $g(u;a)$ need only be semidefinite. For example, let $m = 1$, $g(u;a) = \cos(au)$, and $\theta(a)$ the uniform probability density on the unit interval. Then the integrated form is

$$G(u) = 1 - \frac{\sin(u)}{u} \text{ if } |u| > 0, \text{ and is zero otherwise.} \tag{4}$$

For other examples, see [8]. This result will be extended to valid space-time radial basis functions.

2.2. Positive Definiteness, Conditional Positive Definiteness, and Conditional Negative Definiteness

Let $G(u)$ be a positive definite function defined on R^m. Letting

$$\gamma(u) = G(0) - G(u),$$

then $\gamma(u)$ is conditionally negative definite; moreover, $\gamma(0) = 0$. Thus, there is a subclass of conditionally negative definite functions that correspond to positive definite functions. If $g(u)$ is conditionally negative definite and asymptotically bounded, then there is a corresponding positive definite function. A number of the standard radial basis functions are conditionally negative definite instead of being conditionally positive definite. This has no effect on either the form of the interpolator or on the system of equations used to determine the coefficients in the interpolator. As an example, $\gamma(u) = \|u\|^\alpha$, $0 < \alpha < 2$, is conditionally negative definite but it is not asymptotically bounded. The normalization condition, $g(0) = 0$, does not affect either the interpolator or the system of equations. The multiquadric

$$g(u) = \left[c^2 + \|u\|^2\right]^{0.5}$$

could be replaced by

$$\gamma(u) = \left[c^2 + \|u\|^2\right]^{0.5} - |c|.$$

In the geostatistics literature, normalized conditionally negative definite functions are known as variograms and can be modeled from the data [9,10].

Letting $Z(s)$ be a second-order stationary random function defined on R^m, then the covariance function $C(h_s) = \operatorname{Cov}[Z(s+h_s), Z(s)]$ exists and is only a function of the separation vector h_s. The covariance function is positive definite, and conversely any positive definite function can be associated with a second-order stationary random function. As noted above, covariance functions (which are positive definite) determine an associated conditionally negative definite function; this conditionally negative definite function is usually called the *variogram*. In the following sections, the emphasis will be almost entirely on covariance functions and the associated variograms. At the very end it is shown that the constructions can be extended to include conditionally positive definite functions of higher order (or conditionally negative definite).

2.3. Dimension Splitting

While there is a natural split between space and time in $R^m \times T$, the methodology will work equally well for splitting R^m into the product of several lower-dimensional spaces. That is, let $R^{m_1}, \dots, R^{m_d}$ be Euclidean spaces of dimensions $m_1, \dots, m_d$. Let ${}_1v, \dots, {}_dv$ be in the respective spaces. Then consider a function on $R^{m_1} \times \cdots \times R^{m_d}$ which is not based on a metric on the product space. Let $g_i({}_iv)$, $i = 1, \dots, d$, be positive definite on the respective spaces R^{m_i}. Then

$$\prod_i^d g_i({}_iv)$$

will be positive definite on the product space. Products of fewer factors may be only semidefinite but can be added to the above product and hence, very general positive definite product-sums can be generated.

2.4. Differentiability and Integrability

One of the advantages of separating the dependence of the radial basis function on the space coordinates from that on the time coordinate is that the differentiability and integrability properties are then also separated. For example, in using a radial basis function representation in solving a partial differential equation, differentiability must be considered for each coordinate separately. Many standard radial basis functions are not differentiable, but by constructing a radial basis function using dimension splitting, differentiability can be obtained for some coordinates and not for others. This could be especially important in the case of radial basis functions defined on space-time; metric models do not provide this option.

In the case of problems set in space-time, there is a crucial difference between averaging in space and averaging in time. Hence, separability of dependence is very important.

3. SPATIAL TEMPORAL MODELS

3.1. Metric Models

Let $\phi(\cdot)$ be a conditionally positive definite function defined on R, and let Q be an $m \times m$ positive definite matrix. For any such choice of Q, $\phi(s^\top Qs)$ will be conditionally positive definite of the same order on R^m. This construction would extend equally well to $R^m \times T$ by letting Q be an $(m+1) \times (m+1)$ positive definite matrix. For example, let Q be diagonal with all diagonal entries equal to one except the last, set that entry equal to c^2, and then $(s,t)^\top Q(s,t) = \|s\|^2 + c^2|t|^2$, $\|s\|^2$ being the usual Euclidean norm. Hence, any known valid radial basis function on R^m can be extended to a basis function on $R^m \times T$ by an appropriate choice of Q. This construction may be useful in other contexts. Covariance functions are not only bounded; most models asymptotically tend to zero. If they are of compact support, then the radius of this compact set is called the *range*. For other models such as the exponential and the Gaussian, there is an *effective range*. For a covariance function C, $C(0)$ is not only the variance of the random function, but it is also a bound on the associated variogram. In the geostatistics literature this value is referred to as the *sill* of the variogram. When a covariance or variogram is obtained in the form $C(h_s^\top Qh_s)$, $\gamma(h_s^\top Qh_s)$, the covariance or variogram is said to have a geometric anisotropy. The extension of radial basis functions from space to space-time by the use of a metric corresponds to a space-time model with a geometric anisotropy.

The variogram can be defined for a random function independently of the covariance and under weaker conditions. If

$$E[Z(s+h_s) - Z(s)] = 0, \qquad \forall\, s, h_s,$$

and

$$0.5\mathrm{Var}\,[Z(s+h_s) - Z(s)] \text{ exists}, \qquad \forall\, s, h_s \text{ and depends only on } h_s,$$

the random function is said to be intrinsic stationary. Then

$$\gamma(h_s) = 0.5\mathrm{Var}\,[Z(s+h_s) - Z(s)]$$

is the variogram and will satisfy the above relationship with the covariance if the covariance exists. Brownian motion is an example of a random function for which the variogram exists but the covariance does not; i.e., it is not second-order stationary.

The covariance function might be interpreted as quantifying similarity, and the variogram quantifies dissimilarity. The variance of higher-order differences will lead to conditionally positive definite functions of higher orders; the coefficients in the generalized increments must satisfy certain conditions, however. For general discussion on geostatistics, see [11,12].

3.2. Product Space-Time Models

Let C_s be a positive-definite function in R^m and C_t be a positive-definite function on T; then the product model [13–18]

$$C_{st}(h_s, h_t) = C_s(h_s)C_t(h_t) \tag{5}$$

can be rewritten in terms of variograms as follows:

$$\gamma_{st}(h_s, h_t) = C_t(0)\gamma_s(h_s) + C_s(0)\gamma_t(h_t) - \gamma_s(h_s)\gamma_t(h_t). \tag{6}$$

There are several advantages to writing the product model in terms of the variograms:

- although the sum of two variograms, i.e., conditionally negative definite functions, is generally semidefinite and the product of two such functions will not be of the same type, when the sum and product are combined one can obtain a valid model;
- the "marginal" variograms are of interest.

Since for any variogram, $\gamma(0) = 0$, there are important special cases of (6):

(1) $\gamma_{st}(h_s, 0) = C_t(0)\gamma_s(h_s)$, and
(2) $\gamma_{st}(0, h_t) = C_s(0)\gamma_t(h_t)$.

Moreover, $C_t(0)$ is the sill of $\gamma_t(h_t)$ and $C_s(0)$ is the sill of $\gamma_s(h_s)$, and hence the two variograms completely determine the product model. If data is used to "model" these components separately, then the product model is fully determined.

EXAMPLE 1. Given the following variograms:

$$\gamma_s(h_s;b) = 1 - \frac{\|h_s\|^2}{1+\|h_s\|^2/b}, \qquad b > 0, \tag{7}$$

$$\gamma_t(h_t;c,d) = 1 - e^{-h_t/c}\cos\left(\frac{h_t}{d}\right), \qquad c,d > 0, \tag{8}$$

one obtains a nonmetric space-time variogram, i.e., a space-time radial basis function

$$\gamma_{st}(h_s,h_t) = 1 - \frac{\|h_s\|^2}{1+\|h_s\|^2/b} e^{-|h_t|/c}\cos\left(\frac{|h_t|}{d}\right).$$

EXAMPLE 2. Let

$$\begin{aligned} \gamma_s(h_s;b,\alpha) &= 1 - e^{-\|h_s\|^\alpha/b}, \qquad 1 \le \alpha \le 2, \quad b > 0, \\ \gamma_t(h_t;c) &= \begin{cases} 1.5\dfrac{h_t}{c} - 0.5\left(\dfrac{h_t}{c}\right)^3, & c > 0, \quad h_t \le c, \\ 1, & h_t > c. \end{cases} \end{aligned} \tag{9}$$

For both of these variograms, the sill is one. Note that the covariance corresponding to the time variogram has compact support.

In both of these examples, the space variogram is isotropic but a geometric anisotropy could still be introduced into the space variogram. Neither of these examples corresponds to the use of a metric on space-time.

3.3. Product-Sum Space Time Models

The form of the product covariance and the corresponding variogram suggests the use of a more general form, namely the product-sum [19,20]. As before, let $C_s(h_s)$ be a spatial covariance and $C_t(h_t)$ a temporal covariance. Then

$$C_{st}(h_s,h_t) = k_1 C_s(h_s)C_t(h_t) + k_2 C_s(h_s) + k_3 C_t(h_t) \tag{10}$$

is a space-time covariance for any $k_1 > 0$ and $k_2 \ge 0$, $k_3 \ge 0$. Rewritten in terms of variograms, it becomes

$$\gamma_{st}(h_s,h_t) = [k_2 + k_1 C_t(0)]\gamma_s(h_s) + [k_3 + k_1 C_s(0)]\gamma_t(h_t) - k_1\gamma_s(h_s)\gamma_t(h_t). \tag{11}$$

In this case, the marginal variograms are

$$\gamma_{st}(h_s,0) = [k_2 + k_1 C_t(0)]\gamma_s(h_s)$$

and

$$\gamma_{st}(0,h_t) = [k_3 + k_1 C_s(0)]\gamma_t(h_t).$$

Thus, only the proportionality constants are changed. By fitting the marginals to the data, the coefficients and the two component variograms can be determined.

For an example of a product-sum model applied to environmental data, see [21–23].

EXAMPLE 3. Let

$$\begin{aligned} \gamma_s(h_s;b,\alpha) &= 1 - \left(1 + \frac{\|h_s\|^2}{b^2}\right)^{-\alpha}, \qquad \alpha > 0, \quad b > 0, \\ \gamma_t(h_t;c) &= \begin{cases} \dfrac{h_t}{c}, & c > 0, \quad h_t \le c, \\ 1, & h_t > c. \end{cases} \end{aligned} \tag{12}$$

Note that the time model is only valid in one dimension, and hence, it could be used for a time model but not for a general radial basis function.

3.4. The Cressie-Huang Construction

If $C(h_s, h_t)$ is integrable, then it can be written in the form [24]

$$C(h_s, h_t) = \int_{R^k} e^{ih_s'\omega} \rho(\omega; h_t) k(\omega)\, d\omega, \tag{13}$$

where $\rho(\omega, \cdot)$ is a continuous autocorrelation function for each ω in R^k,

$$\int_{R_+} \rho(\omega; h_t)\, dh_t < \infty, \tag{14}$$

$$k(\omega) > 0, \qquad \text{and} \qquad \int_{\Re^k} k(\omega)\, d\omega < \infty. \tag{15}$$

EXAMPLE 4. Let

$$\rho(\omega; h_t) = e^{-\|\omega\|^2 h_t^2}$$

and

$$k(\omega) = e^{-c\|\omega\|}.$$

This results in a space-time covariance model of the form

$$\left[\frac{1}{(h_t^2 + c)^d}\right] \left[1 + \frac{\|h_s\|^2}{(h_t^2 + c)^2}\right]^{-(d+1)/2}.$$

3.5. Integrated Product-Sum Models

As noted above, given a family of variograms or covariances, i.e., radial basis functions, dependent on a parameter and a probability density function, then the integral is again a function of the same type. This construction can be extended to both the product and the product-sum construction. The integrated product is almost a generalization of the Cressie-Huang construction. The following theorem is found in [25].

THEOREM 1. *Let $\mu(a)$ be a positive measure over $U \subseteq \Re$, and let $C_s(h_s; a)$ and $C_t(h_t; a)$ be valid covariance functions, respectively, in $D \subset R^n$ and $T \subset R_+$, for each $a \in V \subseteq U$.*

(a) *If $C_s(h_s; a) \cdot C_t(h_t; a)$ is integrable with respect to the measure μ over V for each h_s and h_t, given $k > 0$, then*

$$C_{s,t}(h_s, h_t) = \int_V kC_s(h_s; a) C_t(h_t; a)\, d\mu(a) \tag{16}$$

is a valid covariance function in $D \times T$.

(b) *Likewise, if $k_1 C_s(h_s; a) C_t(h_t; a) + k_2 C_s(h_s; a) + k_3 C_t(h_t; a)$ is integrable with respect to the measure μ over V for each h_s and h_t, given $k_1 > 0$, $k_2 \geq 0$, and $k_3 \geq 0$, then*

$$C_{s,t}(h_s, h_t) = \int_V [k_1 C_s(h_s; a) C_t(h_t; a) + k_2 C_s(h_s; a) + k_3 C_t(h_t; a)]\, d\mu(a) \tag{17}$$

is a valid covariance function in $D \times T$.

Since the product and the product-sum covariance models can be written in terms of the associated variograms, it follows that

$$\gamma_{s,t}(h_s, h_t) = \int_V k[C_t(0; a)\gamma_s(h_s; a) + C_s(0; a)\gamma_t(h_t; a) - \gamma_s(h_s; a)\gamma_t(h_t; a)]\, d\mu(a) \tag{18}$$

and

$$\begin{aligned}\gamma_{s,t}(h_s, h_t) = \int_V & [(k_2 + k_1 C_t(0; a))\gamma_s(h_s; a) \\ & + (k_3 + k_1 C_s(0; a))\gamma_t(h_t; a) - k_1\gamma_s(h_s; a)\gamma_t(h_t; a)]\, d\mu(a)\end{aligned} \tag{19}$$

are valid space-time variograms assuming that $\gamma_s(h_s; a)$ and $\gamma_t(h_t; a)$ are valid spatial and temporal variogram models for each choice of $a \in V$, and $C_s(0; a)$ and $C_t(0; a)$ are the corresponding sill values.

The following example is taken from [25].

EXAMPLE 5. Let

$$\begin{aligned} C_s(h_s; a, b, \alpha) &= e^{-a\|h_s\|^\alpha/b}, & 1 \le \alpha \le 2, \quad a > 0, \quad b > 0, \\ C_t(h_t; a, c, \delta) &= e^{-ah_t^\delta/c}, & 1 \le \delta \le 2, \quad c > 0, \\ \phi(a, n, \beta) &= \frac{\beta^{n+1}}{\Gamma(n+1)}\, a^n e^{-\beta a}, & n \ge 0, \qquad \beta > 0. \end{aligned}$$

Since $C_s(h_s; a, b, \alpha)$ and $C_t(h_t; a, c, \delta)$ are, respectively, valid spatial and temporal covariance models for each choice of a over the interval $V = [0; +\infty[$, the integrability conditions are satisfied. Two new classes of nonseparable space-time covariances can be obtained

$$\begin{aligned} C_{s,t}(h_s, h_t; \boldsymbol{\theta}_1) &= \int_V k e^{-a\|h_s\|^\alpha/b} \cdot e^{-ah_t^\delta/c} \cdot \frac{\beta^{n+1}}{\Gamma(n+1)}\, a^n e^{-\beta a}\, da \\ &= \frac{k\beta^{n+1}}{\Gamma(n+1)} \int_V a^n e^{-a\left(\|h_s\|^\alpha/b + h_t^\delta/c + \beta\right)}\, da = \frac{k\beta^{n+1}}{\left(\|h_s\|^\alpha/b + h_t^\delta/c + \beta\right)^{n+1}}, \end{aligned} \tag{20}$$

where $\boldsymbol{\theta}_1 = (b, c, n, k, \alpha, \beta, \delta)$;

$$\begin{aligned} C_{s,t}(h_s, h_t; \boldsymbol{\theta}_2) &= \int_V \left[k_1 e^{-a\|h_s\|^\alpha/b} \cdot e^{-ah_t^\delta/c} + k_2 e^{-a\|h_s\|^\alpha/b} + k_3 e^{-ah_t^\delta/c}\right] \frac{\beta^{n+1}}{\Gamma(n+1)} a^n e^{-\beta a}\, da \\ &= k_1 \frac{\beta^{n+1}}{\left(\|h_s\|^\alpha/b + h_t^\delta/c + \beta\right)^{n+1}} + k_2 \frac{\beta^{n+1}}{\left(\|h_s\|^\alpha/b + \beta\right)^{n+1}} \\ &\quad + k_3 \frac{\beta^{n+1}}{\left(h_t^\delta/c + \beta\right)^{n+1}}, \end{aligned} \tag{21}$$

where $\boldsymbol{\theta}_2 = (b, c, n, k_1, k_2, k_3, \alpha, \beta, \delta)$.

Note that, when $\alpha = \delta = 1$ and $n = 0$, $(\|h_s\|/b + h_t/c)$ in (20) and (21) might correspond to a space-time metric and $\beta/(\|h_s\|^\alpha/b + h_t^\delta/c + \beta)$ would belong to a well-known family of covariance models, namely

$$C(h; w_1, w_2) = \frac{w_1}{w_2 + \|h\|}.$$

3.6. A More General Construction

Let $\gamma_{st}(h_s, h_t)$ be any valid space-time variogram, obtained by whatever construction. Moreover, let $\gamma_{s1}(h_s)$ be any valid space variogram and $\gamma_{t1}(h_t)$ any valid temporal variogram. If $K_1 > 0$, $K_2 \ge 0$, $K_3 \ge 0$, then

$$K_1\gamma_{st}(h_s, h_t) + K_2\gamma_{s1}(h_s) + K_3\gamma_{t1}(h_t)$$

is a valid space-time variogram. Although all of the constructions described above result in space-time variograms that correspond to covariances (and hence, are bounded), neither $\gamma_{s1}(h_s)$ nor $\gamma_{t1}(h_t)$ has to be bounded in this general construction. Obviously, the product of any two space-time covariances is again a valid space-time covariance.

4. NUMERICAL RESULTS

To illustrate the differences in the different constructions for space-time radial basis functions, a metric and a nonmetric space-time model has been used to interpolate a function defined in space-time. The function to be interpolated is

$$F(x, y, t) = \frac{1}{(x^2 + y^2 + 1)} \left(1 + \sqrt{t}\right).$$

Note that F is the product of a function $F_1(t)$ in one-space and a function $F_2(x, y)$ in two-space. A regular grid A in (x, y, t) of $13 \times 13 \times 49$ data points has been selected, where

$$A = \{(x_i, y_j, t_k),\ i, j = 1, \dots, 13;\ k = 1, \dots, 49\},$$

with

$$\begin{aligned} x_i &= (i-1) \times 0.1, & i &= 1, \dots, 13, \\ y_j &= (j-1) \times 0.1, & j &= 1, \dots, 13, \\ t_k &= (k-1) \times 0.025, & k &= 1, \dots, 49. \end{aligned}$$

Two subsets of A, denoted as B and C, have been also been selected,

$$B = \{(x_i, y_j, t_k),\ i, j = 1, \dots, 7;\ k = 1, \dots, 25\},$$

with

$$\begin{aligned} x_i &= (i-1) \times 0.2, & i &= 1, \dots, 7, \\ y_j &= (j-1) \times 0.2, & j &= 1, \dots, 7, \\ t_k &= (k-1) \times 0.05, & k &= 1, \dots, 25, \end{aligned}$$

and

$$C = \{(x_i, y_j, t_k),\ i, j = 1, \dots, 5;\ k = 1, \dots, 49\},$$

with

$$\begin{aligned} x_i &= (i-1) \times 0.3, & i &= 1, \dots, 5, \\ y_j &= (j-1) \times 0.3, & j &= 1, \dots, 5, \\ t_k &= (k-1) \times 0.025, & k &= 1, \dots, 49. \end{aligned}$$

The subsets B and C are considered as "control" data sets, while $D = A - B$ and $E = A - C$ are the "test" data sets. The functional values at the points in B, C will be used to interpolate the function at the points in D, E, respectively. Two different space-time models will be used—one a metric model and one a product model. By comparing the interpolated values at the points in D, E with the computed values, one obtains a measure of the efficacy of the different space-time models. The two space-time radial basis functions are

$$\gamma_1(h) = \left(1 - e^{-\|h\|}\right), \qquad \|h\| = \sqrt{\left(h_x^2 + h_y^2 + h_t^2\right)}, \tag{22}$$

$$\gamma_2(h_{xy}, h_t) = \left(1 - e^{-\|h_{xy}\|}\right)\left(1 - e^{-h_t}\right), \qquad \|h_{xy}\| = \sqrt{\left(h_x^2 + h_y^2\right)}. \tag{23}$$

4.0.1. Using γ_2 and set D

The maximum absolute error was 0.0545, the mean error was 0.0031, and the standard deviation of the errors was 0.0047. When the errors were normalized by the true value, the maximum normalized error was 0.0499, the mean normalized error was 0.0037, and the standard deviation of the normalized errors was 0.0065.

4.0.2. Using γ_2 and set E

The maximum absolute error was 0.0367, the mean error was 0.0065, and the standard deviation of the errors was 0.0064. When the errors were normalized by the true value, the maximum normalized error was 0.0204, the mean normalized error was 0.0067, and the standard deviation of the normalized errors was 0.0048.

4.0.3. Using γ_1 and set D

The maximum error was 0.0516, the mean error was 0.002, and the standard deviation of the errors was 0.0065. When the errors were normalized by the true value, the maximum normalized error was 0.0585, the mean normalized error was 0.004, and the standard deviation of the normalized errors was 0.0095.

4.0.4. Using γ_1 and set E

The maximum error was 0.0586, the mean error was 0.0055, and the standard deviation of the errors was 0.0032. When the errors were normalized by the true value, the maximum normalized error was 0.0715, the mean normalized error was 0.0075, and the standard deviation of the normalized errors was 0.0122.

5. SUMMARY AND CONCLUSIONS

Neither the form of the radial basis function interpolator nor the equations that determine the coefficients in the interpolator require that the basis function be "radial". By allowing zonal anisotropies in the basis function, it is possible to either split the higher-dimensional Euclidean space into independent lower-dimensional spaces or to allow for nonmetric space-time radial basis functions. The difficulty in generating nonmetric radial basis functions is to ensure that they are definite and not simply semidefinite. For example, the sum of a multiquadratic in space and a multiquadratic in time would only be semidefinite. There are two general constructions for generating nonmetric space-time radial basis functions; one is based on the sum and product of a positive definite function defined on space and one defined on time, and the second construction is an integral form of the first construction. Several examples of these constructions have been given. There is a numerical example contrasting the use of a metric and a nonmetric model for the radial basis function.

The use of cross-validation to choose the radial basis function, as described in [26], is directly extendable to the use of space-time radial basis functions. The generalization of the radial basis function interpolator to allow for filtering noise in the data, as described in [27,28], is likewise directly extendable to space-time radial basis functions.

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